Logic of Autonomous Dynamical Systems

André Platzer

Karlsruhe Institute of Technology

Computer Science Department
Carnegie Mellon University

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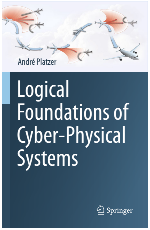
http://keymaeraX.org/



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Outline

Outline (Introduction to Dynamical Systems)

Cyber-Physical Systems Analysis: Aircraft Example

Which control decisions are safe for aircraft collision avoidance?

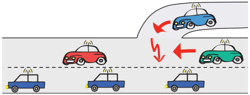
Cyber-Physical Systems

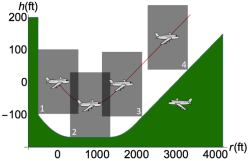
CPSs combine cyber capabilities with physical capabilities
to solve problems that neither part could solve alone.

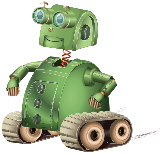
Cyber-Physical Systems Promise Transformative Impact!

Prospects: Safety & Efficiency

(Autonomous) cars | (Auto)Pilot support | Robots near humans







Cyber-Physical Systems

CPSs combine cyber capabilities with physical capabilities
to solve problems that neither part could solve alone.

Can you trust a computer to control physics?

Can you trust a computer to control physics?

1. Depends on how it has been programmed
2. And on what will happen if it malfunctions

Rationale

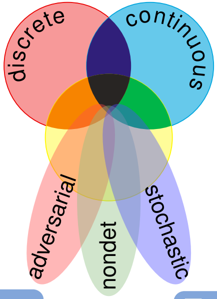
1. Safety guarantees require analytic foundations.
2. A common foundational core helps all application domains.
3. Foundations revolutionized digital computer science & our society.
4. Need even stronger foundations when software reaches out into our physical world.

CPSs deserve proofs as safety evidence!

CPSs are Multi-Dynamical Systems

CPS Dynamics

CPS are characterized by multiple facets of dynamical systems.



CPS Compositions

CPS combines multiple simple dynamical effects.

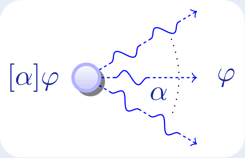
Descriptive simplification

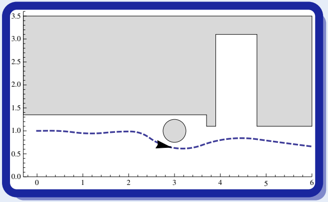
Tame Parts

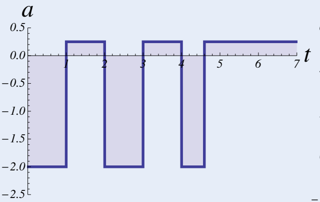
Exploiting compositionality tames CPS complexity.

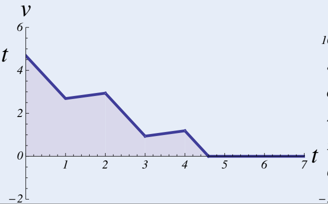
Analytic simplification

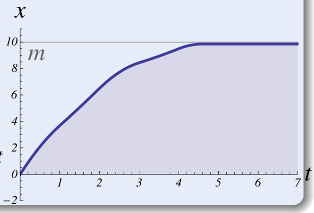
Concept (Differential Dynamic Logic) (JAR'08,LICS'12)











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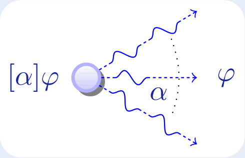

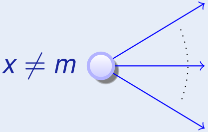

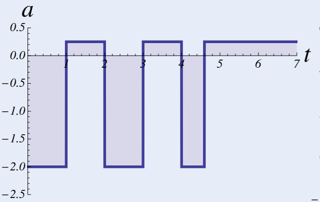

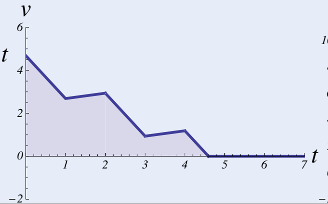

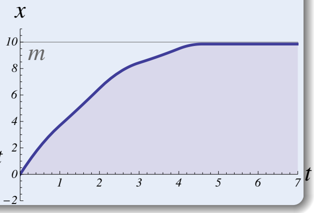

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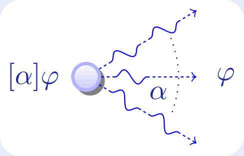

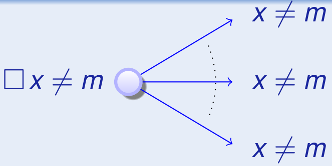

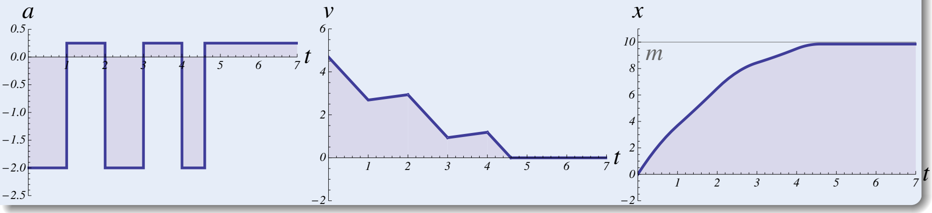

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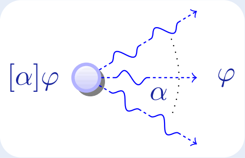

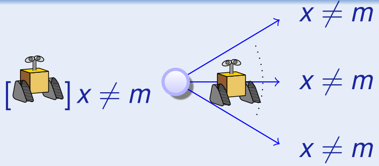

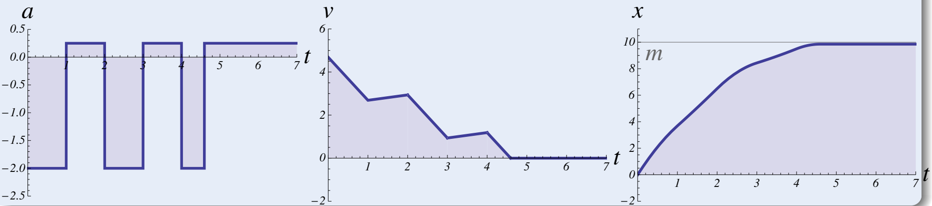

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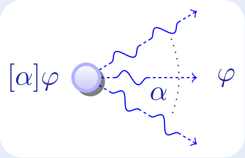

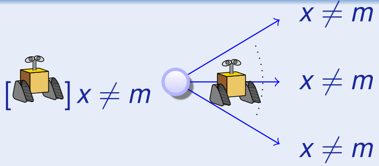

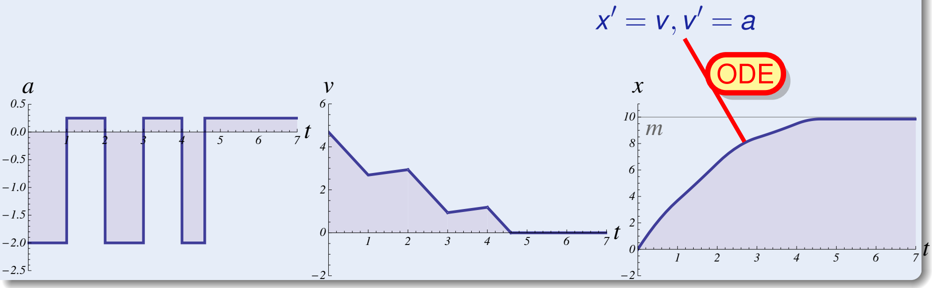

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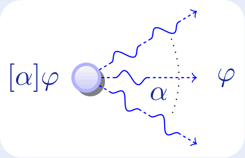

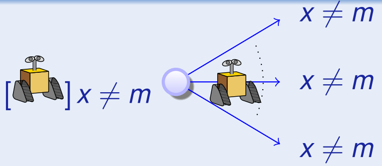

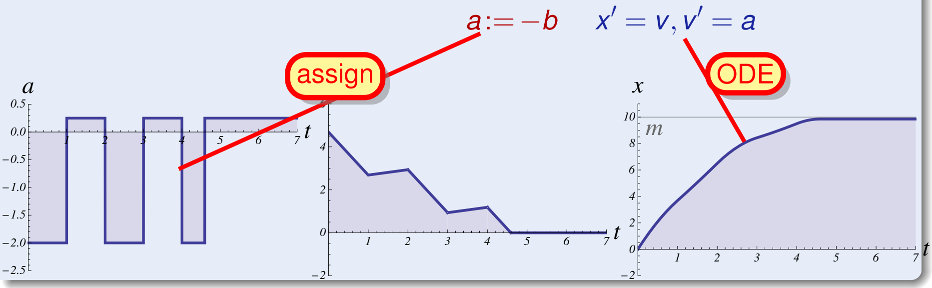

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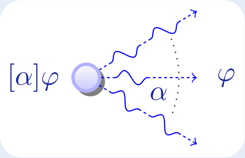
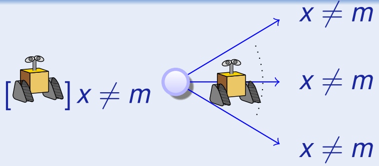
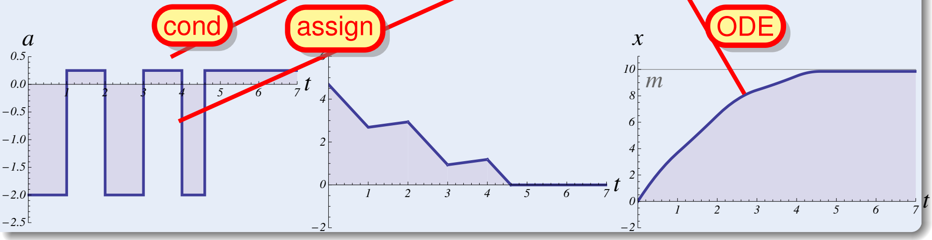




$(\text{if}(SB(x,m)) \quad a := -b) \quad x' = v, v' = a$



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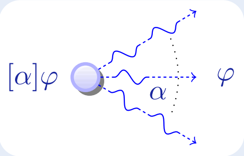


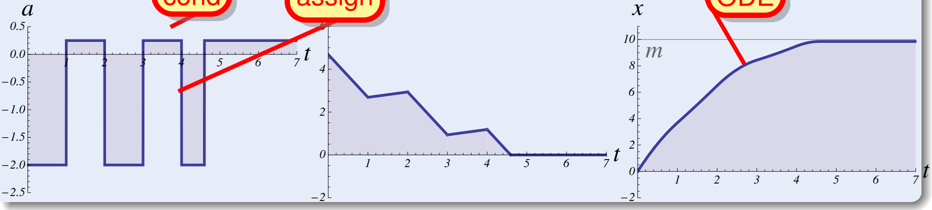
seq. compose

cond

assign

ODE

$(\text{if}(SB(x,m))\ \ a := -b)\ ;\ x' = v, v' = a$



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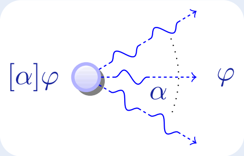


seq. compose

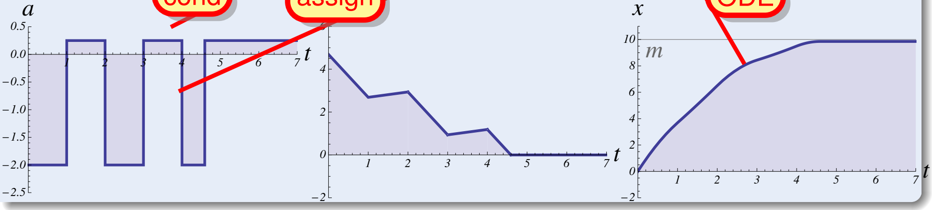
nondet. repeat

$$\big((\text{if}(SB(x,m))\ \ a := -b)\ ;\ x' = v, v' = a\big)^*$$

cond

assign

ODE



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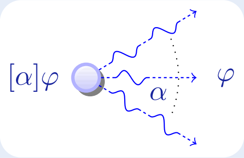
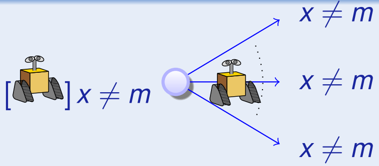
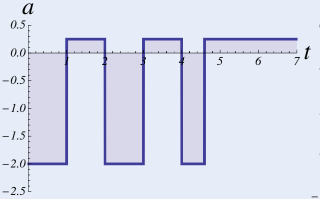
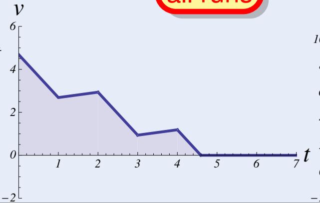


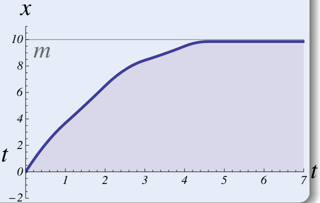


$$\left[\left(\mathrm{if}(\mathrm{SB}(x,m))\quad a:=-b\right);\ x'=v, v'=a\right)^*\Big]\underbrace{x\neq m}_{\text{post}}$$

all runs







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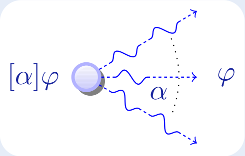
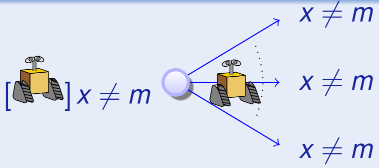
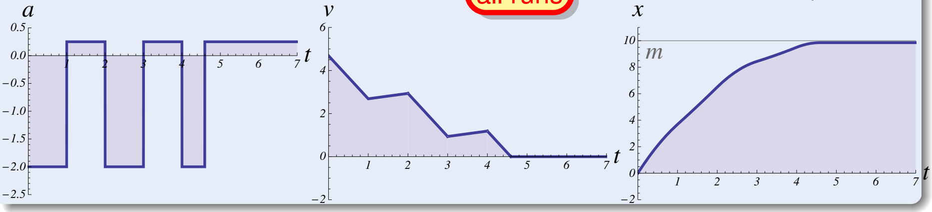




$$\underbrace{x \neq m \land b > 0}_{\text{init}} \rightarrow \Big[\big((\text{if}(\text{SB}(x,m))\quad a := -b)\;;\; x' = v, v' = a\big)^*\Big]\underbrace{x \neq m}_{\text{post}}$$

all runs



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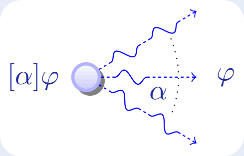
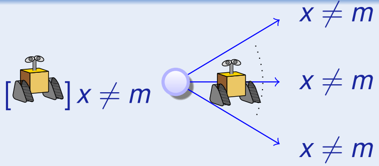
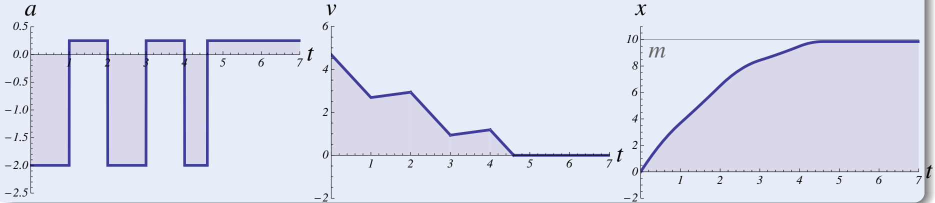




nondet. choice

$$\underbrace{x \neq m \land b > 0}_{\text{init}} \rightarrow \left[\left(\left(?\neg SB(x,m) \cup a := -b\right) ; x' = v, v' = a\right)^*\right] \underbrace{x \neq m}_{\text{post}}$$



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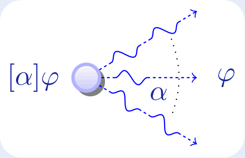
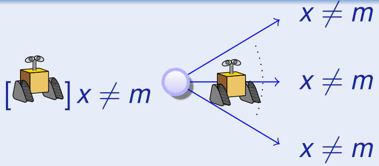
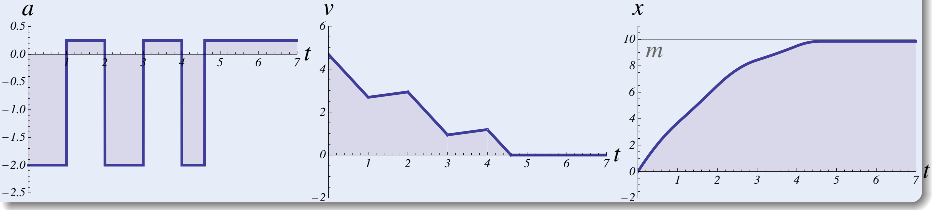




test

nondet. choice

$$\underbrace{x \neq m \wedge b > 0}_{\text{init}} \rightarrow \left[\left(\left(?\neg SB(x,m) \cup a := -b\right) ;\ x' = v, v' = a\right)^*\right]\underbrace{x \neq m}_{\text{post}}$$



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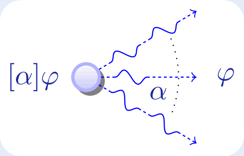
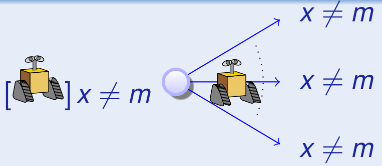




$$\underbrace{x \neq m \land b > 0}_{\text{init}} \rightarrow \Big[\big((?\neg \mathrm{SB}(x,m) \cup a := -b)\ ;\ x' = v, v' = a\big)^*\Big]\underbrace{x \neq m}_{\text{post}}$$

hybrid program dynamics

Outline (Modeling Dynamical Systems)

Hybrid Programs: Syntax

Definition (Syntax of hybrid program α)

$$\alpha, \beta \ ::= \ x := e \mid ?Q \mid x' = f(x) \,\&\, Q \mid \alpha \cup \beta \mid \alpha;\beta \mid \alpha^*$$

Hybrid Programs: Syntax

Definition (Syntax of hybrid program α)

$$\alpha, \beta ::= x := e \mid ?Q \mid x' = f(x) \,\&\, Q \mid \alpha \cup \beta \mid \alpha ; \beta \mid \alpha^*$$

- $x := e$: Discrete Assign
- $?Q$: Test Condition
- $x' = f(x) \,\&\, Q$: Differential Equation
- $\alpha \cup \beta$: Nondet. Choice
- $\alpha ; \beta$: Seq. Compose
- α^*: Nondet. Repeat

Hybrid Programs: Syntax

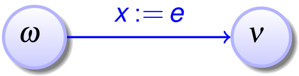
Definition (Syntax of hybrid program α)

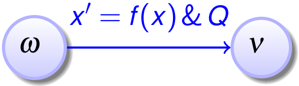
$$\alpha,\beta \;::=\; x:=e \mid ?Q \mid x'=f(x)\,\&\,Q \mid \alpha\cup\beta \mid \alpha;\beta \mid \alpha^*$$

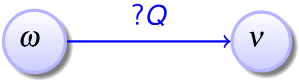
Discrete Assign | Test Condition | Differential Equation | Nondet. Choice | Seq. Compose | Nondet. Repeat

Like regular expressions. Everything nondeterministic

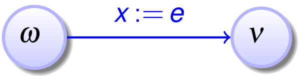
Hybrid Programs: Semantics

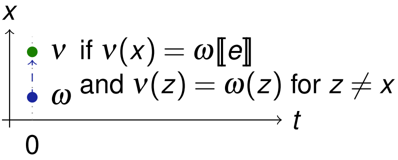


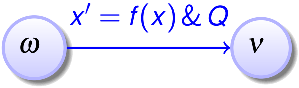


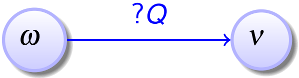


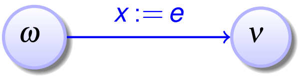
Hybrid Programs: Semantics



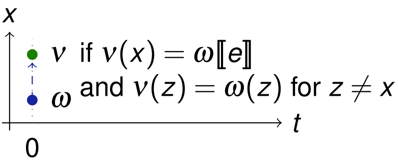




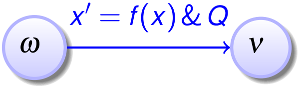




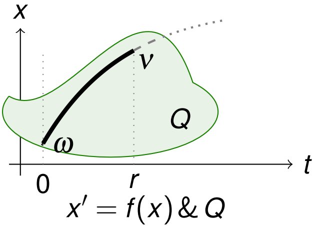
ω
x := e
ν



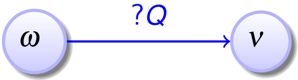
x
ν if ν(x) = ω⟦e⟧
ω and ν(z) = ω(z) for z ≠ x
t
0



ω
x′ = f(x) & Q
ν

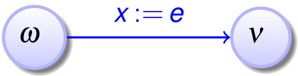


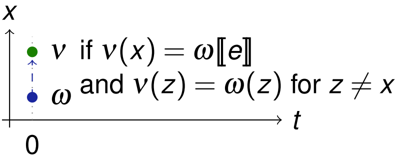
x
ν
Q
ω
t
0
r
x′ = f(x) & Q

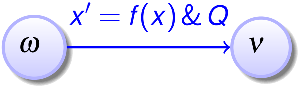


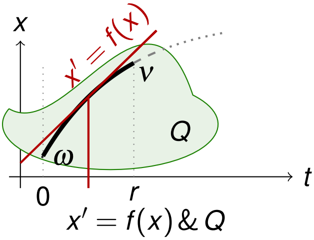
ω
?Q
ν

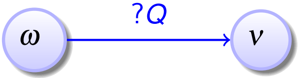
Hybrid Programs: Semantics



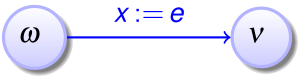


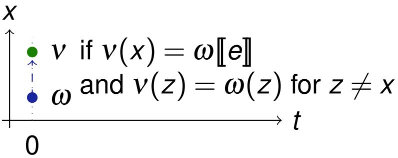


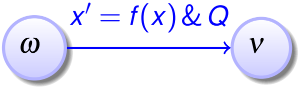


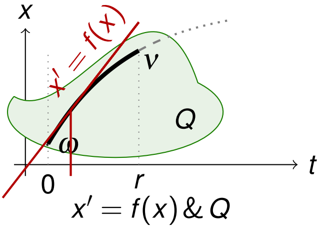


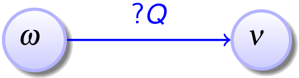
Hybrid Programs: Semantics

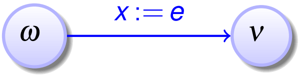


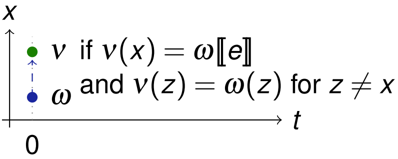


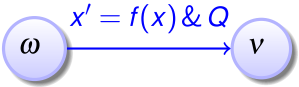


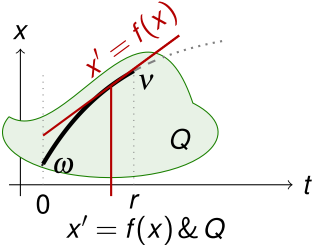


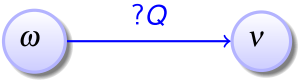



ω
$x := e$
ν

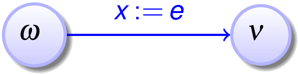

x
ν if $\nu(x) = \omega[\![e]\!]$
ω and $\nu(z) = \omega(z)$ for $z \neq x$
t
0

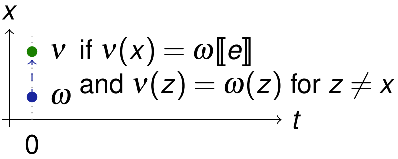

ω
$x' = f(x) \,\&\, Q$
ν

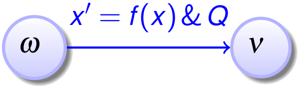

x
$x' = f(x)$
ν
Q
ω
t
0
r
$x' = f(x) \,\&\, Q$

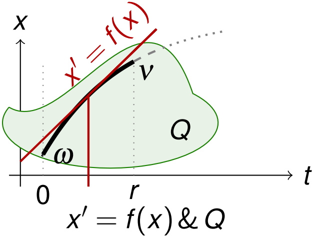

ω
$?Q$
ν

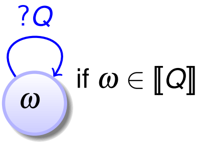
Hybrid Programs: Semantics

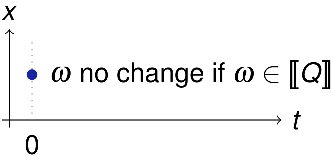




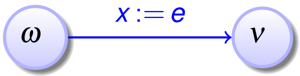


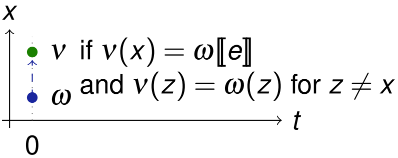


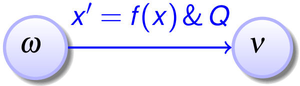


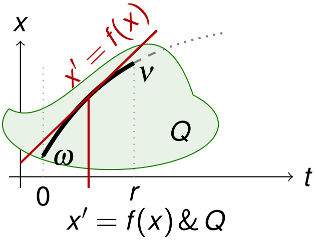


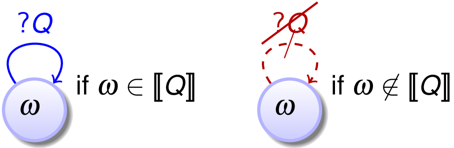
Hybrid Programs: Semantics

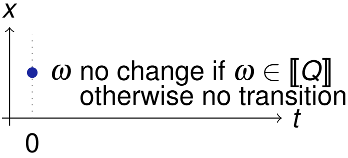




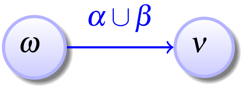


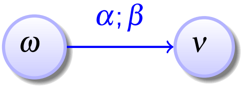


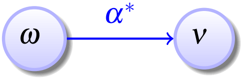




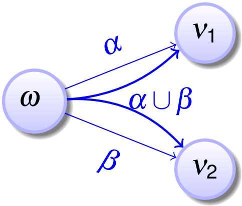
Hybrid Programs: Semantics

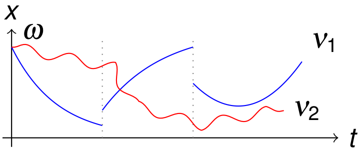


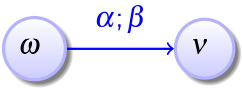


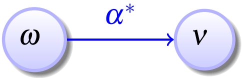


Hybrid Programs: Semantics

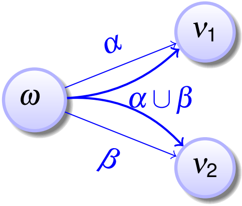


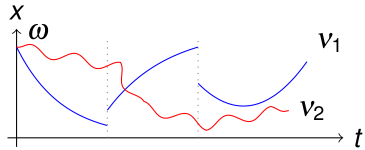


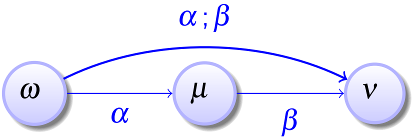


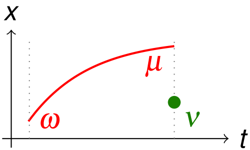


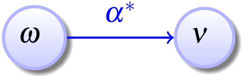
Hybrid Programs: Semantics



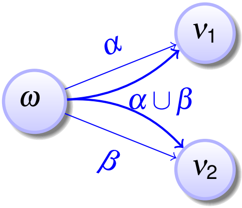


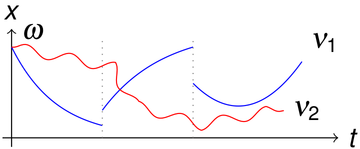


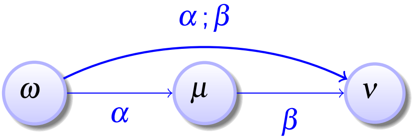


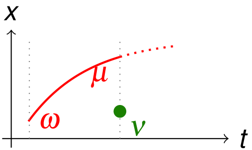


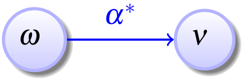
Hybrid Programs: Semantics



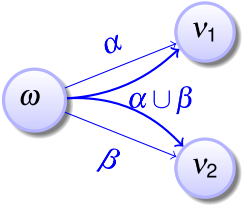


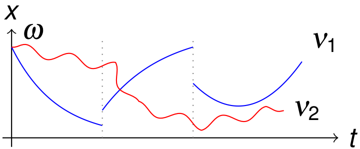


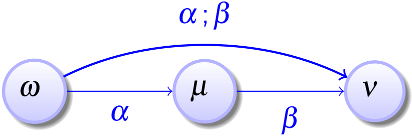


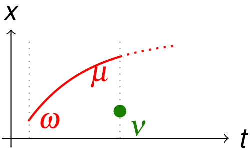


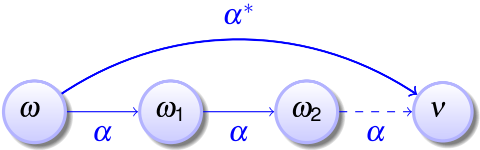
Hybrid Programs: Semantics

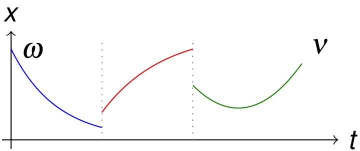




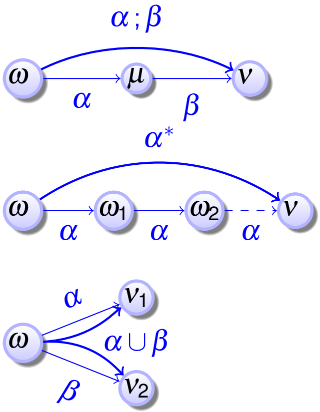




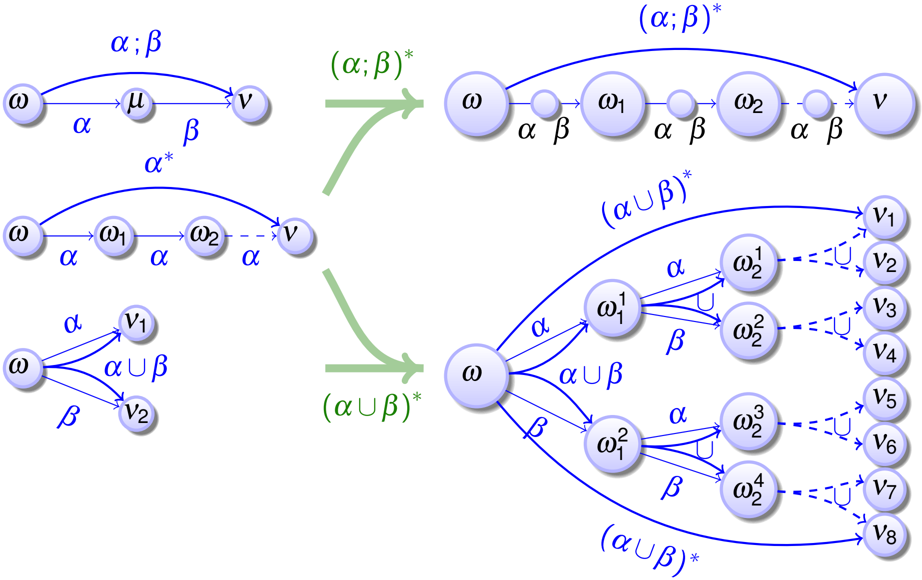




Plug-in for Semantics of Composed Hybrid Programs



Plug-in for Semantics of Composed Hybrid Programs



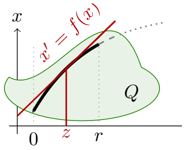
Definition (Syntax of hybrid program α)

$$\alpha,\beta ::= x := e \mid ?Q \mid x' = f(x) \,\&\, Q \mid \alpha \cup \beta \mid \alpha;\beta \mid \alpha^*$$

Definition (Semantics of hybrid programs) $(\llbracket\cdot\rrbracket : \mathrm{HP} \to \wp(\mathscr{S} \times \mathscr{S}))$

$$\begin{aligned}
\llbracket x := e \rrbracket &= \{(\omega,\nu) : \nu = \omega \text{ except } \nu\llbracket x\rrbracket = \omega\llbracket e\rrbracket\} \\
\llbracket ?Q \rrbracket &= \{(\omega,\omega) : \omega \in \llbracket Q\rrbracket\} \\
\llbracket x' = f(x) \rrbracket &= \{(\varphi(0),\varphi(r)) : \varphi \models x' = f(x) \text{ for some duration } r \geq 0\} \\
\llbracket \alpha \cup \beta \rrbracket &= \llbracket\alpha\rrbracket \cup \llbracket\beta\rrbracket \\
\llbracket \alpha;\beta \rrbracket &= \llbracket\alpha\rrbracket \circ \llbracket\beta\rrbracket = \{(\omega,\nu) : (\omega,\mu) \in \llbracket\alpha\rrbracket \text{ and } (\mu,\nu) \in \llbracket\beta\rrbracket\} \\
\llbracket \alpha^* \rrbracket &= \llbracket\alpha\rrbracket^* = \bigcup_{n\in\mathbb{N}} \llbracket\alpha^n\rrbracket \qquad \alpha^n \equiv \underbrace{\alpha;\alpha;\alpha;\ldots;\alpha}_{n \text{ times}}
\end{aligned}$$

compositional



Definition (Syntax of hybrid program α)

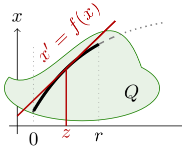
$$\alpha, \beta \ ::= \ x := e \mid ?Q \mid x' = f(x) \,\&\, Q \mid \alpha \cup \beta \mid \alpha ; \beta \mid \alpha^*$$

Definition (Semantics of hybrid programs) ($[\![\cdot]\!] : \mathrm{HP} \to \wp(\mathscr{S} \times \mathscr{S})$)

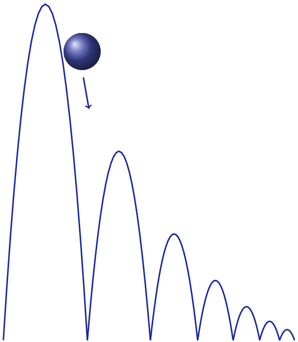
$$\begin{aligned}
[\![x := e]\!] &= \{(\omega, \nu) \ : \ \nu = \omega \text{ except } \nu[\![x]\!] = \omega[\![e]\!]\} \\
[\![?Q]\!] &= \{(\omega, \omega) \ : \ \omega \in [\![Q]\!]\} \\
[\![x' = f(x)]\!] &= \{(\varphi(0), \varphi(r)) \ : \ \varphi \models x' = f(x) \text{ for some duration } r \geq 0\} \\
[\![\alpha \cup \beta]\!] &= [\![\alpha]\!] \cup [\![\beta]\!] \\
[\![\alpha ; \beta]\!] &= [\![\alpha]\!] \circ [\![\beta]\!] \\
[\![\alpha^*]\!] &= [\![\alpha]\!]^* = \bigcup_{n \in \mathbb{N}} [\![\alpha^n]\!]
\end{aligned}$$

compositional

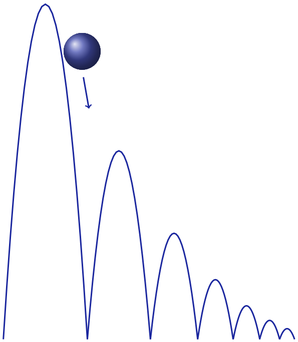
1. $\varphi(z)(x') = \frac{\mathrm{d}\varphi(t)(x)}{\mathrm{d}t}(z)$ exists at all times $0 \leq z \leq r$
2. $\varphi(z) \in [\![x' = f(x) \wedge Q]\!]$ for all times $0 \leq z \leq r$
3. $\varphi(z) = \varphi(0)$ except at x, x'



Quantum the Acrophobic Bouncing Ball

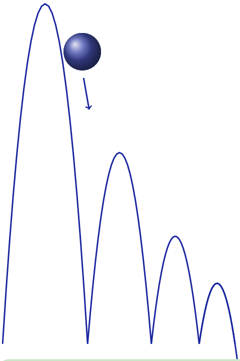


Example (Quantum the Bouncing Ball)



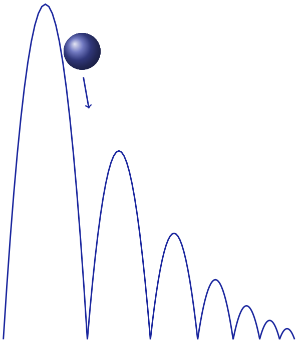
Example (Quantum the Bouncing Ball)

$$\{x' = v, v' = -g\}$$



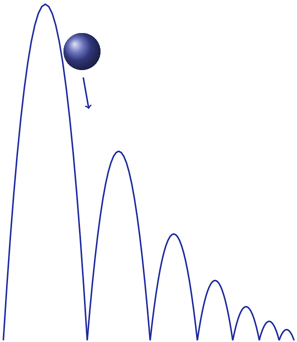
Example (Quantum the Bouncing Ball)

$$\{x' = v, v' = -g\}$$



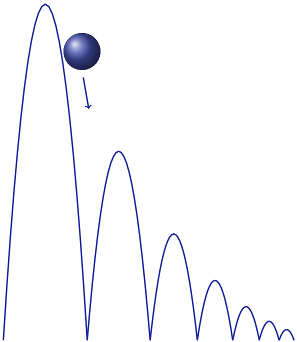
Example (Quantum the Bouncing Ball)

$$\{x' = v, v' = -g \,\&\, x \geq 0\}$$



Example (Quantum the Bouncing Ball)

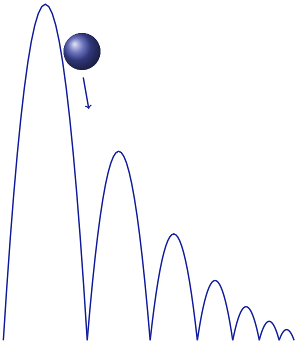
$$\{x' = v, v' = -g \,\&\, x \geq 0\};$$
$$\text{if}(x = 0)\ \ v := -cv$$



Example (Quantum the Bouncing Ball)

$$\big(\{x' = v, v' = -g \,\&\, x \geq 0\};$$
$$\text{if}(x = 0)\ \ v := -cv\big)^*$$

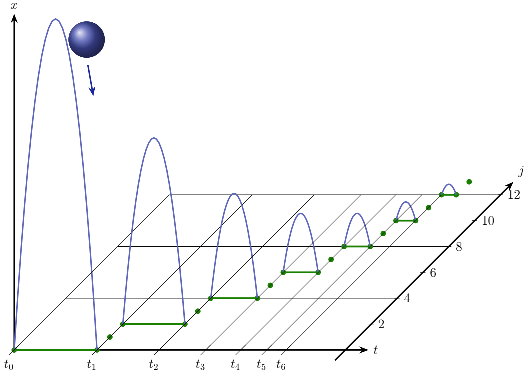
Quantum Discovered a Crack in the Fabric of Time



Example (Quantum the Bouncing Ball)

$$\big(\{x' = v, v' = -g \,\&\, x \geq 0\};\\ \text{if}(x = 0)\ \ v := -cv\big)^*$$

Quantum Discovered a Crack in the Fabric of Time



Example (Quantum the Bouncing Ball)

$$\big(\{x' = v, v' = -g \,\&\, x \geq 0\};$$
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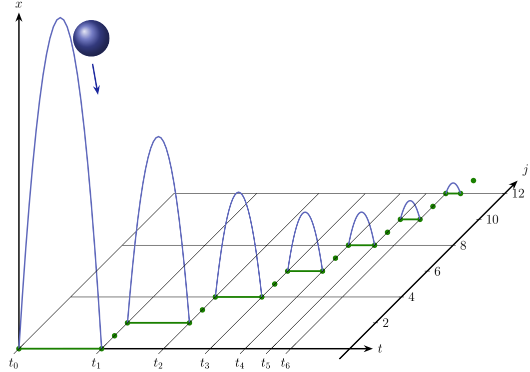
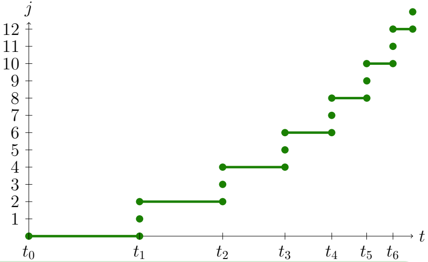
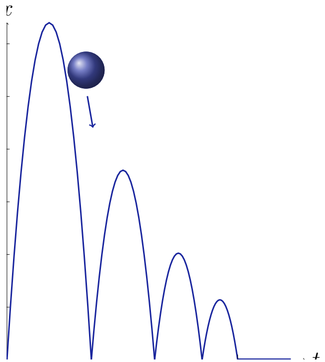
Quantum Discovered a Crack in the Fabric of Time





Example (Quantum the Bouncing Ball)

$$\Big(\{x' = v, v' = -g \,\&\, x \geq 0\};\ \text{if}(x = 0)\ \ v := -cv\Big)^*$$



$\text{if}(Q)\,\alpha\,\text{else}\,\beta \equiv$

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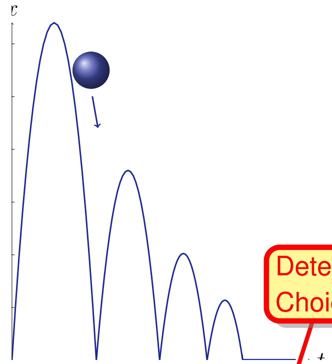


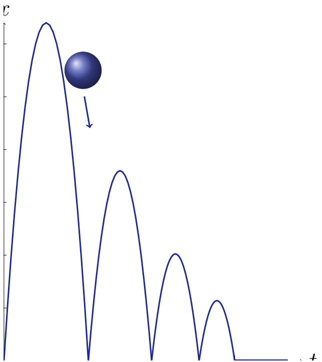
$$\text{if}(Q)\,\alpha\,\text{else}\,\beta \equiv (?Q;\alpha) \cup (?\neg Q;\beta)$$

Determ. Choice

Nondet. Choice

Example (Quantum the Bouncing Ball)

$$(\{x' = v, v' = -g \,\&\, x \geq 0\};$$
$$\text{if}(x = 0)\,(v := -cv \cup v := 0))^*$$



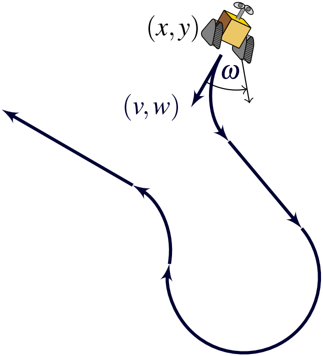
Nondet. Assign

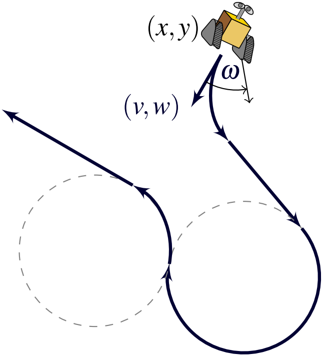
Test Limits

Example (Quantum the Bouncing Ball)

$$\big(\{x' = v, v' = -g \,\&\, x \geq 0\};$$
$$\text{if}(x = 0)\,(c := *; ?c \geq 0; v := -cv)\big)^*$$

Runaround Robot





Example (Runaround Robot)

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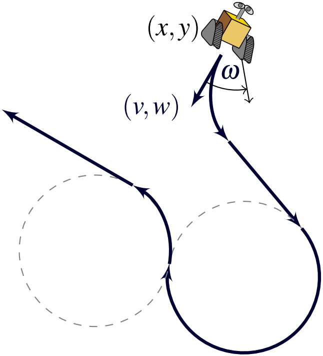

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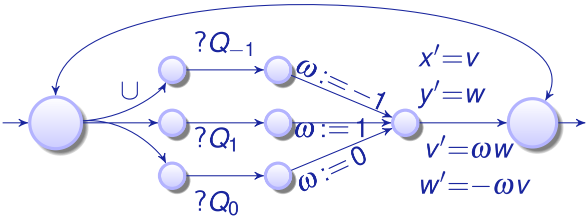

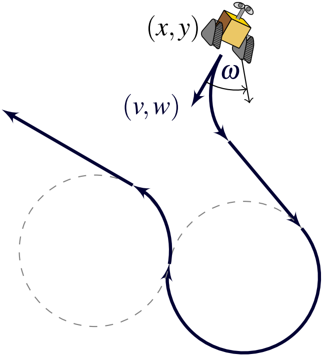

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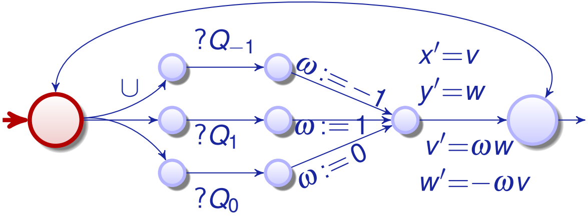

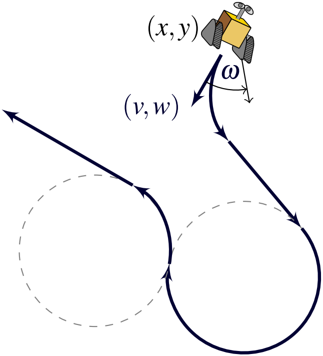

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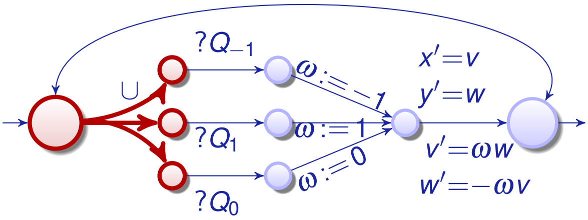

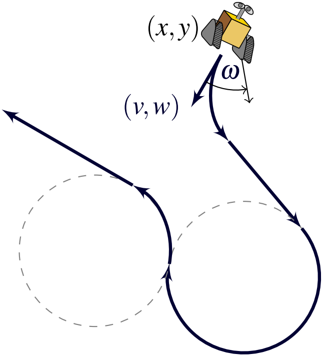

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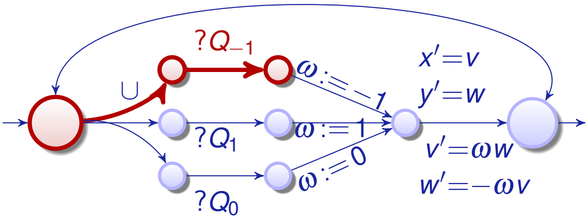

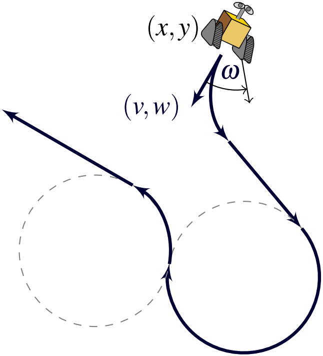

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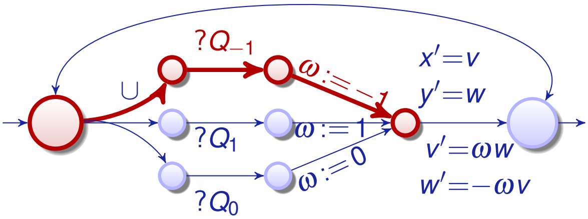

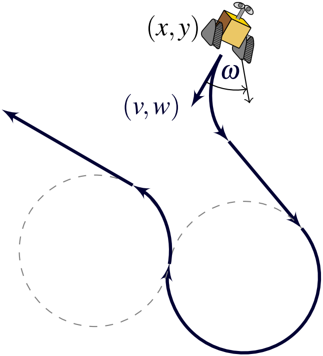

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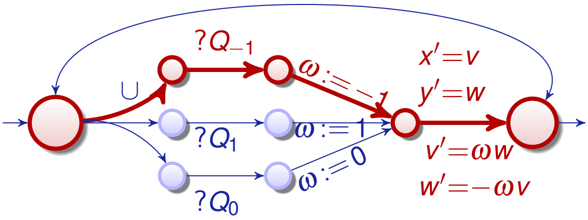

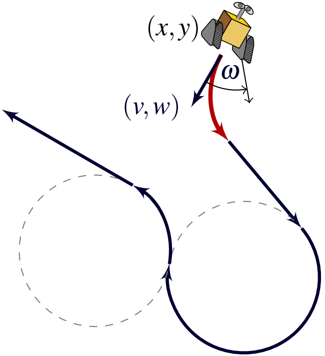

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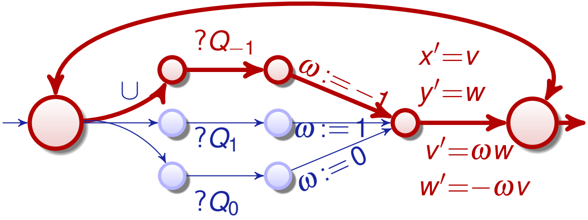

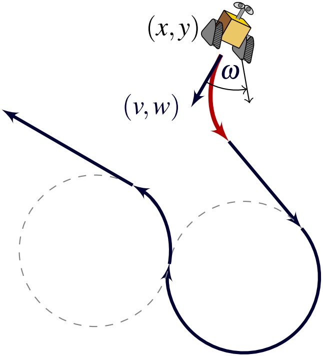

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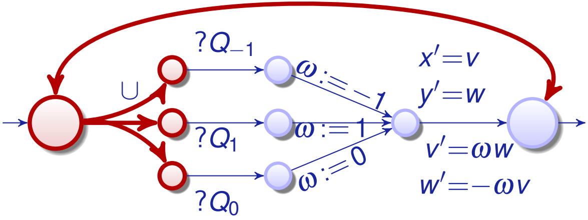

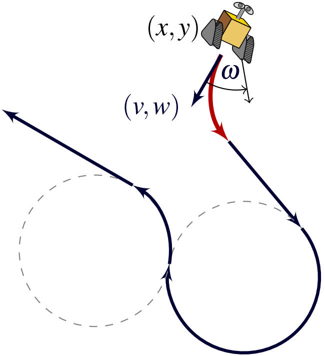

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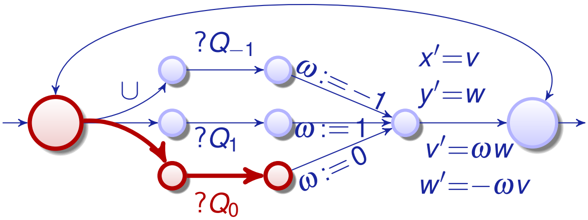

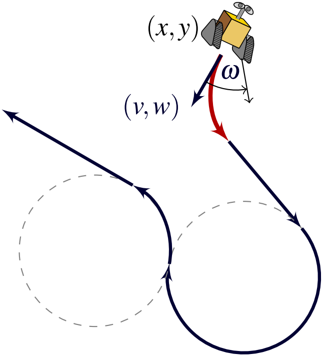

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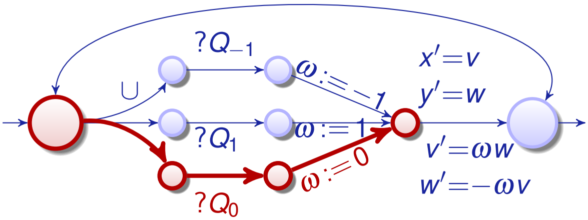

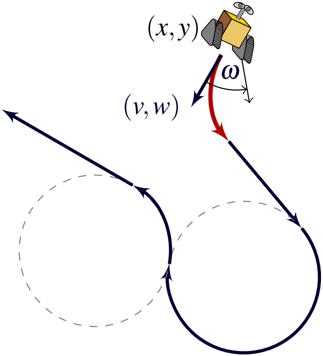

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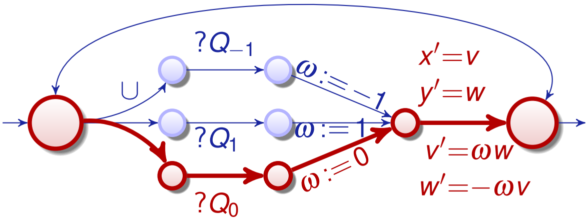

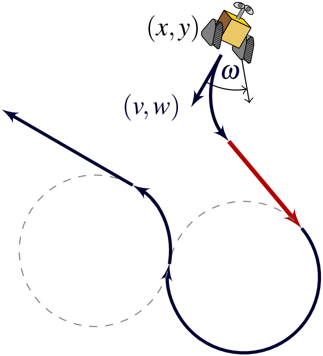

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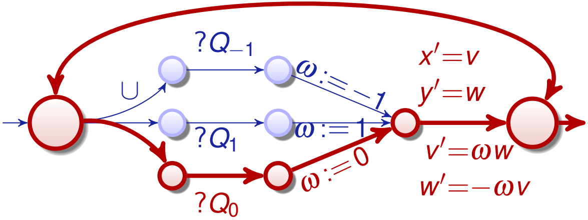

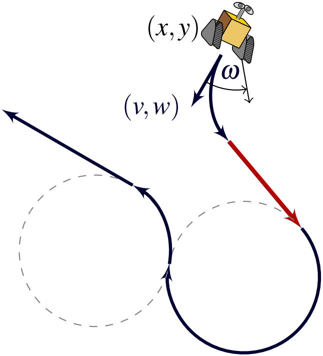

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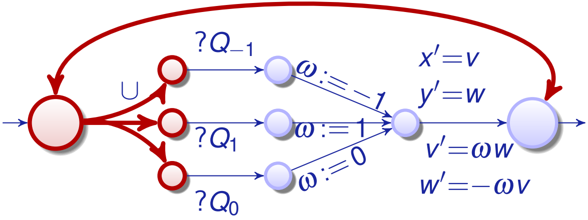

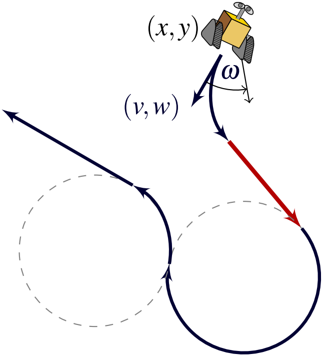

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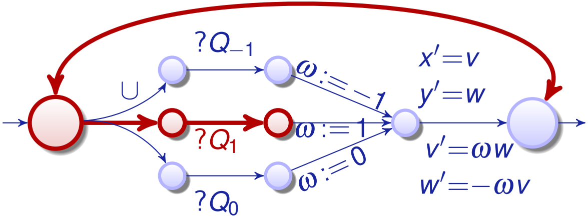

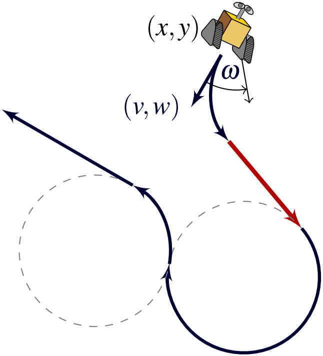

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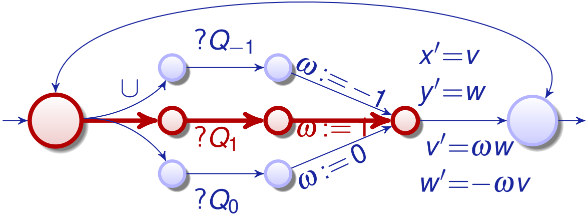

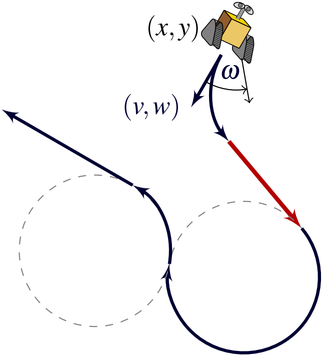

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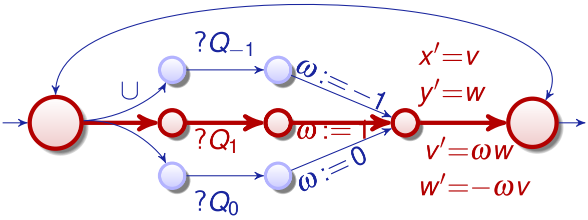

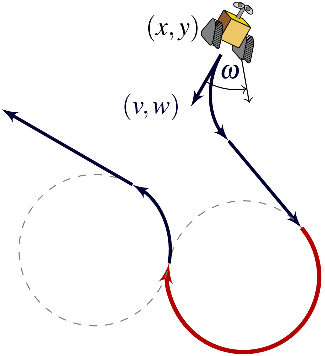

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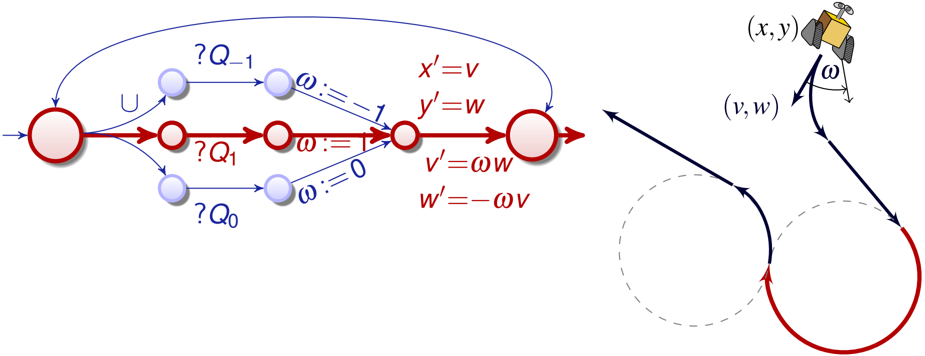

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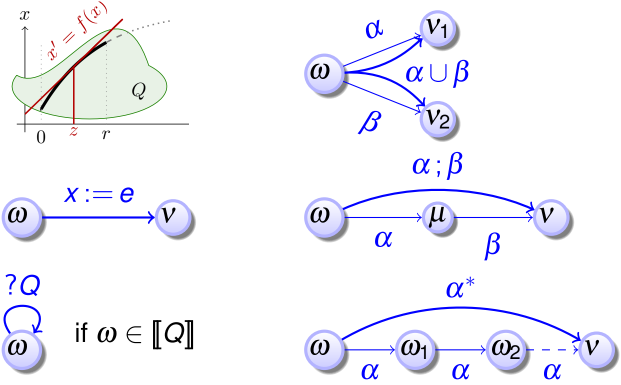
Summary: Modeling Dynamical Systems

Definition (Hybrid program)

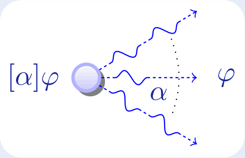
$$\alpha, \beta \ ::= \ x := e \mid ?Q \mid x' = f(x) \,\&\, Q \mid \alpha \cup \beta \mid \alpha ; \beta \mid \alpha^*$$

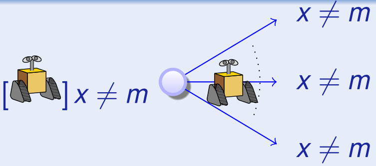


Programming CPS = program cyber + program physics + mutual care

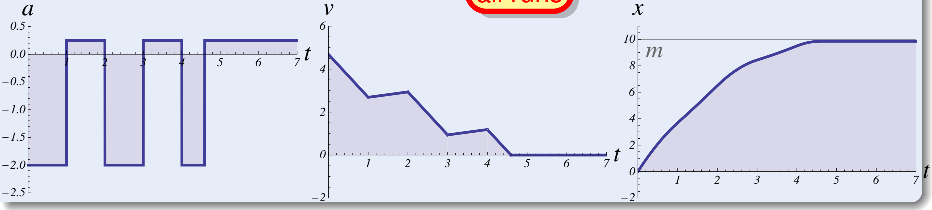
Outline (Specifying Dynamical Systems)

Concept (Differential Dynamic Logic) (JAR'08,LICS'12)





$$\underbrace{x \neq m \wedge b > 0}_{\text{init}} \rightarrow \underbrace{\left[\left(\left(\text{if}(SB(x,m))\quad a := -b\right) ; \; x' = v, v' = a\right)^*\right]}_{\text{all runs}} \underbrace{x \neq m}_{\text{post}}$$



Differential Dynamic Logic: Syntax

Definition (Syntax of differential dynamic logic)

The *formulas* of *differential dynamic logic* are defined by the grammar:

$$P, Q ::= e \geq \tilde{e} \mid \neg P \mid P \wedge Q \mid P \vee Q \mid P \rightarrow Q \mid \forall x\, P \mid \exists x\, P \mid [\alpha]P \mid \langle \alpha \rangle P$$

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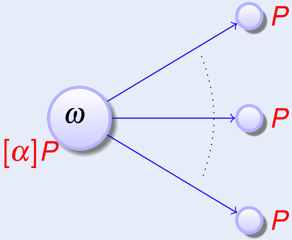
Not | And | Or | Imply | All reals | Some real | All runs | Some runs

Definition (dL Formulas)



Differential Dynamic Logic dL: Semantics

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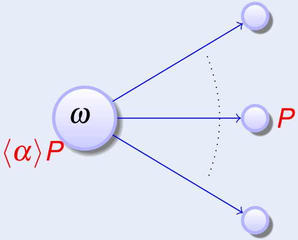

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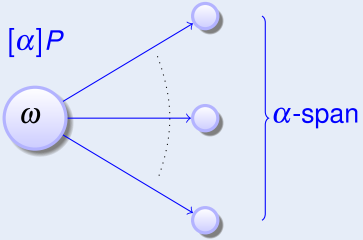

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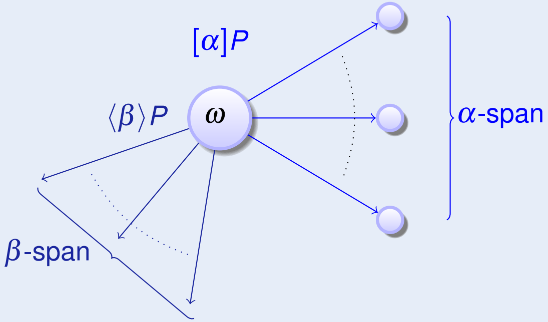

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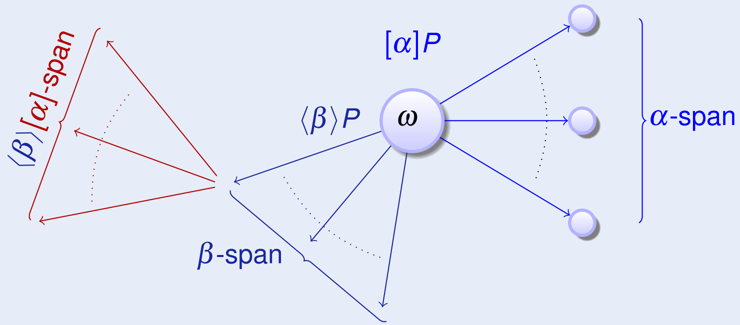

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Definition (dL semantics) ($[\![\cdot]\!] : \mathrm{Fml} \rightarrow \wp(\mathscr{S})$)

$$\begin{aligned}
[\![e \geq \tilde{e}]\!] &= \{\omega : \omega[\![e]\!] \geq \omega[\![\tilde{e}]\!]\} \\
[\![\neg P]\!] &= [\![P]\!]^{\complement} = \mathscr{S} \setminus [\![P]\!] \\
[\![P \wedge Q]\!] &= [\![P]\!] \cap [\![Q]\!] \\
[\![P \vee Q]\!] &= [\![P]\!] \cup [\![Q]\!] \\
[\![P \rightarrow Q]\!] &= [\![P]\!]^{\complement} \cup [\![Q]\!] \\
[\![\langle\alpha\rangle P]\!] &= [\![\alpha]\!] \circ [\![P]\!] = \{\omega : \nu \in [\![P]\!] \text{ for some } \nu : (\omega, \nu) \in [\![\alpha]\!]\} \\
[\![[\alpha]P]\!] &= [\![\neg\langle\alpha\rangle\neg P]\!] = \{\omega : \nu \in [\![P]\!] \text{ for all } \nu : (\omega, \nu) \in [\![\alpha]\!]\} \\
[\![\exists x\, P]\!] &= \{\omega : \omega_x^r \in [\![P]\!] \text{ for some } r \in \mathbb{R}\} \\
[\![\forall x\, P]\!] &= \{\omega : \omega_x^r \in [\![P]\!] \text{ for all } r \in \mathbb{R}\}
\end{aligned}$$

$$\omega_x^d(y) = \begin{cases} d & \text{if } y = x \\ \omega(y) & \text{if } y \neq x \end{cases}$$

$[\![P]\!]$ the set of states in which formula P is true

$\omega \models P$ formula P is true in state ω, alias $\omega \in [\![P]\!]$

$\models P$ formula P is valid, i.e., true in all states ω, i.e., $[\![P]\!] = \mathscr{S}$

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$$
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[\![e \geq \tilde{e}]\!] &= \{\omega : \omega[\![e]\!] \geq \omega[\![\tilde{e}]\!]\} \\
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[\![P \wedge Q]\!] &= [\![P]\!] \cap [\![Q]\!] \\
[\![P \vee Q]\!] &= [\![P]\!] \cup [\![Q]\!] \\
[\![P \rightarrow Q]\!] &= [\![P]\!]^{\complement} \cup [\![Q]\!] \\
[\![\langle\alpha\rangle P]\!] &= [\![\alpha]\!] \circ [\![P]\!] = \{\omega : \nu \in [\![P]\!] \text{ for some } \nu : (\omega,\nu) \in [\![\alpha]\!]\} \\
[\![[\alpha]P]\!] &= [\![\neg\langle\alpha\rangle\neg P]\!] = \{\omega : \nu \in [\![P]\!] \text{ for all } \nu : (\omega,\nu) \in [\![\alpha]\!]\} \\
[\![\exists x\, P]\!] &= \{\omega : \omega_x^r \in [\![P]\!] \text{ for some } r \in \mathbb{R}\} \\
[\![\forall x\, P]\!] &= \{\omega : \omega_x^r \in [\![P]\!] \text{ for all } r \in \mathbb{R}\}
\end{aligned}
$$

$$
\omega_x^d(y) = \begin{cases} d & \text{if } y = x \\ \omega(y) & \text{if } y \neq x \end{cases}
$$

Differential Dynamic Logic: Syntax & Semantics

$[\![P]\!]$ the set of states in which formula P is true

$\omega \models P$ formula P is true in state ω, alias $\omega \in [\![P]\!]$

$\models P$ formula P is valid, i.e., true in all states ω, i.e., $[\![P]\!] = \mathscr{S}$

$\exists d\,[x:=1;x'=d]x\geq 0$ and $[x:=x+1;x'=d]x\geq 0$ and $\langle x'=d\rangle x\geq 0$

Definition (dL semantics) $([\![\cdot]\!] : \mathrm{Fml} \to \wp(\mathscr{S}))$

$$\begin{aligned}
[\![e \geq \tilde{e}]\!] &= \{\omega : \omega[\![e]\!] \geq \omega[\![\tilde{e}]\!]\} \\
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\end{aligned}$$

$$\omega_x^d(y) = \begin{cases} d & \text{if } y=x \\ \omega(y) & \text{if } y\neq x \end{cases}$$

Differential Dynamic Logic: Syntax & Semantics

$[\![P]\!]$	the set of states in which formula P is true
$\omega \models P$	formula P is true in state ω, alias $\omega \in [\![P]\!]$
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$\models \exists d\,[x:=1;x'=d]x\geq 0$ and $\not\models [x:=x+1;x'=d]x\geq 0$ and $\not\models \langle x'=d\rangle x\geq 0$

Definition (dL semantics) $([\![\cdot]\!] : \mathrm{Fml} \to \wp(\mathscr{S}))$

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[\![e \geq \tilde{e}]\!] &= \{\omega : \omega[\![e]\!] \geq \omega[\![\tilde{e}]\!]\}\\
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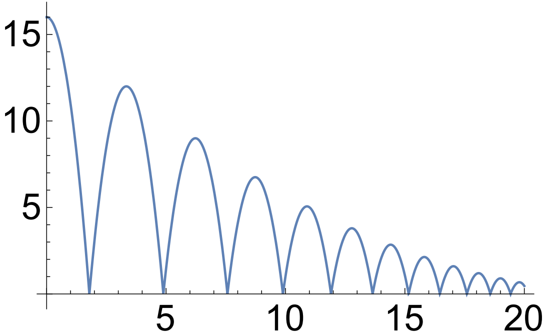
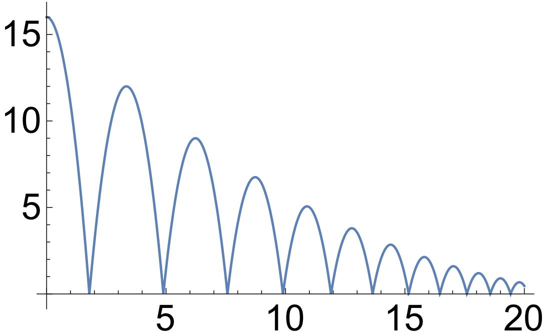
\end{aligned}
$$

$$\omega_x^d(y) = \begin{cases} d & \text{if } y=x \\ \omega(y) & \text{if } y\neq x \end{cases}$$



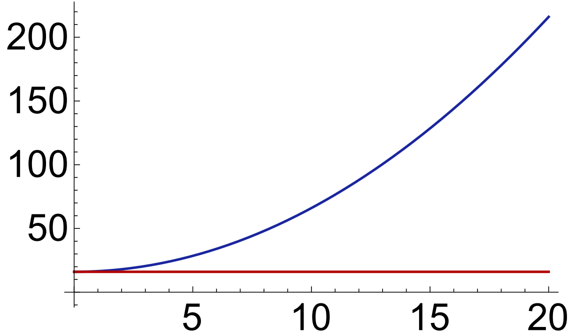
Example (Bouncing Ball)

$$\big(\{x' = v, v' = -g \,\&\, x \geq 0\};$$
$$\text{if}(x = 0)\, v := -cv\big)^*$$



Example (Bouncing Ball)

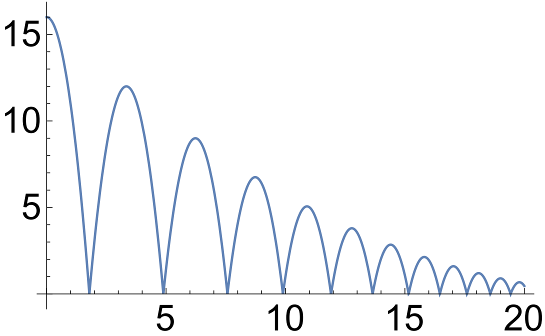
$$H = x \geq 0 \quad \rightarrow [(\{x' = v, v' = -g \,\&\, x \geq 0\}; \; \mathrm{if}(x = 0)\, v := -cv)^*]\, 0 \leq x \leq H$$



Not if $g < 0$ in anti-gravity

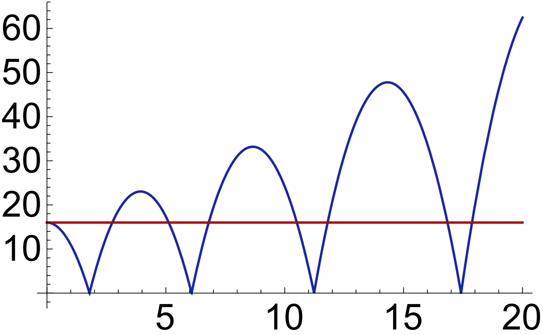
Example (Bouncing Ball)

$$H{=}x{\geq}0 \quad \rightarrow [(\{x' = v, v' = -g \,\&\, x \geq 0\}; \text{if}(x = 0)\, v := -cv)^*]\, 0{\leq}x{\leq}H$$



Example (Bouncing Ball)

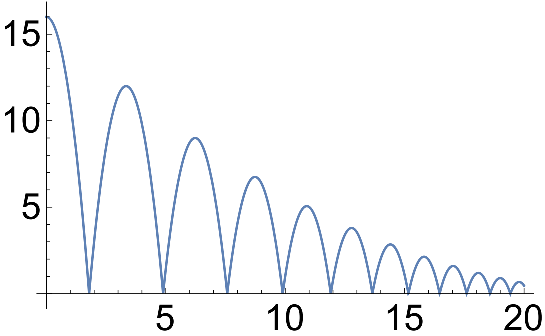
$$H = x \geq 0 \land g > 0 \to \left[\left(\{x' = v, v' = -g \,\&\, x \geq 0\};\ \text{if}(x = 0)\, v := -cv\right)^*\right] 0 \leq x \leq H$$



Not if $c > 1$ for anti-damping

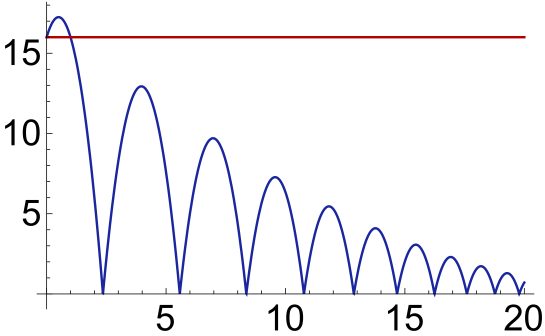
Example (Bouncing Ball)

$$H = x \geq 0 \wedge g > 0 \rightarrow [(\{x' = v, v' = -g \,\&\, x \geq 0\}; \; \text{if}(x = 0)\, v := -cv)^*]\, 0 \leq x \leq H$$



Example (Bouncing Ball)

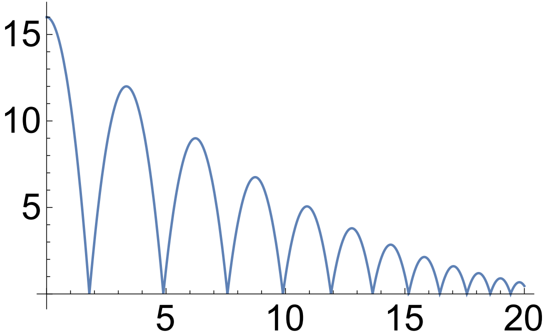
$$1 \geq c \geq 0 \land H = x \geq 0 \land g > 0 \rightarrow \left[\left(\{x' = v, v' = -g \,\&\, x \geq 0\}; \text{if}(x = 0)\, v := -cv\right)^*\right] 0 \leq x \leq H$$



Not if $v > 0$ initial climbing

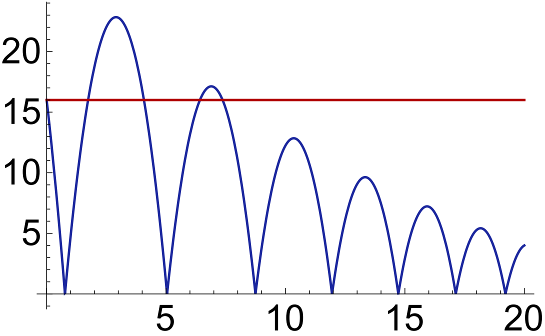
Example (Bouncing Ball)

$$1\geq c\geq 0 \land H=x\geq 0 \land g>0 \to \left[\left(\{x'=v, v'=-g \,\&\, x\geq 0\};\ \text{if}(x=0)\, v:=-cv\right)^*\right] 0\leq x\leq H$$



Example (Bouncing Ball)

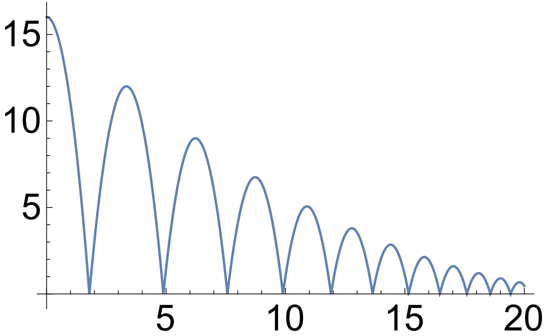
$$v \leq 0 \land 1 \geq c \geq 0 \land H = x \geq 0 \land g > 0 \rightarrow \left[\left(\{x' = v, v' = -g \,\&\, x \geq 0\}; \text{if}(x = 0)\, v := -cv\right)^*\right] 0 \leq x \leq H$$



Not if $v \ll 0$ initial dribbling

Example (Bouncing Ball)

$$v \le 0 \land 1 \ge c \ge 0 \land H = x \ge 0 \land g > 0 \rightarrow [(\{x' = v, v' = -g \,\&\, x \ge 0\}; \text{if}(x = 0)\, v := -cv)^*]\, 0 \le x \le H$$



Example (Bouncing Ball)

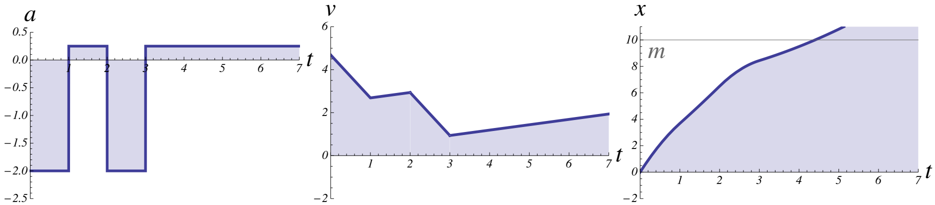
$$v=0 \land 1\geq c\geq 0 \land H=x\geq 0 \land g>0 \rightarrow \left[\left(\{x' = v, v' = -g \,\&\, x \geq 0\};\ \text{if}(x = 0)\, v := -cv\right)^*\right] 0\leq x\leq H$$

Ex: Car Control Programs

Repeat control decisions

Example (Single car car_s)

$$\big((\ a := A\ \cup a := -b);\ \{x' = v, v' = a\}\big)^*$$

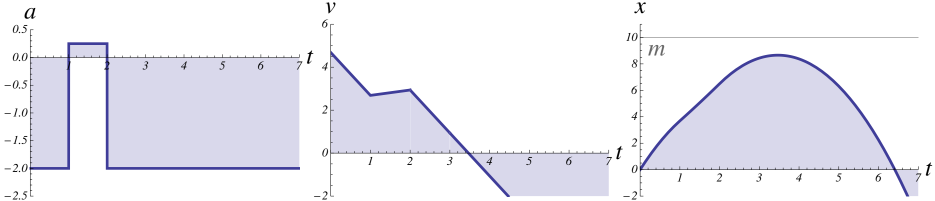


Ex: Car Control Programs

How does this model brake?

Example (Single car car_s)

$$\big((a := A \cup a := -b);\ \{x' = v, v' = a\}\big)^*$$

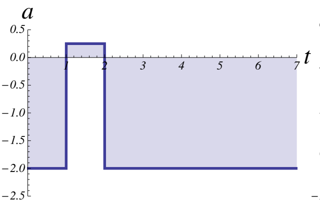


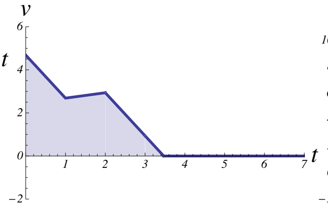
Ex: Car Control Programs

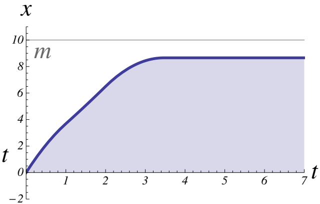
Velocity bound $v \geq 0$ in evolution domain

Example (Single car car_s)

$$\big((\ a := A \ \cup a := -b);\ \{x' = v, v' = a \,\&\, v \geq 0\}\big)^*$$





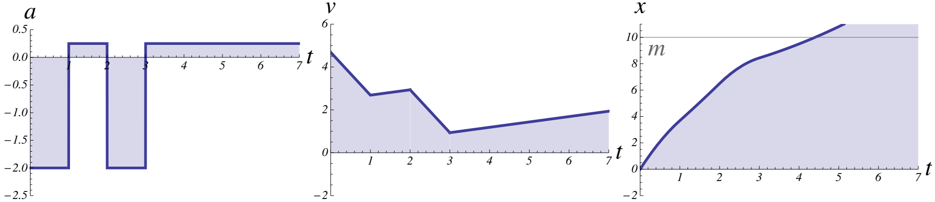


Ex: Car Control Programs

Acceleration not always safe

Example (Single car car_s)

$$\left((a := A \cup a := -b);\ \{x' = v, v' = a \,\&\, v \geq 0\} \right)^*$$

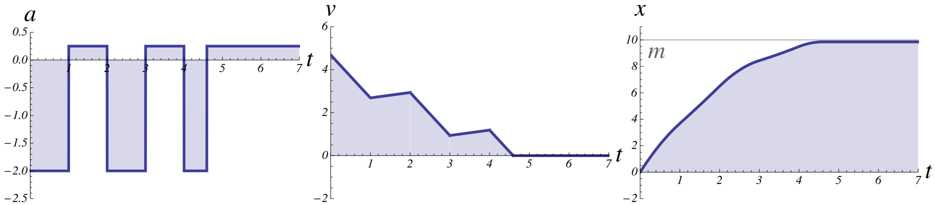


Ex: Car Control Programs

Acceleration condition $?Q$

Example (Single car car_s)

$$\left(((?Q; a := A) \cup a := -b);\ \{x' = v, v' = a \,\&\, v \geq 0\}\right)^*$$



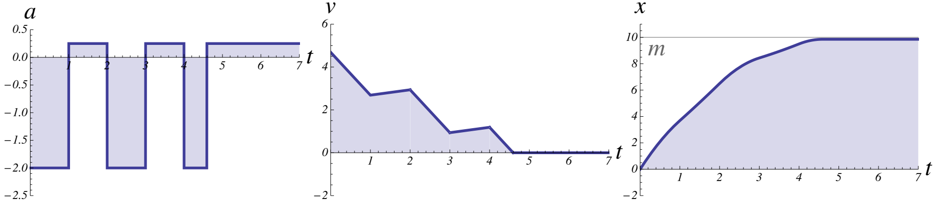
$Q \equiv$

Example (Single car car_ε time-triggered)

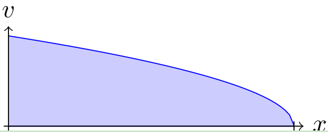
$$(((?Q; a:=A) \cup a:=-b); t:=0; \{x'=v, v'=a, t'=1 \,\&\, v \geq 0 \land t \leq \varepsilon\})^*$$

Example (Safely stays before traffic light m)

$$A \geq 0 \land b > 0 \rightarrow [car_\varepsilon]\, x \leq m$$



$Q \equiv$

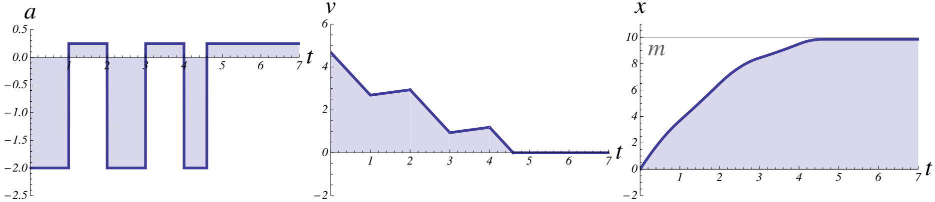


Example (Single car car_ε time-triggered)

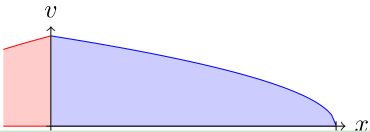
$$\big(((?Q; a := A) \cup a := -b); t := 0; \{x' = v, v' = a, t' = 1 \,\&\, v \geq 0 \land t \leq \varepsilon\}\big)^*$$

Example (Safely stays before traffic light m)

$$v^2 \leq 2b(m - x) \land A \geq 0 \land b > 0 \rightarrow [car_\varepsilon]\, x \leq m$$



$$Q \equiv 2b(m-x) \geq v^2 + (A+b)(A\varepsilon^2 + 2\varepsilon v)$$

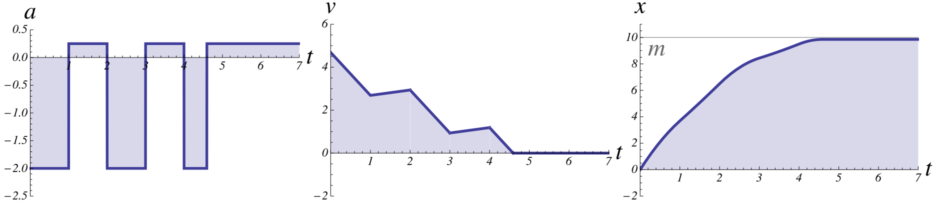


Example (Single car car_ε time-triggered)

$$(((?Q; a := A) \cup a := -b); t := 0; \{x' = v, v' = a, t' = 1 \,\&\, v \geq 0 \wedge t \leq \varepsilon\})^*$$

Example (Safely stays before traffic light m)

$$v^2 \leq 2b(m-x) \wedge A \geq 0 \wedge b > 0 \rightarrow [car_\varepsilon]\, x \leq m$$



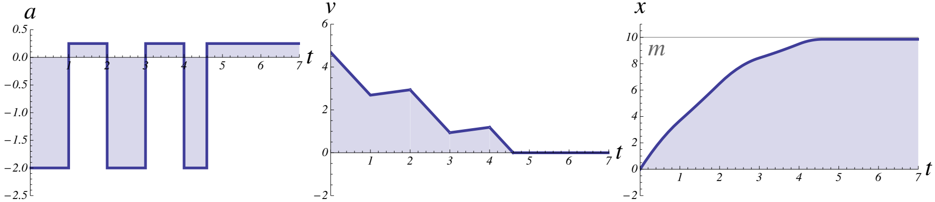
$Q \equiv 2b(m-x) \geq v^2 + (A+b)(A\varepsilon^2 + 2\varepsilon v)$

Example (Single car car_ε time-triggered)

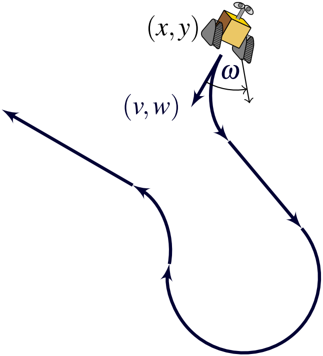
$$\Big(\big((?Q; a := A) \cup a := -b\big); t := 0; \{x' = v, v' = a, t' = 1 \,\&\, v \geq 0 \land t \leq \varepsilon\}\Big)^*$$

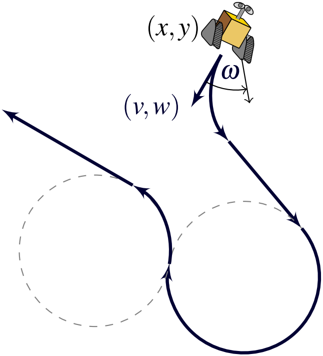
Example (Live, can move everywhere)

$$\varepsilon > 0 \land A > 0 \land b > 0 \rightarrow \forall p \exists m \langle car_\varepsilon \rangle\, x \geq p$$



Exercise: Runaround Robot

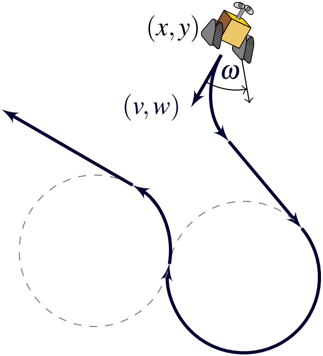




Example (Runaround Robot)

$$\big((\omega := -1 \cup \omega := 1 \cup \omega := 0);$$
$$\{x' = v, y' = w, v' = \omega w, w' = -\omega v\}\big)^*$$

Exercise: Runaround Robot



Example (Runaround Robot)

$$(x,y) \neq o \rightarrow \Big[\big((\omega := -1 \cup \omega := 1 \cup \omega := 0);$$
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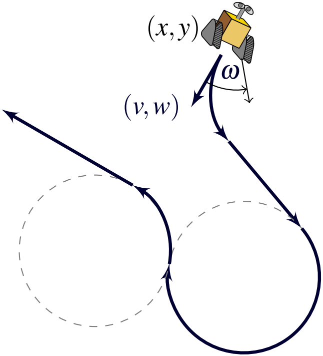


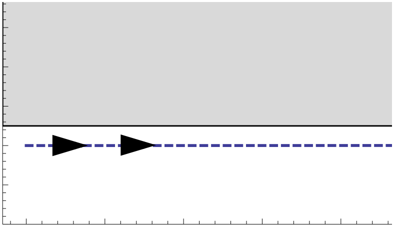
Example (Runaround Robot)

$$(x, y) \neq o \rightarrow \big[\big(\big(?Q_{-1}; \omega := -1 \cup ?Q_1; \omega := 1 \cup ?Q_0; \omega := 0\big); \{x' = v, y' = w, v' = \omega w, w' = -\omega v\}\big)^*\big]\,(x, y) \neq o$$

Exercise: TwoCar Control One Car to Follow Another

Model two cars and control one car to safely follow the leader car.

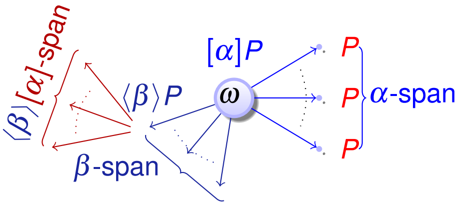
- A maximum acceleration (magnitude)
- B maximum braking (magnitude)
- T maximum reaction time
- x, v, a position, velocity, acceleration of follower car to be controlled
- likewise for lead car, uncontrolled
- motion on a straight line



Summary: Specifying Dynamical Systems

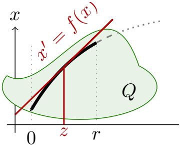
Definition (Differential dynamic logic)

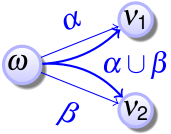
$$P, Q ::= e \geq \tilde{e} \mid \neg P \mid P \wedge Q \mid P \vee Q \mid P \rightarrow Q \mid \forall x\, P \mid \exists x\, P \mid [\alpha]P \mid \langle \alpha \rangle P$$

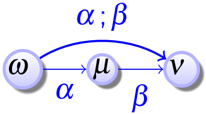


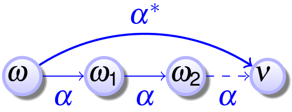
Definition (Hybrid program)

$$\alpha, \beta \;::=\; x := e \mid ?Q \mid x' = f(x) \,\&\, Q \mid \alpha \cup \beta \mid \alpha ; \beta \mid \alpha^*$$







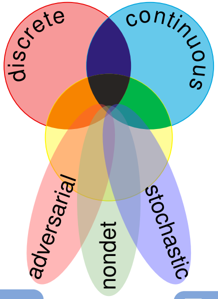


Outline (Proving Dynamical Systems)

CPSs are Multi-Dynamical Systems

CPS Dynamics

CPS are characterized by multiple facets of dynamical systems.



CPS Compositions

CPS combines multiple simple dynamical effects.

Descriptive simplification

Tame Parts

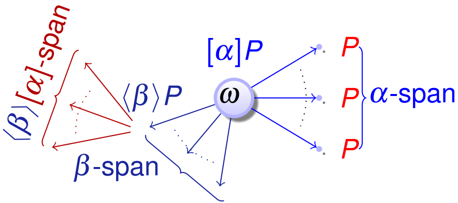
Exploiting compositionality tames CPS complexity.

Analytic simplification

Summary: Specifying Dynamical Systems

Definition (Differential dynamic logic)

$$P, Q ::= e \geq \tilde{e} \mid \neg P \mid P \wedge Q \mid P \vee Q \mid P \rightarrow Q \mid \forall x\, P \mid \exists x\, P \mid [\alpha]P \mid \langle\alpha\rangle P$$



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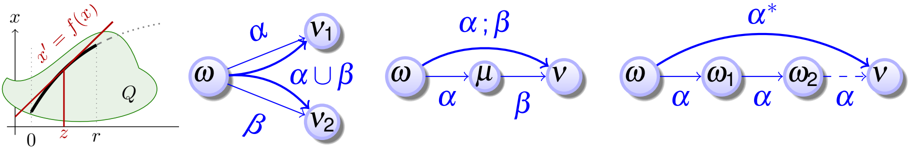
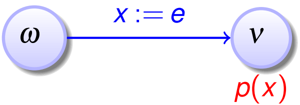
$$\alpha, \beta \ ::= \ x := e \mid ?Q \mid x' = f(x) \,\&\, Q \mid \alpha \cup \beta \mid \alpha;\beta \mid \alpha^*$$



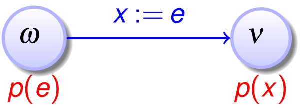
Dynamic Axioms for Dynamical Systems

$[:=]\ \ [x := e]p(x) \leftrightarrow$



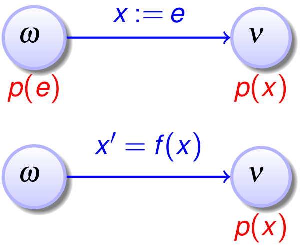
Dynamic Axioms for Dynamical Systems

$[:=]$ $[x := e]p(x) \leftrightarrow p(e)$



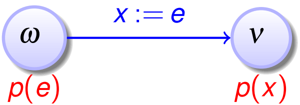
$[:=]$ $[x := e]p(x) \leftrightarrow p(e)$

$[']$ $[x' = f(x)]p(x) \leftrightarrow$

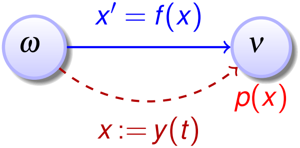


Dynamic Axioms for Dynamical Systems

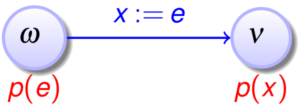
$[:=]\ [x := e]p(x) \leftrightarrow p(e)$



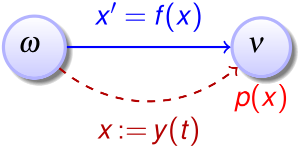
$['] \ [x' = f(x)]p(x) \leftrightarrow \qquad [x := y(t)]p(x)$



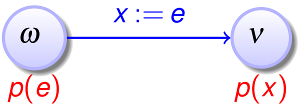
$[:=]\ [x := e]p(x) \leftrightarrow p(e)$



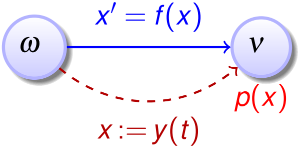
$[']\ [x' = f(x)]p(x) \leftrightarrow \forall t{\geq}0\, [x := y(t)]p(x)$



$[:=]\ [x := e]p(x) \leftrightarrow p(e)$

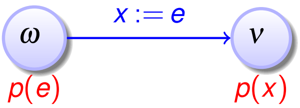


$['] \ [x' = f(x)]p(x) \leftrightarrow \forall t \geq 0\, [x := y(t)]p(x)$

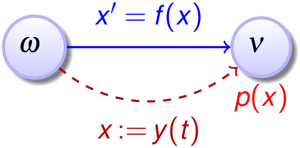


$['] \ [x' = f(x) \,\&\, q(x)]p(x) \leftrightarrow \forall t \geq 0\, \big([x := y(t)]p(x)\big)$

$[:=]\ [x := e]p(x) \leftrightarrow p(e)$



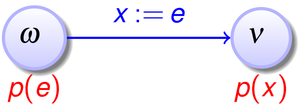
$[']\ [x' = f(x)]p(x) \leftrightarrow \forall t{\geq}0\, [x := y(t)]p(x)$



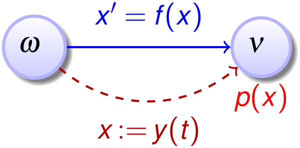
$[']\ [x' = f(x) \,\&\, q(x)]p(x) \leftrightarrow \forall t{\geq}0\, \big(\forall 0{\leq}s{\leq}t\, q(y(s)) \rightarrow [x := y(t)]p(x)\big)$

Dynamic Axioms for Dynamical Systems

[:=] $[x := e]p(x) \leftrightarrow p(e)$

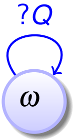


['] $[x' = f(x)]p(x) \leftrightarrow \forall t{\geq}0\, [x := y(t)]p(x)$



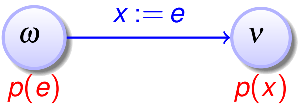
['] $[x' = f(x)\,\&\, q(x)]p(x) \leftrightarrow \forall t{\geq}0\, \big(\forall 0{\leq}s{\leq}t\; q(y(s)) \rightarrow [x := y(t)]p(x)\big)$

[?] $[?Q]P \leftrightarrow$

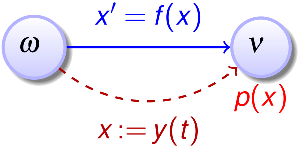


if $\omega \in [\![Q]\!]$

[:=] $[x := e]p(x) \leftrightarrow p(e)$

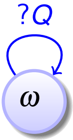


['] $[x' = f(x)]p(x) \leftrightarrow \forall t{\geq}0\,[x := y(t)]p(x)$



['] $[x' = f(x)\,\&\,q(x)]p(x) \leftrightarrow \forall t{\geq}0\,\big(\forall 0{\leq}s{\leq}t\; q(y(s)) \rightarrow [x := y(t)]p(x)\big)$

[?] $[?Q]P \leftrightarrow (Q \rightarrow P)$

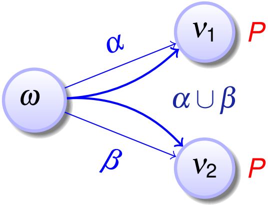


if $\omega \in [\![Q]\!]$

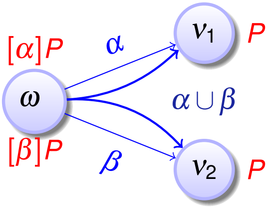
Dynamic Axioms for Dynamical Systems

compositional semantics $\Rightarrow$ compositional proofs

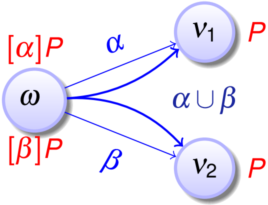
[∪] $[\alpha \cup \beta]P \leftrightarrow$



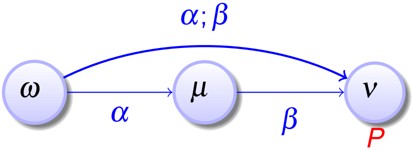
[∪] $[\alpha \cup \beta]P \leftrightarrow [\alpha]P \land [\beta]P$



$[\cup]$ $[\alpha \cup \beta]P \leftrightarrow [\alpha]P \wedge [\beta]P$

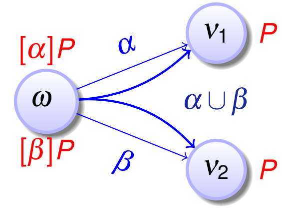


$[;]$ $[\alpha;\beta]P \leftrightarrow$

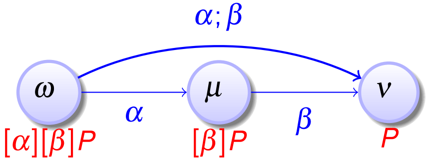


Dynamic Axioms for Dynamical Systems

$[\cup]$ $[\alpha \cup \beta]P \leftrightarrow [\alpha]P \wedge [\beta]P$

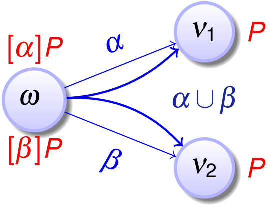


$[;]$ $[\alpha;\beta]P \leftrightarrow [\alpha][\beta]P$

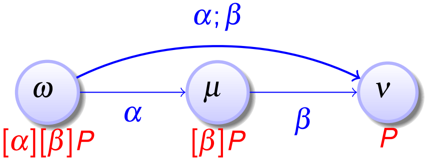


Dynamic Axioms for Dynamical Systems

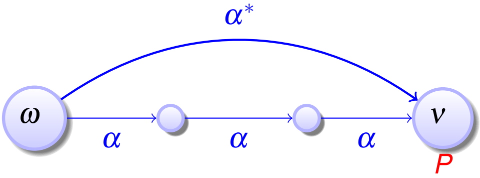
$[\cup]$ $[\alpha \cup \beta]P \leftrightarrow [\alpha]P \land [\beta]P$



$[;]$ $[\alpha;\beta]P \leftrightarrow [\alpha][\beta]P$

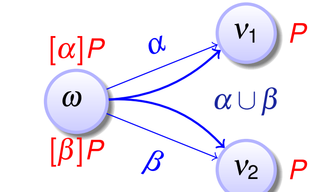


$[^*]$ $[\alpha^*]P \leftrightarrow$

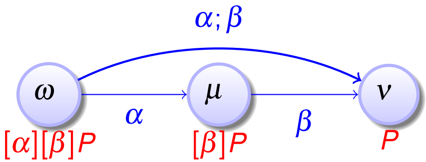


Dynamic Axioms for Dynamical Systems

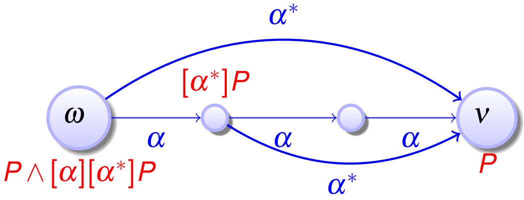
$[\cup]$ $[\alpha \cup \beta]P \leftrightarrow [\alpha]P \land [\beta]P$



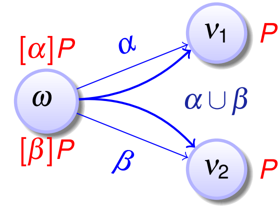
$[;]$ $[\alpha;\beta]P \leftrightarrow [\alpha][\beta]P$



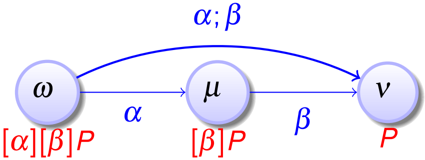
$[^*]$ $[\alpha^*]P \leftrightarrow P \land [\alpha][\alpha^*]P$



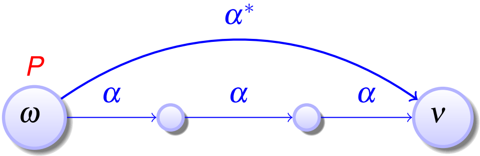
$[\cup]$ $[\alpha \cup \beta]P \leftrightarrow [\alpha]P \wedge [\beta]P$



$[;]$ $[\alpha;\beta]P \leftrightarrow [\alpha][\beta]P$

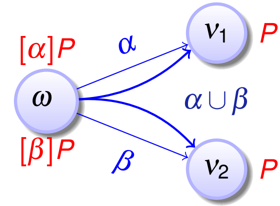


I $[\alpha^*]P \leftrightarrow P \wedge$

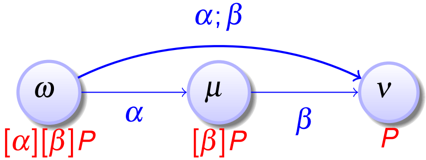


Dynamic Axioms for Dynamical Systems

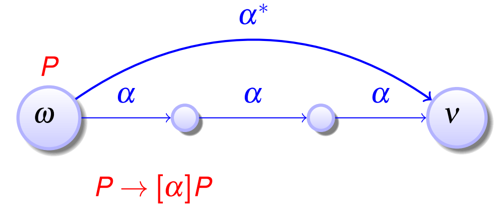
[∪] $[\alpha \cup \beta]P \leftrightarrow [\alpha]P \wedge [\beta]P$



[;] $[\alpha;\beta]P \leftrightarrow [\alpha][\beta]P$

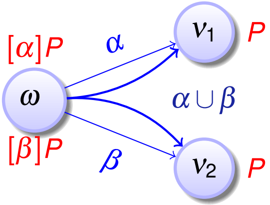


I $[\alpha^*]P \leftrightarrow P \wedge \quad (P \rightarrow [\alpha]P)$

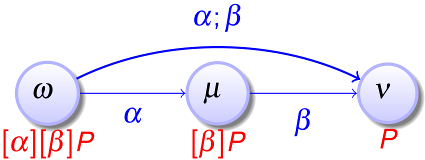


Dynamic Axioms for Dynamical Systems

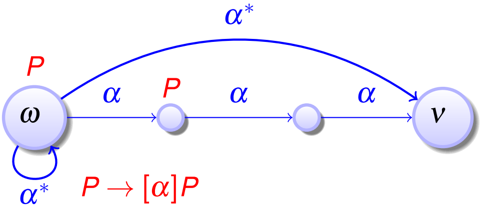
[∪] $[\alpha \cup \beta]P \leftrightarrow [\alpha]P \land [\beta]P$



[;] $[\alpha;\beta]P \leftrightarrow [\alpha][\beta]P$

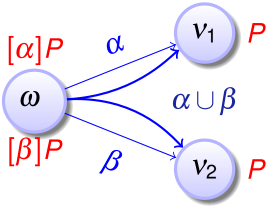


I $[\alpha^*]P \leftrightarrow P \land [\alpha^*](P \rightarrow [\alpha]P)$

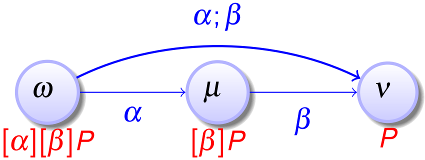


Dynamic Axioms for Dynamical Systems

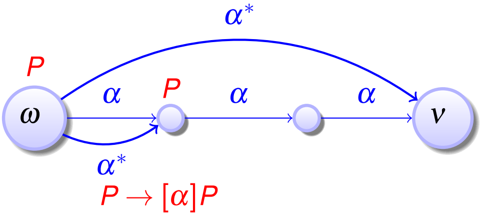
[∪] $[\alpha \cup \beta]P \leftrightarrow [\alpha]P \land [\beta]P$



[;] $[\alpha;\beta]P \leftrightarrow [\alpha][\beta]P$

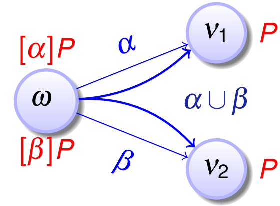


I $[\alpha^*]P \leftrightarrow P \land [\alpha^*](P \rightarrow [\alpha]P)$

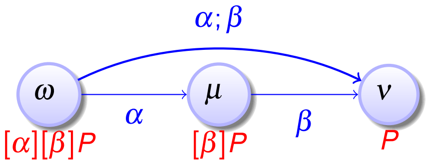


Dynamic Axioms for Dynamical Systems

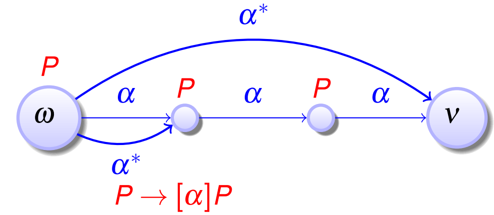
[∪] $[\alpha \cup \beta]P \leftrightarrow [\alpha]P \land [\beta]P$



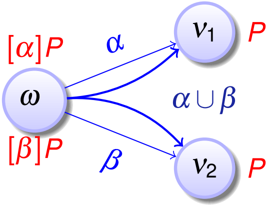
[;] $[\alpha;\beta]P \leftrightarrow [\alpha][\beta]P$



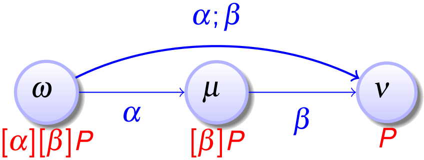
I $[\alpha^*]P \leftrightarrow P \land [\alpha^*](P \to [\alpha]P)$



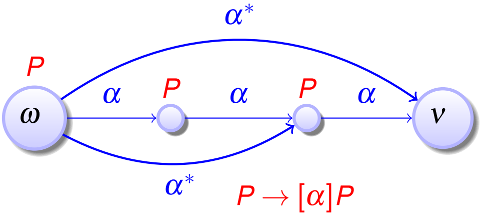
$[\cup]$ $[\alpha \cup \beta]P \leftrightarrow [\alpha]P \land [\beta]P$



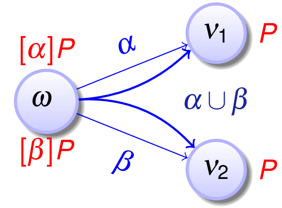
$[;]$ $[\alpha;\beta]P \leftrightarrow [\alpha][\beta]P$



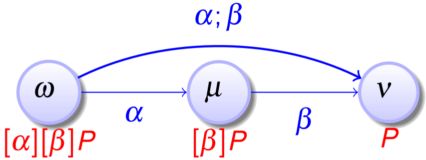
I $[\alpha^*]P \leftrightarrow P \land [\alpha^*](P \to [\alpha]P)$



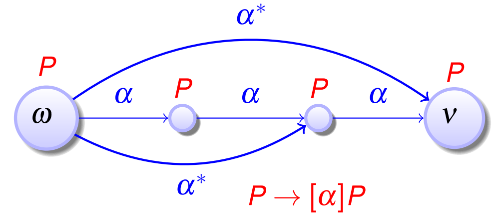
[∪] $[\alpha \cup \beta]P \leftrightarrow [\alpha]P \wedge [\beta]P$



[;] $[\alpha;\beta]P \leftrightarrow [\alpha][\beta]P$

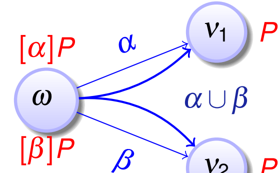


I $[\alpha^*]P \leftrightarrow P \wedge [\alpha^*](P \rightarrow [\alpha]P)$

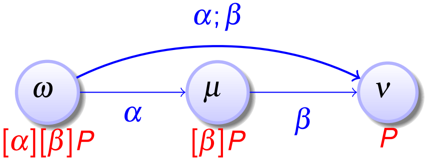


Dynamic Axioms for Dynamical Systems

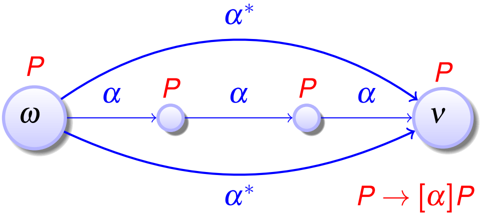
[∪] $[\alpha \cup \beta]P \leftrightarrow [\alpha]P \wedge [\beta]P$



[;] $[\alpha;\beta]P \leftrightarrow [\alpha][\beta]P$



I $[\alpha^*]P \leftrightarrow P \wedge [\alpha^*](P \rightarrow [\alpha]P)$

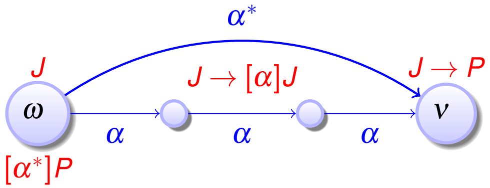


Proof Rule: Loop Invariants

G $\dfrac{P}{[\alpha]P}$ I $[\alpha^*]P \leftrightarrow P \land [\alpha^*](P \rightarrow [\alpha]P)$ M[·] $\dfrac{P \rightarrow Q}{[\alpha]P \rightarrow [\alpha]Q}$

Lemma (Loop invariant rule is derived syntactically)

$$\text{loop}\ \frac{\Gamma \rightarrow J, \Delta \quad J \rightarrow [\alpha]J \quad J \rightarrow P}{\Gamma \rightarrow [\alpha^*]P, \Delta}$$



Sequent notation $\Gamma \rightarrow \Delta$ means $\left(\bigwedge_{A \in \Gamma} A\right) \rightarrow \left(\bigvee_{B \in \Delta} B\right)$ for sets Γ, Δ

Proof Rule: Loop Invariants

$$\text{G}\ \frac{P}{[\alpha]P} \qquad \text{I}\ [\alpha^*]P \leftrightarrow P \wedge [\alpha^*](P \to [\alpha]P) \qquad \text{M}[\cdot]\ \frac{P \to Q}{[\alpha]P \to [\alpha]Q}$$

Lemma (Loop invariant rule is derived syntactically)

$$\textit{loop}\ \frac{\Gamma \to J, \Delta \quad J \to [\alpha]J \quad J \to P}{\Gamma \to [\alpha^*]P, \Delta}$$

Proof (Derived rule).

$$\text{cut}\ \dfrac{\Gamma \to J, \Delta \qquad \text{I}\ \dfrac{\text{G}\ \dfrac{J \to [\alpha]J}{J \to J \wedge [\alpha^*](J \to [\alpha]J)}}{J \to [\alpha^*]J} \qquad \text{M}[\cdot]\ \dfrac{J \to P}{[\alpha^*]J \to [\alpha^*]P}}{\Gamma \to [\alpha^*]P, \Delta}$$

□

Proof Rule: Loop Invariants

$$\text{G}\ \frac{P}{[\alpha]P} \qquad \text{I}\ [\alpha^*]P \leftrightarrow P \wedge [\alpha^*](P \rightarrow [\alpha]P) \qquad \text{M}[\cdot]\ \frac{P \rightarrow Q}{[\alpha]P \rightarrow [\alpha]Q}$$

Lemma (Loop invariant rule is derived syntactically)

$$\text{loop}\ \frac{\Gamma \rightarrow J, \Delta \quad J \rightarrow [\alpha]J \quad J \rightarrow P}{\Gamma \rightarrow [\alpha^*]P, \Delta}$$

Proof (Derived rule).

$$\text{cut}\ \dfrac{\Gamma \rightarrow J, \Delta \qquad \text{I}\ \dfrac{\text{G}\ \dfrac{J \rightarrow [\alpha]J}{J \rightarrow J \wedge [\alpha^*](J \rightarrow [\alpha]J)}}{J \rightarrow [\alpha^*]J} \qquad \text{M}[\cdot]\ \dfrac{J \rightarrow P}{[\alpha^*]J \rightarrow [\alpha^*]P}}{\Gamma \rightarrow [\alpha^*]P, \Delta}$$

□

Finding invariant J can be a challenge.

Misplaced $[\alpha^*]$ suggests that J needs to carry along info about α^* history.

Differential Dynamic Logic dL: Axiomatization

equations of truth

[:=] $[x := e]P(x) \leftrightarrow P(e)$

[?] $[?Q]P \leftrightarrow (Q \rightarrow P)$

[′] $[x' = f(x)]P \leftrightarrow \forall t{\geq}0\, [x := y(t)]P$ $\quad (y'(t) = f(y))$

[∪] $[\alpha \cup \beta]P \leftrightarrow [\alpha]P \wedge [\beta]P$

[;] $[\alpha;\beta]P \leftrightarrow [\alpha][\beta]P$

[*] $[\alpha^*]P \leftrightarrow P \wedge [\alpha][\alpha^*]P$

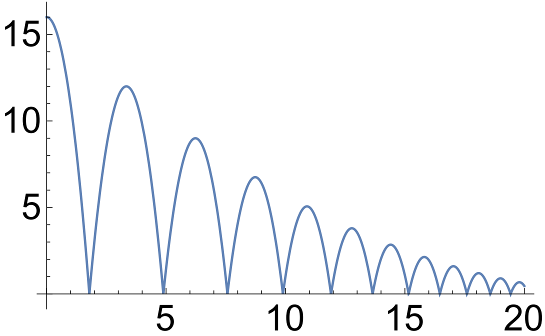
K $[\alpha](P \rightarrow Q) \rightarrow ([\alpha]P \rightarrow [\alpha]Q)$

I $[\alpha^*]P \leftrightarrow P \wedge [\alpha^*](P \rightarrow [\alpha]P)$

C $[\alpha^*]\forall v{>}0\,(P(v) \rightarrow \langle\alpha\rangle P(v{-}1)) \rightarrow \forall v\,(P(v) \rightarrow \langle\alpha^*\rangle \exists v{\leq}0\, P(v))$

laws of logic of
laws of physics

LICS'12,JAR'17



Example (Bouncing Ball)

$$v=0 \land 1 \geq c \geq 0 \land H=x \geq 0 \land g>0 \rightarrow \left[\left(\{x'=v, v'=-g \,\&\, x \geq 0\};\ \text{if}(x=0)\, v:=-cv\right)^*\right] 0 \leq x \leq H$$

$$\frac{}{A \rightarrow [\big(\text{grav}; (?x{=}0; v{:=}{-}cv \cup ?x{\neq}0)\big)^*]B(x,v)}$$

$$A \equiv 0 \leq x \land x = H \land v = 0 \land g > 0 \land 1 \geq c \geq 0$$

$$B(x,v) \equiv 0 \leq x \land x \leq H$$

$$\text{grav} \equiv \{x' = v, v' = -g \,\&\, x \geq 0\}$$

$$\text{loop}\ \frac{A \rightarrow \mathrm{j}(x,v) \qquad \dfrac{}{\mathrm{j}(x,v) \rightarrow [\text{grav}; (?x{=}0; v{:=}{-}cv \cup ?x{\neq}0)]\mathrm{j}(x,v)} \qquad \mathrm{j}(x,v) \rightarrow B(x,v)}{A \rightarrow [(\text{grav}; (?x{=}0; v{:=}{-}cv \cup ?x{\neq}0))^*]B(x,v)}$$

$$A \equiv 0 \leq x \wedge x = H \wedge v = 0 \wedge g > 0 \wedge 1 \geq c \geq 0$$

$$B(x,v) \equiv 0 \leq x \wedge x \leq H$$

$$\text{grav} \equiv \{x' = v, v' = -g \,\&\, x \geq 0\}$$

$$\text{loop}\ \dfrac{A \to j(x,v) \qquad \dfrac{j(x,v) \to [\text{grav}; (?x=0; v:=-cv \cup ?x\neq 0)]j(x,v)}{j(x,v) \to [\text{grav}; (?x=0; v:=-cv \cup ?x\neq 0)]j(x,v)} \qquad j(x,v) \to B(x,v)}{A \to [\big(\text{grav}; (?x=0; v:=-cv \cup ?x\neq 0)\big)^*]B(x,v)}$$

$$A \equiv 0 \leq x \land x = H \land v = 0 \land g > 0 \land 1 \geq c \geq 0$$

$$B(x,v) \equiv 0 \leq x \land x \leq H$$

$$\text{grav} \equiv \{x' = v, v' = -g \,\&\, x \geq 0\}$$

$$
\dfrac{A \rightarrow \mathrm{j}(x,v) \qquad \dfrac{\overline{\mathrm{j}(x,v) \rightarrow [\mathrm{grav}][?x=0;\, v:=-cv \cup ?x\neq 0]\mathrm{j}(x,v)}}{\dfrac{\mathrm{j}(x,v) \rightarrow [\mathrm{grav};\, (?x=0;\, v:=-cv \cup ?x\neq 0)]\mathrm{j}(x,v)}{\mathrm{j}(x,v) \rightarrow [\mathrm{grav};\, (?x=0;\, v:=-cv \cup ?x\neq 0)]\mathrm{j}(x,v)}}\,[;] \qquad \mathrm{j}(x,v) \rightarrow B(x,v)}{A \rightarrow [\big(\mathrm{grav};\, (?x=0;\, v:=-cv \cup ?x\neq 0)\big)^*]B(x,v)}\,\text{loop}
$$

$$A \equiv 0 \leq x \wedge x = H \wedge v = 0 \wedge g > 0 \wedge 1 \geq c \geq 0$$

$$B(x,v) \equiv 0 \leq x \wedge x \leq H$$

$$\mathrm{grav} \equiv \{x' = v, v' = -g \,\&\, x \geq 0\}$$

$$
\dfrac{\dfrac{j(x,v)\rightarrow[\text{grav}]j(x,v) \qquad \overline{j(x,v)\rightarrow[?x=0;\,v:=-cv\cup ?x\neq 0]j(x,v)}}{j(x,v)\rightarrow[\text{grav}][?x=0;\,v:=-cv\cup ?x\neq 0]j(x,v)}\text{MR}}{\dfrac{A\rightarrow j(x,v) \qquad \dfrac{j(x,v)\rightarrow[\text{grav};(?x=0;\,v:=-cv\cup ?x\neq 0)]j(x,v)}{j(x,v)\rightarrow[\text{grav};(?x=0;\,v:=-cv\cup ?x\neq 0)]j(x,v)} \qquad j(x,v)\rightarrow B(x,v)}{A\rightarrow[\big(\text{grav};(?x=0;\,v:=-cv\cup ?x\neq 0)\big)^*]B(x,v)}\text{loop}}[;]
$$

$$A \equiv 0 \leq x \wedge x = H \wedge v = 0 \wedge g > 0 \wedge 1 \geq c \geq 0$$

$$B(x,v) \equiv 0 \leq x \wedge x \leq H$$

$$\text{grav} \equiv \{x' = v, v' = -g \,\&\, x \geq 0\}$$

$$
\text{loop}\;\dfrac{A\to \mathrm{j}(x,v) \qquad \text{[;]}\,\dfrac{\text{MR}\,\dfrac{\mathrm{j}(x,v)\to[\mathrm{grav}]\mathrm{j}(x,v) \quad [\cup]\,\dfrac{\dfrac{}{\mathrm{j}(x,v)\to[?x=0;\,v:=-cv]\mathrm{j}(x,v)\land[?x\neq 0]\mathrm{j}(x,v)}}{\mathrm{j}(x,v)\to[?x=0;\,v:=-cv\cup ?x\neq 0]\mathrm{j}(x,v)}}{\mathrm{j}(x,v)\to[\mathrm{grav}][?x=0;\,v:=-cv\cup ?x\neq 0]\mathrm{j}(x,v)}}{\dfrac{\mathrm{j}(x,v)\to[\mathrm{grav};(?x=0;\,v:=-cv\cup ?x\neq 0)]\mathrm{j}(x,v)}{\mathrm{j}(x,v)\to[\mathrm{grav};(?x=0;\,v:=-cv\cup ?x\neq 0)]\mathrm{j}(x,v)}} \qquad \mathrm{j}(x,v)\to B(x,v)}{A\to\big[\big(\mathrm{grav};(?x=0;\,v:=-cv\cup ?x\neq 0)\big)^*\big]B(x,v)}
$$

$$A \equiv 0\le x\land x=H\land v=0\land g>0\land 1\ge c\ge 0$$

$$B(x,v)\equiv 0\le x\land x\le H$$

$$\mathrm{grav}\equiv\{x'=v,v'=-g\,\&\,x\ge 0\}$$

$$\wedge\mathrm{R}\ \dfrac{\dfrac{}{j(x,v)\rightarrow[?x=0;\, v:=-cv]j(x,v)} \qquad \dfrac{}{j(x,v)\rightarrow[?x\neq 0]j(x,v)}}{j(x,v)\rightarrow[?x=0;\, v:=-cv]j(x,v) \wedge [?x\neq 0]j(x,v)}$$

$$[\cup]\ \dfrac{j(x,v)\rightarrow[?x=0;\, v:=-cv]j(x,v) \wedge [?x\neq 0]j(x,v)}{j(x,v)\rightarrow[?x=0;\, v:=-cv \cup ?x\neq 0]j(x,v)}$$

$$\mathrm{MR}\ \dfrac{j(x,v)\rightarrow[\mathrm{grav}]j(x,v) \qquad j(x,v)\rightarrow[?x=0;\, v:=-cv \cup ?x\neq 0]j(x,v)}{j(x,v)\rightarrow[\mathrm{grav}][?x=0;\, v:=-cv \cup ?x\neq 0]j(x,v)}$$

$$[;]\ \dfrac{j(x,v)\rightarrow[\mathrm{grav}][?x=0;\, v:=-cv \cup ?x\neq 0]j(x,v)}{j(x,v)\rightarrow[\mathrm{grav};\,(?x=0;\, v:=-cv \cup ?x\neq 0)]j(x,v)}$$

$$\mathrm{loop}\ \dfrac{A\rightarrow j(x,v) \qquad j(x,v)\rightarrow[\mathrm{grav};\,(?x=0;\, v:=-cv \cup ?x\neq 0)]j(x,v) \qquad j(x,v)\rightarrow B(x,v)}{A\rightarrow\big[\big(\mathrm{grav};\,(?x=0;\, v:=-cv \cup ?x\neq 0)\big)^*\big]B(x,v)}$$

$$A \equiv 0 \leq x \wedge x = H \wedge v = 0 \wedge g > 0 \wedge 1 \geq c \geq 0$$

$$B(x,v) \equiv 0 \leq x \wedge x \leq H$$

$$\mathrm{grav} \equiv \{x' = v,\, v' = -g \,\&\, x \geq 0\}$$

$$
\text{loop}\;\dfrac{A\rightarrow \mathrm{j}(x,v) \qquad \dfrac{\text{[;]}\;\dfrac{\text{MR}\;\dfrac{\mathrm{j}(x,v)\rightarrow[\mathrm{grav}]\mathrm{j}(x,v) \qquad [\cup]\;\dfrac{\wedge\mathrm{R}\;\dfrac{\text{[;]}\;\dfrac{\dfrac{}{\mathrm{j}(x,v)\rightarrow[?x=0][v:=-cv]\mathrm{j}(x,v)}}{\mathrm{j}(x,v)\rightarrow[?x=0;\,v:=-cv]\mathrm{j}(x,v)} \qquad \dfrac{}{\mathrm{j}(x,v)\rightarrow[?x\neq0]\mathrm{j}(x,v)}}{\mathrm{j}(x,v)\rightarrow[?x=0;\,v:=-cv]\mathrm{j}(x,v)\wedge[?x\neq0]\mathrm{j}(x,v)}}{\mathrm{j}(x,v)\rightarrow[?x=0;\,v:=-cv\cup ?x\neq0]\mathrm{j}(x,v)}}{\mathrm{j}(x,v)\rightarrow[\mathrm{grav}][?x=0;\,v:=-cv\cup ?x\neq0]\mathrm{j}(x,v)}}{\mathrm{j}(x,v)\rightarrow[\mathrm{grav};(?x=0;\,v:=-cv\cup ?x\neq0)]\mathrm{j}(x,v)}}{\mathrm{j}(x,v)\rightarrow[\mathrm{grav};(?x=0;\,v:=-cv\cup ?x\neq0)]\mathrm{j}(x,v)} \qquad \mathrm{j}(x,v)\rightarrow B(x,v)}{A\rightarrow\big[\big(\mathrm{grav};(?x=0;\,v:=-cv\cup ?x\neq0)\big)^*\big]B(x,v)}
$$

$$A \equiv 0\le x\wedge x=H\wedge v=0\wedge g>0\wedge 1\ge c\ge 0$$

$$B(x,v)\equiv 0\le x\wedge x\le H$$

$$\mathrm{grav}\equiv\{x'=v,v'=-g\,\&\,x\ge 0\}$$

$$\dfrac{}{\mathrm{j}(x,v),\, x=0 \to [v:=-cv]\mathrm{j}(x,v)}$$

$$[?],\!\to\!\mathrm{R}\ \dfrac{\mathrm{j}(x,v),\, x=0 \to [v:=-cv]\mathrm{j}(x,v)}{\mathrm{j}(x,v) \to [?x=0][v:=-cv]\mathrm{j}(x,v)}$$

$$[;]\ \dfrac{\mathrm{j}(x,v) \to [?x=0][v:=-cv]\mathrm{j}(x,v)}{\mathrm{j}(x,v) \to [?x=0;\, v:=-cv]\mathrm{j}(x,v)} \qquad \dfrac{}{\mathrm{j}(x,v) \to [?x\neq 0]\mathrm{j}(x,v)}$$

$$\wedge\mathrm{R}\ \dfrac{\mathrm{j}(x,v) \to [?x=0;\, v:=-cv]\mathrm{j}(x,v) \qquad \mathrm{j}(x,v) \to [?x\neq 0]\mathrm{j}(x,v)}{\mathrm{j}(x,v) \to [?x=0;\, v:=-cv]\mathrm{j}(x,v) \wedge [?x\neq 0]\mathrm{j}(x,v)}$$

$$[\cup]\ \dfrac{\mathrm{j}(x,v) \to [?x=0;\, v:=-cv]\mathrm{j}(x,v) \wedge [?x\neq 0]\mathrm{j}(x,v)}{\mathrm{j}(x,v) \to [?x=0;\, v:=-cv \cup ?x\neq 0]\mathrm{j}(x,v)}$$

$$\mathrm{MR}\ \dfrac{\mathrm{j}(x,v) \to [\mathrm{grav}]\mathrm{j}(x,v) \qquad \mathrm{j}(x,v) \to [?x=0;\, v:=-cv \cup ?x\neq 0]\mathrm{j}(x,v)}{\mathrm{j}(x,v) \to [\mathrm{grav}][?x=0;\, v:=-cv \cup ?x\neq 0]\mathrm{j}(x,v)}$$

$$[;]\ \dfrac{\mathrm{j}(x,v) \to [\mathrm{grav}][?x=0;\, v:=-cv \cup ?x\neq 0]\mathrm{j}(x,v)}{\mathrm{j}(x,v) \to [\mathrm{grav};(?x=0;\, v:=-cv \cup ?x\neq 0)]\mathrm{j}(x,v)}$$

$$\mathrm{loop}\ \dfrac{A \to \mathrm{j}(x,v) \qquad \mathrm{j}(x,v) \to [\mathrm{grav};(?x=0;\, v:=-cv \cup ?x\neq 0)]\mathrm{j}(x,v) \qquad \mathrm{j}(x,v) \to B(x,v)}{A \to [\big(\mathrm{grav};(?x=0;\, v:=-cv \cup ?x\neq 0)\big)^*]B(x,v)}$$

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$$[:=]\ \dfrac{\mathrm{j}(x,v),\, x{=}0 \to \mathrm{j}(x,-cv)}{\mathrm{j}(x,v),\, x{=}0 \to [v:=-cv]\mathrm{j}(x,v)}$$

$$[?],\to\mathrm{R}\ \dfrac{\mathrm{j}(x,v),\, x{=}0 \to [v:=-cv]\mathrm{j}(x,v)}{\mathrm{j}(x,v) \to [?x{=}0][v:=-cv]\mathrm{j}(x,v)}$$

$$[;]\ \dfrac{\mathrm{j}(x,v) \to [?x{=}0][v:=-cv]\mathrm{j}(x,v)}{\mathrm{j}(x,v) \to [?x{=}0;\, v:=-cv]\mathrm{j}(x,v)}$$

$$\dfrac{}{\mathrm{j}(x,v) \to [?x{\neq}0]\mathrm{j}(x,v)}$$

$$\wedge\mathrm{R}\ \dfrac{\mathrm{j}(x,v) \to [?x{=}0;\, v:=-cv]\mathrm{j}(x,v) \qquad \mathrm{j}(x,v) \to [?x{\neq}0]\mathrm{j}(x,v)}{\mathrm{j}(x,v) \to [?x{=}0;\, v:=-cv]\mathrm{j}(x,v) \wedge [?x{\neq}0]\mathrm{j}(x,v)}$$

$$[\cup]\ \dfrac{\mathrm{j}(x,v) \to [?x{=}0;\, v:=-cv]\mathrm{j}(x,v) \wedge [?x{\neq}0]\mathrm{j}(x,v)}{\mathrm{j}(x,v) \to [?x{=}0;\, v:=-cv \cup ?x{\neq}0]\mathrm{j}(x,v)}$$

$$\mathrm{MR}\ \dfrac{\mathrm{j}(x,v) \to [\mathrm{grav}]\mathrm{j}(x,v) \qquad \mathrm{j}(x,v) \to [?x{=}0;\, v:=-cv \cup ?x{\neq}0]\mathrm{j}(x,v)}{\mathrm{j}(x,v) \to [\mathrm{grav}][?x{=}0;\, v:=-cv \cup ?x{\neq}0]\mathrm{j}(x,v)}$$

$$[;]\ \dfrac{\mathrm{j}(x,v) \to [\mathrm{grav}][?x{=}0;\, v:=-cv \cup ?x{\neq}0]\mathrm{j}(x,v)}{\mathrm{j}(x,v) \to [\mathrm{grav};\, (?x{=}0;\, v:=-cv \cup ?x{\neq}0)]\mathrm{j}(x,v)}$$

$$\dfrac{\mathrm{j}(x,v) \to [\mathrm{grav};\, (?x{=}0;\, v:=-cv \cup ?x{\neq}0)]\mathrm{j}(x,v)}{\mathrm{j}(x,v) \to [\mathrm{grav};\, (?x{=}0;\, v:=-cv \cup ?x{\neq}0)]\mathrm{j}(x,v)}$$

$$\mathrm{loop}\ \dfrac{A \to \mathrm{j}(x,v) \qquad \mathrm{j}(x,v) \to [\mathrm{grav};\, (?x{=}0;\, v:=-cv \cup ?x{\neq}0)]\mathrm{j}(x,v) \qquad \mathrm{j}(x,v) \to B(x,v)}{A \to [\big(\mathrm{grav};\, (?x{=}0;\, v:=-cv \cup ?x{\neq}0)\big)^*]B(x,v)}$$

$$A \equiv 0 \le x \wedge x = H \wedge v = 0 \wedge g > 0 \wedge 1 \ge c \ge 0$$

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$$\mathrm{grav} \equiv \{x' = v, v' = -g \,\&\, x \ge 0\}$$

$$
\text{loop}\;\dfrac{A\to j(x,v) \qquad \dfrac{\text{[;]}\;\dfrac{\text{MR}\;\dfrac{j(x,v)\to[\text{grav}]j(x,v) \quad \text{[}\cup\text{]}\;\dfrac{\land\text{R}\;\dfrac{\text{[;]}\;\dfrac{\text{[?]},\to\text{R}\;\dfrac{\text{[:=]}\;\dfrac{j(x,v),\, x{=}0\to j(x,-cv)}{j(x,v),\, x{=}0\to[v:=-cv]j(x,v)}}{j(x,v)\to[?x{=}0][v:=-cv]j(x,v)}}{j(x,v)\to[?x{=}0;\, v:=-cv]j(x,v)} \quad \text{[?]}\;\dfrac{j(x,v),\, x\neq0\to j(x,v)}{j(x,v)\to[{\color{red}?x\neq0}]j(x,v)}}{j(x,v)\to[?x{=}0;\, v:=-cv]j(x,v)\land[?x\neq0]j(x,v)}}{j(x,v)\to[?x{=}0;\, v:=-cv\cup ?x\neq0]j(x,v)}}{j(x,v)\to[\text{grav}][?x{=}0;\, v:=-cv\cup ?x\neq0]j(x,v)}}{j(x,v)\to[\text{grav};(?x{=}0;\, v:=-cv\cup ?x\neq0)]j(x,v)}}{j(x,v)\to[\text{grav};(?x{=}0;\, v:=-cv\cup ?x\neq0)]j(x,v)} \qquad j(x,v)\to B(x,v)}{A\to\big[\big(\text{grav};(?x{=}0;\, v:=-cv\cup ?x\neq0)\big)^*\big]B(x,v)}
$$

$$A\equiv 0\leq x\land x=H\land v=0\land g>0\land 1\geq c\geq 0$$

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$$\text{grav}\equiv\{x'=v,v'=-g\,\&\,x\geq 0\}$$

$$[:=]\;\frac{\mathrm{j}(x,v),\, x=0 \to \mathrm{j}(x,-cv)}{\mathrm{j}(x,v),\, x=0 \to [v:=-cv]\mathrm{j}(x,v)}$$

$$[?],\!\to\!\mathrm{R}\;\frac{\mathrm{j}(x,v),\, x=0 \to [v:=-cv]\mathrm{j}(x,v)}{\mathrm{j}(x,v) \to [?x=0][v:=-cv]\mathrm{j}(x,v)}$$

$$[;]\;\frac{\mathrm{j}(x,v) \to [?x=0][v:=-cv]\mathrm{j}(x,v)}{\mathrm{j}(x,v) \to [?x=0;\, v:=-cv]\mathrm{j}(x,v)}$$

$$[?]\;\frac{\mathrm{j}(x,v),\, x\neq 0 \to \mathrm{j}(x,v)}{\mathrm{j}(x,v) \to [?x\neq 0]\mathrm{j}(x,v)}$$

$$\wedge\mathrm{R}\;\frac{\mathrm{j}(x,v) \to [?x=0;\, v:=-cv]\mathrm{j}(x,v) \qquad \mathrm{j}(x,v) \to [?x\neq 0]\mathrm{j}(x,v)}{\mathrm{j}(x,v) \to [?x=0;\, v:=-cv]\mathrm{j}(x,v) \wedge [?x\neq 0]\mathrm{j}(x,v)}$$

$$[\cup]\;\frac{\mathrm{j}(x,v) \to [?x=0;\, v:=-cv]\mathrm{j}(x,v) \wedge [?x\neq 0]\mathrm{j}(x,v)}{\mathrm{j}(x,v) \to [?x=0;\, v:=-cv \cup ?x\neq 0]\mathrm{j}(x,v)}$$

$$\mathrm{MR}\;\frac{\mathrm{j}(x,v) \to [\mathrm{grav}]\mathrm{j}(x,v) \qquad \mathrm{j}(x,v) \to [?x=0;\, v:=-cv \cup ?x\neq 0]\mathrm{j}(x,v)}{\mathrm{j}(x,v) \to [\mathrm{grav}][?x=0;\, v:=-cv \cup ?x\neq 0]\mathrm{j}(x,v)}$$

$$[;]\;\frac{\mathrm{j}(x,v) \to [\mathrm{grav}][?x=0;\, v:=-cv \cup ?x\neq 0]\mathrm{j}(x,v)}{\mathrm{j}(x,v) \to [\mathrm{grav};\, (?x=0;\, v:=-cv \cup ?x\neq 0)]\mathrm{j}(x,v)}$$

$$\mathrm{loop}\;\frac{A \to \mathrm{j}(x,v) \qquad \dfrac{\mathrm{j}(x,v) \to [\mathrm{grav};\, (?x=0;\, v:=-cv \cup ?x\neq 0)]\mathrm{j}(x,v)}{\mathrm{j}(x,v) \to [\mathrm{grav};\, (?x=0;\, v:=-cv \cup ?x\neq 0)]\mathrm{j}(x,v)} \qquad \mathrm{j}(x,v) \to B(x,v)}{A \to \left[\big(\mathrm{grav};\, (?x=0;\, v:=-cv \cup ?x\neq 0)\big)^*\right]B(x,v)}$$

$$A \equiv 0 \le x \wedge x = H \wedge v = 0 \wedge g > 0 \wedge 1 \ge c \ge 0$$

$$B(x,v) \equiv 0 \le x \wedge x \le H$$

$$\mathrm{grav} \equiv \{x' = v, v' = -g \,\&\, x \ge 0\}$$

$A \to j(x,v)$
$j(x,v) \to [\text{grav}](j(x,v))$
$j(x,v), x=0 \to j(x,(-cv))$
$j(x,v), x\neq 0 \to j(x,v)$
$j(x,v) \to B(x,v)$

$$A \equiv 0 \le x \land x = H \land v = 0 \land g > 0 \land 1 \ge c \ge 0$$
$$B(x,v) \equiv 0 \le x \land x \le H$$
$$\text{grav} \equiv \{x' = v, v' = -g \,\&\, x \ge 0\}$$

$0 \leq x \land x = H \land v = 0 \land g > 0 \land 1 \geq c \geq 0 \rightarrow \mathrm{j}(x,v)$

$\mathrm{j}(x,v) \rightarrow [\{x'=v, v'=-g \,\&\, x \geq 0\}](\mathrm{j}(x,v))$

$\mathrm{j}(x,v), x=0 \rightarrow \mathrm{j}(x,(-cv))$

$\mathrm{j}(x,v), x \neq 0 \rightarrow \mathrm{j}(x,v)$

$\mathrm{j}(x,v) \rightarrow 0 \leq x \land x \leq H$

$$A \equiv 0 \leq x \land x = H \land v = 0 \land g > 0 \land 1 \geq c \geq 0$$

$$B(x,v) \equiv 0 \leq x \land x \leq H$$

$$\mathrm{grav} \equiv \{x' = v, v' = -g \,\&\, x \geq 0\}$$

Proving Quantum the Acrophobic Bouncing Ball

$0 \le x \land x = H \land v = 0 \land g > 0 \land 1 \ge c \ge 0 \rightarrow \mathrm{j}(x,v)$

$\mathrm{j}(x,v) \rightarrow [\{x'=v, v'=-g \,\&\, x\ge 0\}](\mathrm{j}(x,v))$

$\mathrm{j}(x,v), x=0 \rightarrow \mathrm{j}(x,(-cv))$

$\mathrm{j}(x,v), x\neq 0 \rightarrow \mathrm{j}(x,v)$

$\mathrm{j}(x,v) \rightarrow 0 \le x \land x \le H$

2. $\mathrm{j}(x,v) \equiv 0 \le x \land x \le H$

$$A \equiv 0 \le x \land x = H \land v = 0 \land g > 0 \land 1 \ge c \ge 0$$

$$B(x,v) \equiv 0 \le x \land x \le H$$

$$\mathrm{grav} \equiv \{x' = v, v' = -g \,\&\, x \ge 0\}$$

$0 \le x \wedge x = H \wedge v = 0 \wedge g > 0 \wedge 1 \ge c \ge 0 \rightarrow \mathrm{j}(x,v)$

$\mathrm{j}(x,v) \rightarrow [\{x'=v, v'=-g \,\&\, x \ge 0\}](\mathrm{j}(x,v))$

$\mathrm{j}(x,v), x=0 \rightarrow \mathrm{j}(x,(-cv))$

$\mathrm{j}(x,v), x \ne 0 \rightarrow \mathrm{j}(x,v)$

$\mathrm{j}(x,v) \rightarrow 0 \le x \wedge x \le H$

2. $\mathrm{j}(x,v) \equiv 0 \le x \wedge x \le H$ weak: fails ODE if $v \gg 0$

$$A \equiv 0 \le x \wedge x = H \wedge v = 0 \wedge g > 0 \wedge 1 \ge c \ge 0$$

$$B(x,v) \equiv 0 \le x \wedge x \le H$$

$$\text{grav} \equiv \{x' = v, v' = -g \,\&\, x \ge 0\}$$

$0 \le x \wedge x = H \wedge v = 0 \wedge g > 0 \wedge 1 \ge c \ge 0 \rightarrow j(x,v)$

$j(x,v) \rightarrow [\{x'=v, v'=-g \,\&\, x \ge 0\}](j(x,v))$

$j(x,v), x=0 \rightarrow j(x,(-cv))$

$j(x,v), x \ne 0 \rightarrow j(x,v)$

$j(x,v) \rightarrow 0 \le x \wedge x \le H$

1. $j(x,v) \equiv x \ge 0$
2. $j(x,v) \equiv 0 \le x \wedge x \le H$ weak: fails ODE if $v \gg 0$

$$A \equiv 0 \le x \wedge x = H \wedge v = 0 \wedge g > 0 \wedge 1 \ge c \ge 0$$

$$B(x,v) \equiv 0 \le x \wedge x \le H$$

$$\text{grav} \equiv \{x' = v, v' = -g \,\&\, x \ge 0\}$$

Proving Quantum the Acrophobic Bouncing Ball

$0 \leq x \land x = H \land v = 0 \land g > 0 \land 1 \geq c \geq 0 \rightarrow \mathrm{j}(x,v)$

$\mathrm{j}(x,v) \rightarrow [\{x'=v, v'=-g \,\&\, x \geq 0\}](\mathrm{j}(x,v))$

$\mathrm{j}(x,v), x=0 \rightarrow \mathrm{j}(x,(-cv))$

$\mathrm{j}(x,v), x \neq 0 \rightarrow \mathrm{j}(x,v)$

$\mathrm{j}(x,v) \rightarrow 0 \leq x \land x \leq H$

1. $\mathrm{j}(x,v) \equiv x \geq 0$ — weaker: fails postcondition if $x > H$
2. $\mathrm{j}(x,v) \equiv 0 \leq x \land x \leq H$ — weak: fails ODE if $v \gg 0$

$$A \equiv 0 \leq x \land x = H \land v = 0 \land g > 0 \land 1 \geq c \geq 0$$

$$B(x,v) \equiv 0 \leq x \land x \leq H$$

$$\mathrm{grav} \equiv \{x' = v, v' = -g \,\&\, x \geq 0\}$$

$0 \le x \wedge x = H \wedge v = 0 \wedge g > 0 \wedge 1 \ge c \ge 0 \rightarrow j(x,v)$

$j(x,v) \rightarrow [\{x'=v, v'=-g \,\&\, x \ge 0\}](j(x,v))$

$j(x,v), x=0 \rightarrow j(x,(-cv))$

$j(x,v), x \ne 0 \rightarrow j(x,v)$

$j(x,v) \rightarrow 0 \le x \wedge x \le H$

1. $j(x,v) \equiv x \ge 0$ — weaker: fails postcondition if $x > H$
2. $j(x,v) \equiv 0 \le x \wedge x \le H$ — weak: fails ODE if $v \gg 0$
3. $j(x,v) \equiv x = 0 \wedge v = 0$

$$A \equiv 0 \le x \wedge x = H \wedge v = 0 \wedge g > 0 \wedge 1 \ge c \ge 0$$

$$B(x,v) \equiv 0 \le x \wedge x \le H$$

$$\text{grav} \equiv \{x' = v, v' = -g \,\&\, x \ge 0\}$$

$$0 \le x \land x = H \land v = 0 \land g > 0 \land 1 \ge c \ge 0 \to j(x,v)$$
$$j(x,v) \to [\{x'=v, v'=-g \,\&\, x \ge 0\}](j(x,v))$$
$$j(x,v), x=0 \to j(x,(-cv))$$
$$j(x,v), x \ne 0 \to j(x,v)$$
$$j(x,v) \to 0 \le x \land x \le H$$

1. $j(x,v) \equiv x \ge 0$ — weaker: fails postcondition if $x > H$
2. $j(x,v) \equiv 0 \le x \land x \le H$ — weak: fails ODE if $v \gg 0$
3. $j(x,v) \equiv x = 0 \land v = 0$ — strong: fails initial condition if $x > 0$

$$A \equiv 0 \le x \land x = H \land v = 0 \land g > 0 \land 1 \ge c \ge 0$$
$$B(x,v) \equiv 0 \le x \land x \le H$$
$$\text{grav} \equiv \{x' = v, v' = -g \,\&\, x \ge 0\}$$

$$0 \le x \wedge x = H \wedge v = 0 \wedge g > 0 \wedge 1 \ge c \ge 0 \rightarrow j(x,v)$$
$$j(x,v) \rightarrow [\{x'=v, v'=-g \,\&\, x \ge 0\}](j(x,v))$$
$$j(x,v), x=0 \rightarrow j(x,(-cv))$$
$$j(x,v), x \ne 0 \rightarrow j(x,v)$$
$$j(x,v) \rightarrow 0 \le x \wedge x \le H$$

1. $j(x,v) \equiv x \ge 0$ — weaker: fails postcondition if $x > H$
2. $j(x,v) \equiv 0 \le x \wedge x \le H$ — weak: fails ODE if $v \gg 0$
3. $j(x,v) \equiv x = 0 \wedge v = 0$ — strong: fails initial condition if $x > 0$
4. $j(x,v) \equiv x = 0 \vee x = H \wedge v = 0$

$$A \equiv 0 \le x \wedge x = H \wedge v = 0 \wedge g > 0 \wedge 1 \ge c \ge 0$$
$$B(x,v) \equiv 0 \le x \wedge x \le H$$
$$\text{grav} \equiv \{x' = v, v' = -g \,\&\, x \ge 0\}$$

$$0 \leq x \land x = H \land v = 0 \land g > 0 \land 1 \geq c \geq 0 \rightarrow \mathrm{j}(x,v)$$

$$\mathrm{j}(x,v) \rightarrow [\{x'=v, v'=-g \,\&\, x \geq 0\}](\mathrm{j}(x,v))$$

$$\mathrm{j}(x,v), x=0 \rightarrow \mathrm{j}(x,(-cv))$$

$$\mathrm{j}(x,v), x \neq 0 \rightarrow \mathrm{j}(x,v)$$

$$\mathrm{j}(x,v) \rightarrow 0 \leq x \land x \leq H$$

1. $\mathrm{j}(x,v) \equiv x \geq 0$ — weaker: fails postcondition if $x > H$
2. $\mathrm{j}(x,v) \equiv 0 \leq x \land x \leq H$ — weak: fails ODE if $v \gg 0$
3. $\mathrm{j}(x,v) \equiv x = 0 \land v = 0$ — strong: fails initial condition if $x > 0$
4. $\mathrm{j}(x,v) \equiv x = 0 \lor x = H \land v = 0$ — no space for intermediate states

$$A \equiv 0 \leq x \land x = H \land v = 0 \land g > 0 \land 1 \geq c \geq 0$$

$$B(x,v) \equiv 0 \leq x \land x \leq H$$

$$\mathrm{grav} \equiv \{x' = v, v' = -g \,\&\, x \geq 0\}$$

$$0 \le x \land x = H \land v = 0 \land g > 0 \land 1 \ge c \ge 0 \rightarrow j(x,v)$$
$$j(x,v) \rightarrow [\{x'=v, v'=-g \,\&\, x \ge 0\}](j(x,v))$$
$$j(x,v), x=0 \rightarrow j(x,(-cv))$$
$$j(x,v), x \ne 0 \rightarrow j(x,v)$$
$$j(x,v) \rightarrow 0 \le x \land x \le H$$

1. $j(x,v) \equiv x \ge 0$ — weaker: fails postcondition if $x > H$
2. $j(x,v) \equiv 0 \le x \land x \le H$ — weak: fails ODE if $v \gg 0$
3. $j(x,v) \equiv x = 0 \land v = 0$ — strong: fails initial condition if $x > 0$
4. $j(x,v) \equiv x = 0 \lor x = H \land v = 0$ — no space for intermediate states
5. $j(x,v) \equiv 2gx = 2gH - v^2 \land x \ge 0$

$$A \equiv 0 \le x \land x = H \land v = 0 \land g > 0 \land 1 \ge c \ge 0$$
$$B(x,v) \equiv 0 \le x \land x \le H$$
$$\text{grav} \equiv \{x' = v, v' = -g \,\&\, x \ge 0\}$$

$$0 \le x \land x = H \land v = 0 \land g > 0 \land 1 \ge c \ge 0 \rightarrow j(x,v)$$
$$j(x,v) \rightarrow [\{x'=v, v'=-g \,\&\, x \ge 0\}](j(x,v))$$
$$j(x,v), x=0 \rightarrow j(x,(-cv))$$
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1. $j(x,v) \equiv x \ge 0$ — weaker: fails postcondition if $x > H$
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3. $j(x,v) \equiv x = 0 \land v = 0$ — strong: fails initial condition if $x > 0$
4. $j(x,v) \equiv x = 0 \lor x = H \land v = 0$ — no space for intermediate states
5. $j(x,v) \equiv 2gx = 2gH - v^2 \land x \ge 0$ — works: implicitly links v and x

$$A \equiv 0 \le x \land x = H \land v = 0 \land g > 0 \land 1 \ge c \ge 0$$
$$B(x,v) \equiv 0 \le x \land x \le H$$
$$\text{grav} \equiv \{x' = v, v' = -g \,\&\, x \ge 0\}$$

$$0 \leq x \land x = H \land v = 0 \land g > 0 \land 1 \geq c \geq 0 \rightarrow 2gx=2gH-v^2 \land x\geq 0$$
$$2gx=2gH-v^2 \land x\geq 0 \rightarrow [\{x'=v, v'=-g \,\&\, x\geq 0\}](2gx=2gH-v^2 \land x\geq 0)$$
$$2gx=2gH-v^2 \land x\geq 0, x=0 \rightarrow 2gx=2gH-(-cv)^2 \land x\geq 0$$
$$2gx=2gH-v^2 \land x\geq 0, x\neq 0 \rightarrow 2gx=2gH-v^2 \land x\geq 0$$
$$2gx=2gH-v^2 \land x\geq 0 \rightarrow 0 \leq x \land x \leq H$$

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5. $j(x,v) \equiv 2gx=2gH-v^2 \land x\geq 0$ works: implicitly links v and x

$$0 \leq x \land x = H \land v = 0 \land g > 0 \land 1 \geq c \geq 0 \rightarrow 2gx{=}2gH{-}v^2 \land x{\geq}0$$

$$2gx{=}2gH{-}v^2 \land x{\geq}0 \rightarrow [\{x'{=}v, v'{=}{-}g \,\&\, x{\geq}0\}](2gx{=}2gH{-}v^2 \land x{\geq}0)$$

$$2gx{=}2gH{-}v^2 \land x{\geq}0, x{=}0 \rightarrow 2gx{=}2gH{-}({-}cv)^2 \land x{\geq}0$$

$$2gx{=}2gH{-}v^2 \land x{\geq}0, x{\neq}0 \rightarrow 2gx{=}2gH{-}v^2 \land x{\geq}0$$

$$2gx{=}2gH{-}v^2 \land x{\geq}0 \rightarrow 0 \leq x \land x \leq H$$

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5. $\mathrm{j}(x,v) \equiv 2gx{=}2gH{-}v^2 \land x{\geq}0$ works: implicitly links v and x

$0 \le x \land x = H \land v = 0 \land g > 0 \land 1 \ge c \ge 0 \rightarrow 2gx{=}2gH{-}v^2 \land x{\ge}0$

$2gx{=}2gH{-}v^2 \land x{\ge}0 \rightarrow [\{x'{=}v, v'{=}{-}g \,\&\, x{\ge}0\}](2gx{=}2gH{-}v^2 \land x{\ge}0)$

✓ $2gx{=}2gH{-}v^2 \land x{\ge}0, x{=}0 \rightarrow 2gx{=}2gH{-}(-cv)^2 \land x{\ge}0$ if $c = 1 \ldots$

$2gx{=}2gH{-}v^2 \land x{\ge}0, x{\neq}0 \rightarrow 2gx{=}2gH{-}v^2 \land x{\ge}0$

$2gx{=}2gH{-}v^2 \land x{\ge}0 \rightarrow 0 \le x \land x \le H$

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$0 \le x \land x = H \land v = 0 \land g > 0 \land 1 \ge c \ge 0 \rightarrow 2gx{=}2gH{-}v^2 \land x{\ge}0$

$2gx{=}2gH{-}v^2 \land x{\ge}0 \rightarrow [\{x'{=}v, v'{=}{-}g \,\&\, x{\ge}0\}](2gx{=}2gH{-}v^2 \land x{\ge}0)$

✓ $2gx{=}2gH{-}v^2 \land x{\ge}0, x{=}0 \rightarrow 2gx{=}2gH{-}(-cv)^2 \land x{\ge}0$ if $c = 1 \ldots$

$2gx{=}2gH{-}v^2 \land x{\ge}0, x{\ne}0 \rightarrow 2gx{=}2gH{-}v^2 \land x{\ge}0$

$2gx{=}2gH{-}v^2 \land x{\ge}0 \rightarrow 0 \le x \land x \le H$

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$0 \leq x \land x = H \land v = 0 \land g > 0 \land 1 \geq c \geq 0 \rightarrow 2gx{=}2gH{-}v^2 \land x{\geq}0$

$2gx{=}2gH{-}v^2 \land x{\geq}0 \rightarrow [\{x'{=}v, v'{=}{-}g \,\&\, x{\geq}0\}](2gx{=}2gH{-}v^2 \land x{\geq}0)$

✓ $2gx{=}2gH{-}v^2 \land x{\geq}0, x{=}0 \rightarrow 2gx{=}2gH{-}({-}cv)^2 \land x{\geq}0$ if $c = 1 \ldots$

✓ $2gx{=}2gH{-}v^2 \land x{\geq}0, x{\neq}0 \rightarrow 2gx{=}2gH{-}v^2 \land x{\geq}0$

$2gx{=}2gH{-}v^2 \land x{\geq}0 \rightarrow 0 \leq x \land x \leq H$

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5. $j(x,v) \equiv 2gx{=}2gH{-}v^2 \land x{\geq}0$ works: implicitly links v and x

$0 \le x \land x = H \land v = 0 \land g > 0 \land 1 \ge c \ge 0 \rightarrow 2gx{=}2gH{-}v^2 \land x{\ge}0$

$2gx{=}2gH{-}v^2 \land x{\ge}0 \rightarrow [\{x'{=}v, v'{=}{-}g \,\&\, x{\ge}0\}](2gx{=}2gH{-}v^2 \land x{\ge}0)$

✓ $2gx{=}2gH{-}v^2 \land x{\ge}0, x{=}0 \rightarrow 2gx{=}2gH{-}(-cv)^2 \land x{\ge}0$ if $c = 1 \ldots$

✓ $2gx{=}2gH{-}v^2 \land x{\ge}0, x{\ne}0 \rightarrow 2gx{=}2gH{-}v^2 \land x{\ge}0$

$2gx{=}2gH{-}v^2 \land x{\ge}0 \rightarrow 0 \le x \land x \le H$

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$0 \leq x \land x = H \land v = 0 \land g > 0 \land 1 \geq c \geq 0 \rightarrow 2gx{=}2gH{-}v^2 \land x{\geq}0$

$2gx{=}2gH{-}v^2 \land x{\geq}0 \rightarrow [\{x'{=}v, v'{=}{-}g \,\&\, x{\geq}0\}](2gx{=}2gH{-}v^2 \land x{\geq}0)$

✓ $2gx{=}2gH{-}v^2 \land x{\geq}0, x{=}0 \rightarrow 2gx{=}2gH{-}({-}cv)^2 \land x{\geq}0$ if $c = 1 \ldots$

✓ $2gx{=}2gH{-}v^2 \land x{\geq}0, x{\neq}0 \rightarrow 2gx{=}2gH{-}v^2 \land x{\geq}0$

✓ $2gx{=}2gH{-}v^2 \land x{\geq}0 \rightarrow 0 \leq x \land x \leq H$ because $g > 0$

1.
2.
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5. $\mathrm{j}(x,v) \equiv 2gx{=}2gH{-}v^2 \land x{\geq}0$ works: implicitly links v and x

$0 \leq x \land x = H \land v = 0 \land g > 0 \land 1 \geq c \geq 0 \rightarrow 2gx = 2gH - v^2 \land x \geq 0$

$2gx = 2gH - v^2 \land x \geq 0 \rightarrow [\{x' = v, v' = -g \,\&\, x \geq 0\}](2gx = 2gH - v^2 \land x \geq 0)$

✓ $2gx = 2gH - v^2 \land x \geq 0, x = 0 \rightarrow 2gx = 2gH - (-cv)^2 \land x \geq 0$ if $c = 1 \ldots$

✓ $2gx = 2gH - v^2 \land x \geq 0, x \neq 0 \rightarrow 2gx = 2gH - v^2 \land x \geq 0$

✓ $2gx = 2gH - v^2 \land x \geq 0 \rightarrow 0 \leq x \land x \leq H$ because $g > 0$

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✓ $0 \le x \land x = H \land v = 0 \land g > 0 \land 1 \ge c \ge 0 \rightarrow 2gx{=}2gH{-}v^2 \land x{\ge}0$

$2gx{=}2gH{-}v^2 \land x{\ge}0 \rightarrow [\{x'{=}v, v'{=}{-}g \,\&\, x{\ge}0\}](2gx{=}2gH{-}v^2 \land x{\ge}0)$

✓ $2gx{=}2gH{-}v^2 \land x{\ge}0, x{=}0 \rightarrow 2gx{=}2gH{-}(-cv)^2 \land x{\ge}0$ if $c = 1 \ldots$

✓ $2gx{=}2gH{-}v^2 \land x{\ge}0, x{\neq}0 \rightarrow 2gx{=}2gH{-}v^2 \land x{\ge}0$

✓ $2gx{=}2gH{-}v^2 \land x{\ge}0 \rightarrow 0 \le x \land x \le H$ because $g > 0$

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5. $\mathrm{j}(x,v) \equiv 2gx{=}2gH{-}v^2 \land x{\ge}0$ works: implicitly links v and x

$x(t) = H - \frac{g}{2}t^2$ $v(t) = -gt$

✓ $0 \leq x \land x = H \land v = 0 \land g > 0 \land 1 \geq c \geq 0 \rightarrow 2gx{=}2gH{-}v^2 \land x{\geq}0$

$2gx{=}2gH{-}v^2 \land x{\geq}0 \rightarrow [\{x'{=}v, v'{=}{-}g \,\&\, x{\geq}0\}](2gx{=}2gH{-}v^2 \land x{\geq}0)$

✓ $2gx{=}2gH{-}v^2 \land x{\geq}0, x{=}0 \rightarrow 2gx{=}2gH{-}(-cv)^2 \land x{\geq}0$ if $c = 1 \ldots$

✓ $2gx{=}2gH{-}v^2 \land x{\geq}0, x{\neq}0 \rightarrow 2gx{=}2gH{-}v^2 \land x{\geq}0$

✓ $2gx{=}2gH{-}v^2 \land x{\geq}0 \rightarrow 0 \leq x \land x \leq H$ because $g > 0$

1.
2.
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4.
5. $\mathrm{j}(x,v) \equiv 2gx{=}2gH{-}v^2 \land x{\geq}0$ works: implicitly links v and x

$$x(t) = H - \frac{g}{2}t^2 \rightsquigarrow 2gx(t) = 2gH - g^2t^2 \qquad v(t)^2 = g^2t^2 \leftsquigarrow v(t) = -gt$$

$$[']\ \frac{}{\mathrm{j}(x,v) \rightarrow [x'=v, v'=-g \,\&\, x \geq 0]\mathrm{j}(x,v)}$$

$$\dfrac{[;]\quad \overline{\text{j}(x,v) \rightarrow \forall t{\geq}0\,[x:=H-\frac{g}{2}t^2;\, v:=-gt](x{\geq}0 \rightarrow \text{j}(x,v))}}{[']\quad \text{j}(x,v) \rightarrow [x'=v, v'=-g\,\&\, x{\geq}0]\text{j}(x,v)}$$

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$$\dfrac{\dfrac{[:=]\quad \text{j}(x,v) \rightarrow \forall t{\geq}0\,[x:=H-\frac{g}{2}t^2][v:=-gt](x{\geq}0 \rightarrow \text{j}(x,v))}{[;]\quad \text{j}(x,v) \rightarrow \forall t{\geq}0\,[x:=H-\frac{g}{2}t^2;\, v:=-gt](x{\geq}0 \rightarrow \text{j}(x,v))}}{['\,]\quad \text{j}(x,v) \rightarrow [x'=v, v'=-g\,\&\, x{\geq}0]\text{j}(x,v)}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{}{j(x,v) \rightarrow \forall t{\geq}0\,[x{:=}H-\frac{g}{2}t^2](x{\geq}0 \rightarrow j(x,-gt))}[{:=}]}{j(x,v) \rightarrow \forall t{\geq}0\,[x{:=}H-\frac{g}{2}t^2][v{:=}-gt](x{\geq}0 \rightarrow j(x,v))}[{:=}]}{j(x,v) \rightarrow \forall t{\geq}0\,[x{:=}H-\frac{g}{2}t^2;\, v{:=}-gt](x{\geq}0 \rightarrow j(x,v))}[;]}{j(x,v) \rightarrow [x'=v, v'=-g\,\&\, x{\geq}0]j(x,v)}['] $$

$$\forall R \quad \text{j}(x,v) \rightarrow \forall t{\geq}0\,\big(H-\frac{g}{2}t^2\geq 0 \rightarrow \text{j}(H-\tfrac{g}{2}t^2, -gt)\big)$$

$$[:=] \quad \text{j}(x,v) \rightarrow \forall t{\geq}0\,[x:=H-\frac{g}{2}t^2]\big(x\geq 0 \rightarrow \text{j}(x, -gt)\big)$$

$$[:=] \quad \text{j}(x,v) \rightarrow \forall t{\geq}0\,[x:=H-\frac{g}{2}t^2][v:=-gt]\big(x\geq 0 \rightarrow \text{j}(x,v)\big)$$

$$[;] \quad \text{j}(x,v) \rightarrow \forall t{\geq}0\,[x:=H-\frac{g}{2}t^2;\, v:=-gt]\big(x\geq 0 \rightarrow \text{j}(x,v)\big)$$

$$['] \quad \text{j}(x,v) \rightarrow [x'=v, v'=-g\,\&\,x\geq 0]\text{j}(x,v)$$

$$
\begin{array}{ll}
{\scriptstyle \to R} & \overline{\text{j}(x,v) \to t\geq 0 \to H-\frac{g}{2}t^2\geq 0 \to \text{j}(H-\frac{g}{2}t^2,-gt)} \\
{\scriptstyle \forall R} & \overline{\text{j}(x,v) \to \forall t\geq 0\, (H-\frac{g}{2}t^2\geq 0 \to \text{j}(H-\frac{g}{2}t^2,-gt))} \\
{\scriptstyle [:=]} & \overline{\text{j}(x,v) \to \forall t\geq 0\, [x:=H-\frac{g}{2}t^2](x\geq 0 \to \text{j}(x,-gt))} \\
{\scriptstyle [:=]} & \overline{\text{j}(x,v) \to \forall t\geq 0\, [x:=H-\frac{g}{2}t^2][v:=-gt](x\geq 0 \to \text{j}(x,v))} \\
{\scriptstyle [;]} & \overline{\text{j}(x,v) \to \forall t\geq 0\, [x:=H-\frac{g}{2}t^2; v:=-gt](x\geq 0 \to \text{j}(x,v))} \\
{\scriptstyle [']} & \text{j}(x,v) \to [x'=v, v'=-g\,\&\, x\geq 0]\text{j}(x,v)
\end{array}
$$

	$j(x,v), t\geq 0, H-\frac{g}{2}t^2\geq 0 \rightarrow j(H-\frac{g}{2}t^2, -gt)$
$\rightarrow$R	$j(x,v) \rightarrow t\geq 0 \rightarrow H-\frac{g}{2}t^2\geq 0 \rightarrow j(H-\frac{g}{2}t^2, -gt)$
$\forall$R	$j(x,v) \rightarrow \forall t\geq 0\, (H-\frac{g}{2}t^2\geq 0 \rightarrow j(H-\frac{g}{2}t^2, -gt))$
$[:=]$	$j(x,v) \rightarrow \forall t\geq 0\, [x:=H-\frac{g}{2}t^2](x\geq 0 \rightarrow j(x, -gt))$
$[:=]$	$j(x,v) \rightarrow \forall t\geq 0\, [x:=H-\frac{g}{2}t^2][v:=-gt](x\geq 0 \rightarrow j(x,v))$
$[;]$	$j(x,v) \rightarrow \forall t\geq 0\, [x:=H-\frac{g}{2}t^2; v:=-gt](x\geq 0 \rightarrow j(x,v))$
$[']$	$j(x,v) \rightarrow [x'=v, v'=-g \,\&\, x\geq 0]j(x,v)$

$$j(x,v) \equiv 2gx=2gH-v^2 \wedge x\geq 0$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{2gx=2gH-v^2\wedge x\geq 0, H-\frac{g}{2}t^2\geq 0 \rightarrow 2g(H-\frac{g}{2}t^2)=2gH-(gt)^2\wedge(H-\frac{g}{2}t^2)\geq 0}{j(x,v), t\geq 0, H-\frac{g}{2}t^2\geq 0 \rightarrow j(H-\frac{g}{2}t^2,-gt)}}{j(x,v)\rightarrow t\geq 0\rightarrow H-\frac{g}{2}t^2\geq 0\rightarrow j(H-\frac{g}{2}t^2,-gt)}\rightarrow R}{j(x,v)\rightarrow\forall t\geq 0\,(H-\frac{g}{2}t^2\geq 0\rightarrow j(H-\frac{g}{2}t^2,-gt))}\forall R}{j(x,v)\rightarrow\forall t\geq 0\,[x:=H-\frac{g}{2}t^2](x\geq 0\rightarrow j(x,-gt))}[:=]}{j(x,v)\rightarrow\forall t\geq 0\,[x:=H-\frac{g}{2}t^2][v:=-gt](x\geq 0\rightarrow j(x,v))}[:=]}{j(x,v)\rightarrow\forall t\geq 0\,[x:=H-\frac{g}{2}t^2;v:=-gt](x\geq 0\rightarrow j(x,v))}[;]}{j(x,v)\rightarrow[x'=v,v'=-g\,\&\,x\geq 0]j(x,v)}[']}{}$$

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$$\dfrac{\overline{2gx=2gH-v^2 \to 2g(H-\tfrac{g}{2}t^2)=2gH-(gt)^2} \qquad \overline{H-\tfrac{g}{2}t^2\geq 0 \to H-\tfrac{g}{2}t^2\geq 0}}{2gx=2gH-v^2 \wedge x\geq 0,\, H-\tfrac{g}{2}t^2\geq 0 \to 2g(H-\tfrac{g}{2}t^2)=2gH-(gt)^2 \wedge (H-\tfrac{g}{2}t^2)\geq 0}\ \wedge\mathrm{R}$$

$$\mathrm{j}(x,v),\, t\geq 0,\, H-\tfrac{g}{2}t^2\geq 0 \to \mathrm{j}(H-\tfrac{g}{2}t^2, -gt)$$

$\to$R

$$\mathrm{j}(x,v) \to t\geq 0 \to H-\tfrac{g}{2}t^2\geq 0 \to \mathrm{j}(H-\tfrac{g}{2}t^2, -gt)$$

$\forall$R

$$\mathrm{j}(x,v) \to \forall t\geq 0\, (H-\tfrac{g}{2}t^2\geq 0 \to \mathrm{j}(H-\tfrac{g}{2}t^2, -gt))$$

[:=]

$$\mathrm{j}(x,v) \to \forall t\geq 0\, [x:=H-\tfrac{g}{2}t^2](x\geq 0 \to \mathrm{j}(x, -gt))$$

[:=]

$$\mathrm{j}(x,v) \to \forall t\geq 0\, [x:=H-\tfrac{g}{2}t^2][v:=-gt](x\geq 0 \to \mathrm{j}(x,v))$$

[;]

$$\mathrm{j}(x,v) \to \forall t\geq 0\, [x:=H-\tfrac{g}{2}t^2;\, v:=-gt](x\geq 0 \to \mathrm{j}(x,v))$$

[′]

$$\mathrm{j}(x,v) \to [x'=v, v'=-g \,\&\, x\geq 0]\mathrm{j}(x,v)$$

$$\dfrac{\dfrac{\dfrac{*}{2gx=2gH-v^2 \to 2g(H-\frac{g}{2}t^2)=2gH-(gt)^2}\ \mathbb{R} \qquad \dfrac{}{H-\frac{g}{2}t^2\geq 0 \to H-\frac{g}{2}t^2\geq 0}}{2gx=2gH-v^2 \land x\geq 0, H-\frac{g}{2}t^2\geq 0 \to 2g(H-\frac{g}{2}t^2)=2gH-(gt)^2 \land (H-\frac{g}{2}t^2)\geq 0}\ \land\mathrm{R}}{}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{j(x,v), t\geq 0, H-\frac{g}{2}t^2\geq 0 \to j(H-\frac{g}{2}t^2, -gt)}{j(x,v) \to t\geq 0 \to H-\frac{g}{2}t^2\geq 0 \to j(H-\frac{g}{2}t^2, -gt)}\ \to\mathrm{R}}{j(x,v) \to \forall t\geq 0\, (H-\frac{g}{2}t^2\geq 0 \to j(H-\frac{g}{2}t^2, -gt))}\ \forall\mathrm{R}}{j(x,v) \to \forall t\geq 0\, [x:=H-\frac{g}{2}t^2](x\geq 0 \to j(x,-gt))}\ [:=]}{j(x,v) \to \forall t\geq 0\, [x:=H-\frac{g}{2}t^2][v:=-gt](x\geq 0 \to j(x,v))}\ [:=]}{j(x,v) \to \forall t\geq 0\, [x:=H-\frac{g}{2}t^2; v:=-gt](x\geq 0 \to j(x,v))}\ [;]$$

$$\dfrac{j(x,v) \to \forall t\geq 0\, [x:=H-\frac{g}{2}t^2; v:=-gt](x\geq 0 \to j(x,v))}{j(x,v) \to [x'=v, v'=-g \,\&\, x\geq 0]j(x,v)}\ [']$$

$$\mathbb{R}\ \frac{*}{2gx=2gH-v^2 \rightarrow 2g(H-\frac{g}{2}t^2)=2gH-(gt)^2} \qquad \text{id}\ \frac{*}{H-\frac{g}{2}t^2\geq 0 \rightarrow H-\frac{g}{2}t^2\geq 0}$$

$$\wedge\text{R}\ \ 2gx=2gH-v^2 \wedge x\geq 0,\ H-\frac{g}{2}t^2\geq 0 \rightarrow 2g(H-\frac{g}{2}t^2)=2gH-(gt)^2 \wedge (H-\frac{g}{2}t^2)\geq 0$$

$$j(x,v),\ t\geq 0,\ H-\frac{g}{2}t^2\geq 0 \rightarrow j(H-\frac{g}{2}t^2, -gt)$$

$$\rightarrow\text{R}\ \ j(x,v) \rightarrow t\geq 0 \rightarrow H-\frac{g}{2}t^2\geq 0 \rightarrow j(H-\frac{g}{2}t^2, -gt)$$

$$\forall\text{R}\ \ j(x,v) \rightarrow \forall t\geq 0\, (H-\frac{g}{2}t^2\geq 0 \rightarrow j(H-\frac{g}{2}t^2, -gt))$$

$$[:=]\ \ j(x,v) \rightarrow \forall t\geq 0\, [x:=H-\frac{g}{2}t^2](x\geq 0 \rightarrow j(x, -gt))$$

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$$[']\ \ j(x,v) \rightarrow [x'=v, v'=-g\,\&\, x\geq 0]j(x,v)$$

$$\mathbb{R}\ \frac{*}{2gx=2gH-v^2 \rightarrow 2g(H-\frac{g}{2}t^2)=2gH-(gt)^2} \qquad \text{id}\ \frac{*}{H-\frac{g}{2}t^2\geq 0 \rightarrow H-\frac{g}{2}t^2\geq 0}$$

$$\wedge\text{R}\ \frac{\ldots}{2gx=2gH-v^2\wedge x\geq 0,\ H-\frac{g}{2}t^2\geq 0 \rightarrow 2g(H-\frac{g}{2}t^2)=2gH-(gt)^2 \wedge (H-\frac{g}{2}t^2)\geq 0}$$

$$\rightarrow\text{R}\ \frac{j(x,v),\ t\geq 0,\ H-\frac{g}{2}t^2\geq 0 \rightarrow j(H-\frac{g}{2}t^2, -gt)}{j(x,v) \rightarrow t\geq 0 \rightarrow H-\frac{g}{2}t^2\geq 0 \rightarrow j(H-\frac{g}{2}t^2, -gt)}$$

$$\forall\text{R}\ \frac{}{j(x,v) \rightarrow \forall t\geq 0\,(H-\frac{g}{2}t^2\geq 0 \rightarrow j(H-\frac{g}{2}t^2, -gt))}$$

$$[:=]\ \frac{}{j(x,v) \rightarrow \forall t\geq 0\,[x:=H-\frac{g}{2}t^2](x\geq 0 \rightarrow j(x, -gt))}$$

$$[:=]\ \frac{}{j(x,v) \rightarrow \forall t\geq 0\,[x:=H-\frac{g}{2}t^2][v:=-gt](x\geq 0 \rightarrow j(x,v))}$$

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$$[']\ \frac{}{j(x,v) \rightarrow [x'=v, v'=-g\ \&\ x\geq 0]j(x,v)}$$

- Oh no! The solutions assume $x = H, v = 0$ which $j(x,v)$ can't guarantee!

$$\dfrac{\dfrac{\dfrac{*}{2gx=2gH-v^2\to 2g(H-\frac{g}{2}t^2)=2gH-(gt)^2}\,\mathbb{R} \qquad \dfrac{*}{H-\frac{g}{2}t^2\geq 0\to H-\frac{g}{2}t^2\geq 0}\,\text{id}}{2gx=2gH-v^2\wedge x\geq 0, H-\frac{g}{2}t^2\geq 0\to 2g(H-\frac{g}{2}t^2)=2gH-(gt)^2\wedge(H-\frac{g}{2}t^2)\geq 0}\,\wedge R}{}$$

$$\dfrac{j(x,v), t\geq 0, H-\frac{g}{2}t^2\geq 0 \to j(H-\frac{g}{2}t^2, -gt)}{j(x,v) \to t\geq 0 \to H-\frac{g}{2}t^2\geq 0 \to j(H-\frac{g}{2}t^2,-gt)}\,\to R$$

$$\dfrac{}{j(x,v) \to \forall t\geq 0\,(H-\frac{g}{2}t^2\geq 0 \to j(H-\frac{g}{2}t^2,-gt))}\,\forall R$$

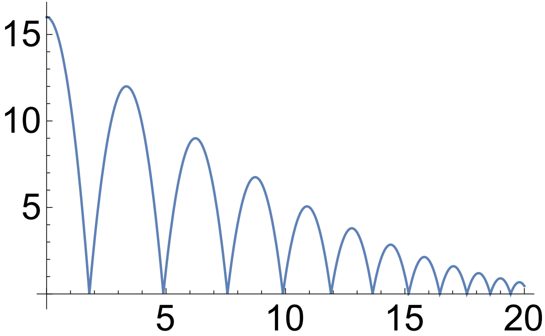
$$\dfrac{}{j(x,v) \to \forall t\geq 0\,[x:=H-\frac{g}{2}t^2](x\geq 0 \to j(x,-gt))}\,[:=]$$

$$\dfrac{}{j(x,v) \to \forall t\geq 0\,[x:=H-\frac{g}{2}t^2][v:=-gt](x\geq 0 \to j(x,v))}\,[:=]$$

$$\dfrac{}{j(x,v) \to \forall t\geq 0\,[x:=H-\frac{g}{2}t^2; v:=-gt](x\geq 0 \to j(x,v))}\,[;]$$

$$\dfrac{}{j(x,v) \to [x'=v, v'=-g\,\&\,x\geq 0]j(x,v)}\,[']$$

- Oh no! The solutions assume $x = H, v = 0$ which $j(x,v)$ can't guarantee!
- Never use solutions without proof! ▸ Todo redo proof with true solution



Example (Bouncing Ball)

$$v{=}0 \land 1{\geq}c{\geq}0 \land H{=}x{\geq}0 \land g{>}0 \rightarrow \left[\left(\{x' = v, v' = -g \,\&\, x \geq 0\};\right.\right.$$
$$\left.\left.\text{if}(x = 0)\, v{:=}{-}cv\right)^*\right] 0{\leq}x{\leq}H$$

The lion's share of understanding comes from understanding what does change (variants/progress measures) and what doesn't change (invariants).

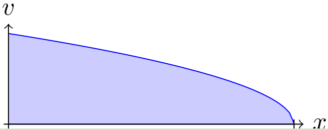
Invariants are a fundamental force of CS

Variants are another fundamental force of CS

"Making something variable is easy.
Controlling duration of constancy is the trick." – Alan J. Perlis

Ex: Car Control Properties

$Q \equiv$

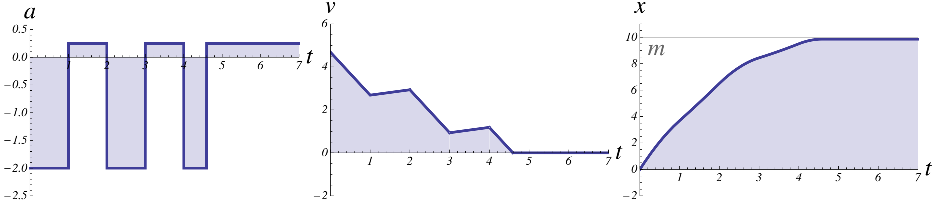


Example (Single car car_ε time-triggered)

$$\Big(\big(((?Q; a:=A) \cup a:=-b); t:=0; \{x' = v, v' = a, t' = 1 \,\&\, v \geq 0 \land t \leq \varepsilon\}\big)^*$$

Example (Safely stays before traffic light m)

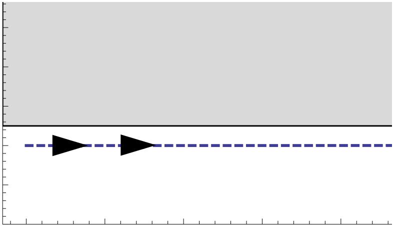
$$v^2 \leq 2b(m-x) \land A \geq 0 \land b > 0 \rightarrow [car_\varepsilon]\, x \leq m$$

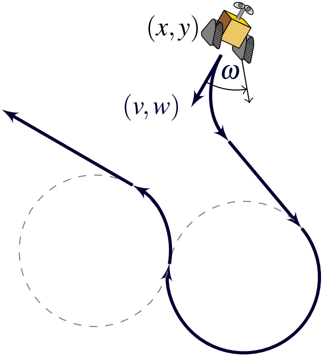


Exercise: TwoCar Control One Car to Follow Another

Model two cars and control one car to safely follow the leader car.

- A maximum acceleration (magnitude)
- B maximum braking (magnitude)
- T maximum reaction time
- x, v, a position, velocity, acceleration of follower car to be controlled
- likewise for lead car, uncontrolled
- motion on a straight line

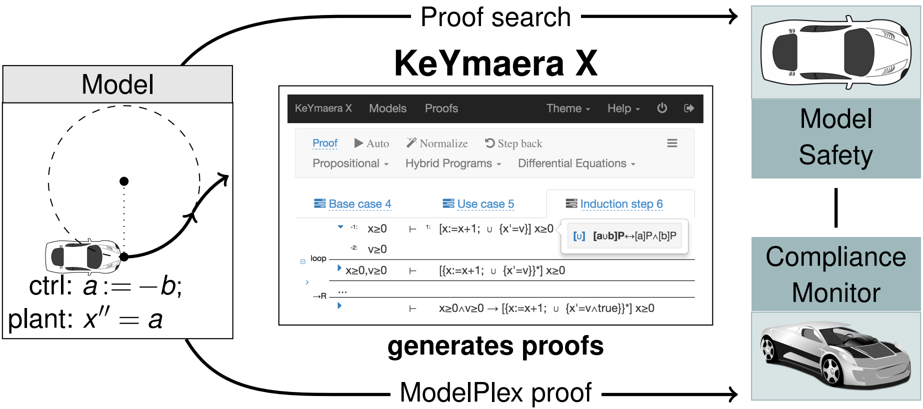




Example (Runaround Robot)

$$(x,y) \neq o \rightarrow \Big[\Big(\big(?Q_{-1};\, \omega:=-1 \cup ?Q_1;\, \omega:=1 \cup ?Q_0;\, \omega:=0\big);$$
$$\{x' = v, y' = w, v' = \omega w, w' = -\omega v\}\Big)^*\Big]\,(x,y) \neq o$$

KeYmaera X aXiomatic Tactical Theorem Prover for CPS



Trustworthy

Uniform substitution

Sound & complete

Small core: 1700 LOC

Flexible

Proof automation

Interactive UI

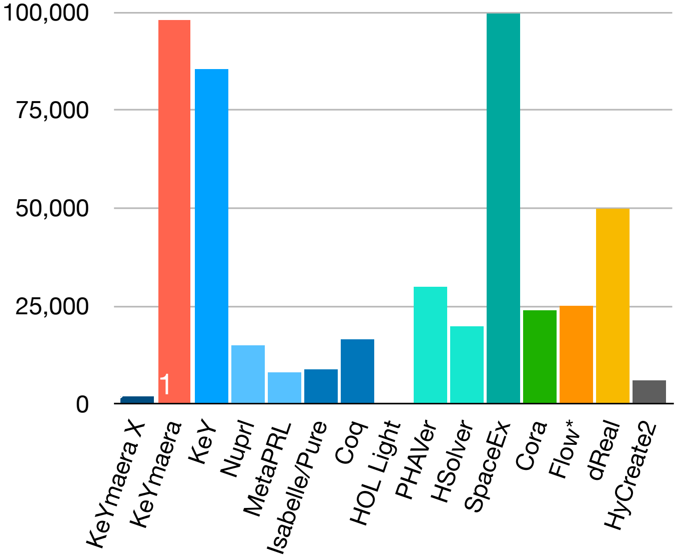
Programmable

Customizable

Scala+Java API

Command line

REST API



Disclaimer: Self-reported estimates of the soundness-critical lines of code + rules

Theorem (Soundness) replace all occurrences of $p(\cdot)$

$$\text{US}\ \frac{\phi}{\sigma(\phi)}$$

provided $FV(\sigma|_{\Sigma(\theta)}) \cap BV(\otimes(\cdot)) = \emptyset$ *for each operation* $\otimes(\theta)$ *in* ϕ

i.e. bound variables $U = \mathrm{BV}(\otimes(\cdot))$ of **no** operator $\otimes$
are free in the substitution on its argument θ (U-admissible)

$$\text{US}\ \frac{[a \cup b]p(\bar{x}) \leftrightarrow [a]p(\bar{x}) \wedge [b]p(\bar{x})}{[x := x+1 \cup x' = 1]x \geq 0 \leftrightarrow [x := x+1]x \geq 0 \wedge [x' = 1]x \geq 0}$$

JAR'17

Uniform Substitution

Theorem (Soundness) replace all occurrences of $p(\cdot)$

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$$\frac{[v:=f]p(v) \leftrightarrow p(f)}{[v:=-x][x'=v]\,x \geq 0 \leftrightarrow [x'=-x]\,x \geq 0}$$

JAR'17

Theorem (Soundness) replace all occurrences of $p(\cdot)$

Modular interface: Prover vs. Logic

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i.e. bound variables $U = \mathrm{BV}(\otimes(\cdot))$ of **no** operator $\otimes$ are free in the substitution on its argument θ (U-admissible)

If you bind a free variable, you go to logic jail!

$$\frac{[v := f]p(v) \leftrightarrow p(f)}{[v := -x][x' = v]\,x \geq 0 \leftrightarrow [x' = -x]\,x \geq 0}$$

Clash

JAR'17

Theorem (Soundness) replace all occurrences of $p(\cdot)$

Modular interface: Prover vs. Logic

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$$\frac{\langle x' = f(x), y' = a(x)y\rangle\, x \geq 1 \leftrightarrow \langle x' = f(x)\rangle\, x \geq 1}{\langle x' = x^2, y' = zy\,y\rangle\, x \geq 1 \leftrightarrow \langle x' = x^2\rangle\, x \geq 1}$$

JAR'17

Uniform Substitution

Theorem (Soundness) — replace all occurrences of $p(\cdot)$

Modular interface: Prover vs. Logic

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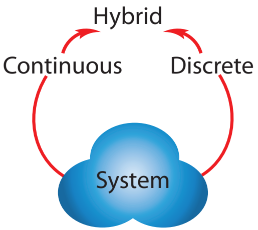
Clash

JAR'17

Complete Proof Theory of Hybrid Systems

Theorem (Sound & Complete) (JAR'08, LICS'12, JAR'17)

dL *calculus is a sound & complete axiomatization of hybrid systems relative to either differential equations* ***or*** *to discrete dynamics.*



Complete Proof Theory of Hybrid Systems

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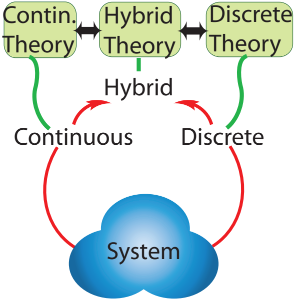

[:=] $[x := e]P(x) \leftrightarrow P(e)$

equations of truth

[?] $[?Q]P \leftrightarrow (Q \rightarrow P)$

[′] $[x' = f(x)]P \leftrightarrow \forall t{\geq}0\, [x{:=}y(t)]P$ $\quad (y'(t) = f(y))$

[∪] $[\alpha \cup \beta]P \leftrightarrow [\alpha]P \wedge [\beta]P$

[;] $[\alpha;\beta]P \leftrightarrow [\alpha][\beta]P$

[*] $[\alpha^*]P \leftrightarrow P \wedge [\alpha][\alpha^*]P$

K $[\alpha](P \rightarrow Q) \rightarrow ([\alpha]P \rightarrow [\alpha]Q)$

laws of logic of
laws of physics

I $[\alpha^*]P \leftrightarrow P \wedge [\alpha^*](P \rightarrow [\alpha]P)$

C $[\alpha^*]\forall v{>}0\, (P(v) \rightarrow \langle\alpha\rangle P(v{-}1)) \rightarrow \forall v\, (P(v) \rightarrow \langle\alpha^*\rangle \exists v{\leq}0\, P(v))$

LICS'12,JAR'17

Outline (Proving ODEs)

Summary: Proving Dynamical Systems

[:=] $[x := e]P(x) \leftrightarrow P(e)$

equations of truth

[?] $[?Q]P \leftrightarrow (Q \rightarrow P)$

['] $[x' = f(x)]P \leftrightarrow \forall t{\geq}0\,[x{:=}y(t)]P$ $\quad (y'(t) = f(y))$

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laws of logic of
laws of physics

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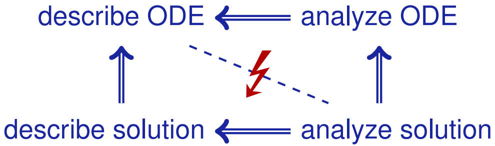
LICS'12,JAR'17

Differential Equation Axiomatization

- Classical approach: ① Given ODE ② Solve ODE ③ Analyze solution
- Descriptive power of ODEs: ODE much easier than its solution
- ⚡ Analyzing ODEs via their solutions undoes their descriptive power!



Poincaré 1881

1. Logical foundations of differential equation invariants LICS'18,JACM'20
2. Decide invariance by dL proof

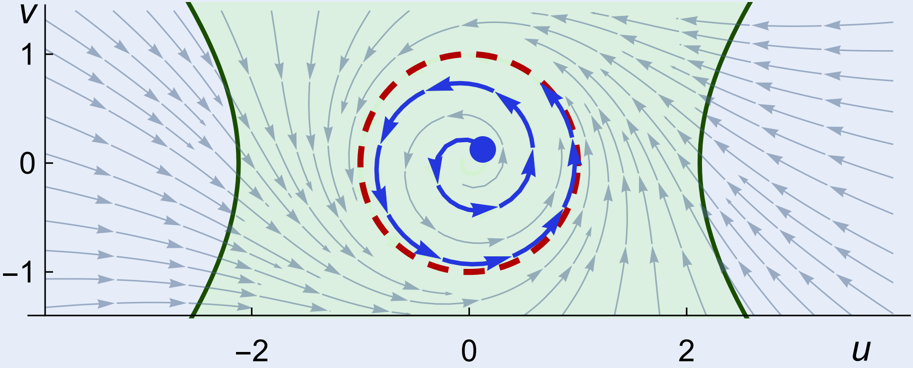
$$x'' = -x \qquad \text{has } x(t) = \sin(t) = t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} + \frac{t^9}{9!} - \dots$$

$$x''(t) = e^{t^2} \qquad \text{has no elementary closed-form solution}$$

Concept (Differential Dynamic Logic) (JAR'08,LICS'12)

$$u^2 \le v^2 + \frac{9}{2} \to [u' = -v + \frac{u}{4}(1-u^2-v^2), v' = u + \frac{v}{4}(1-u^2-v^2)]\, u^2 \le v^2 + \frac{9}{2}$$

$$u^2+v^2 = 1 \to [u' = -v + \frac{u}{4}(1-u^2-v^2), v' = u + \frac{v}{4}(1-u^2-v^2)]\, u^2+v^2 = 1$$



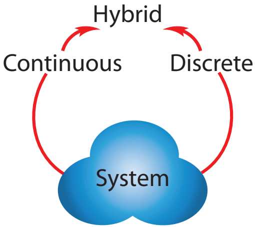
Differential Equation Axiomatization

Theorem (Invariant Completeness) (LICS'18,JACM'20)

dL *calculus is a sound & complete axiomatization of arithmetic invariants of differential equations. They are decidable with a derived axiom.*

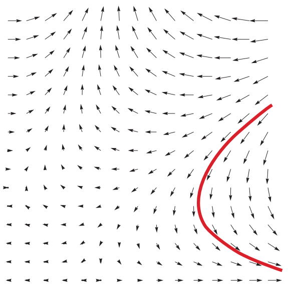
Theorem (Sound & Complete) (JAR'08, LICS'12, JAR'17)

dL *calculus is a sound & complete axiomatization of hybrid systems relative to either differential equations* ***or*** *to discrete dynamics.*

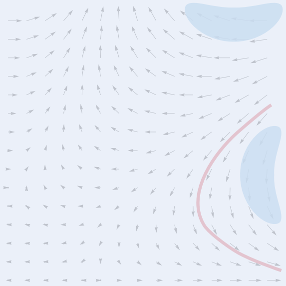


Differential Invariants for Differential Equations

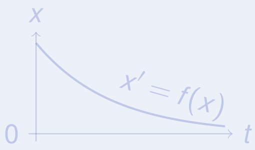
Differential Invariant

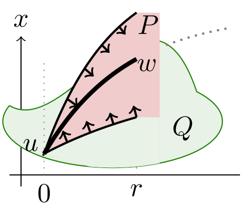


Differential Cut

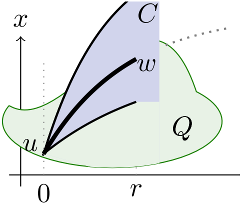


Differential Ghost





$x' = f(x) \,\&\, Q$



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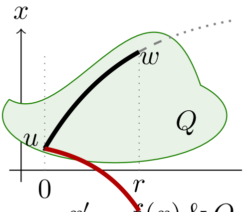
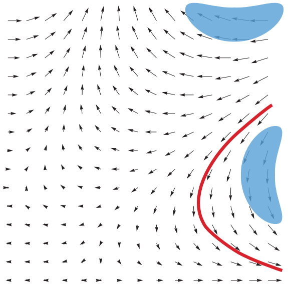


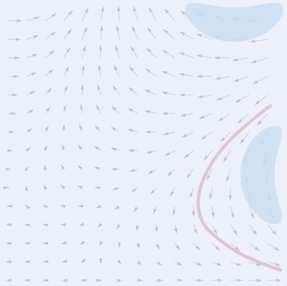
$x' = f(x) \,\&\, Q$

Differential Invariants for Differential Equations

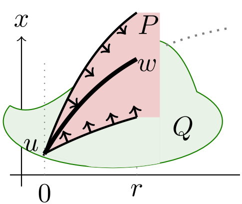
Differential Invariant



Differential Cut



Differential Ghost



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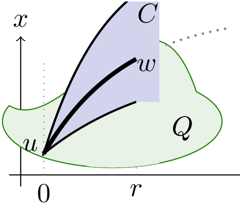

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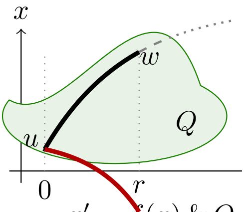
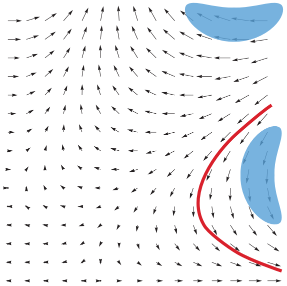
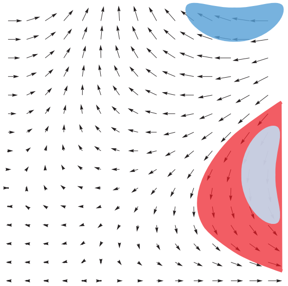


$x' = f(x) \,\&\, Q$

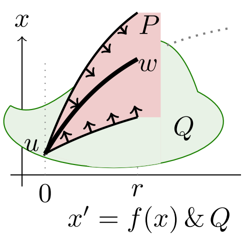
Differential Invariant

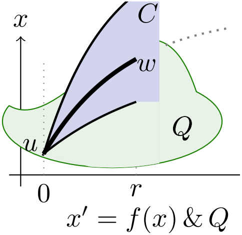


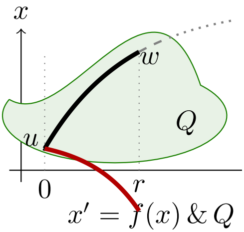
Differential Cut



Differential Ghost

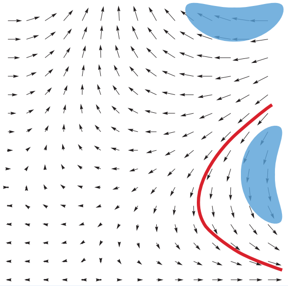




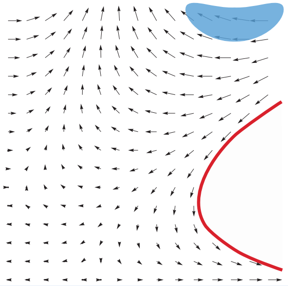


Differential Invariants for Differential Equations

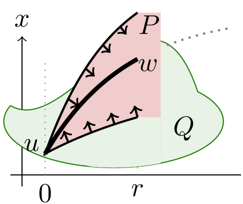
Differential Invariant



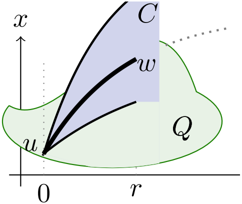
Differential Cut



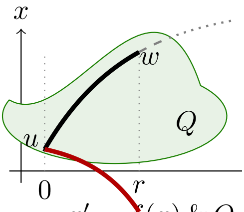
Differential Ghost



$x' = f(x) \,\&\, Q$



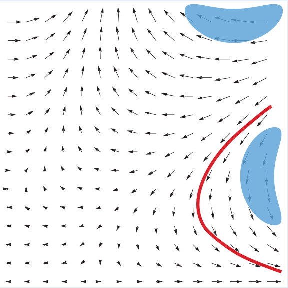
$x' = f(x) \,\&\, Q$

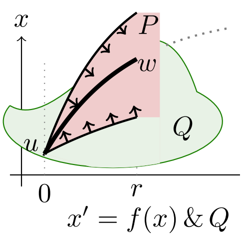


$x' = f(x) \,\&\, Q$

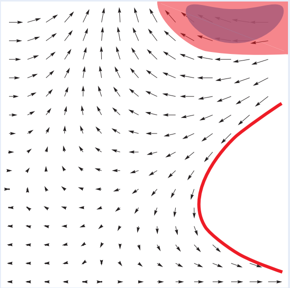
Differential Invariants for Differential Equations

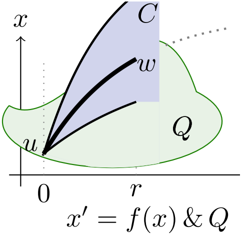
Differential Invariant



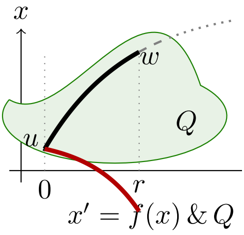


Differential Cut



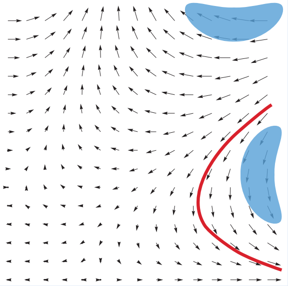


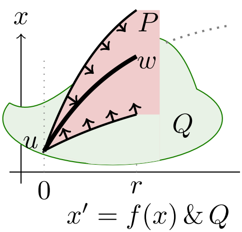
Differential Ghost



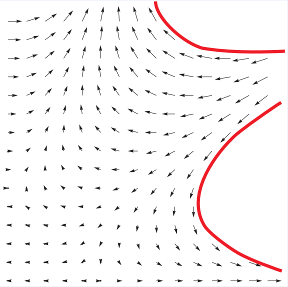
Differential Invariants for Differential Equations

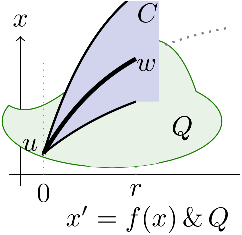
Differential Invariant



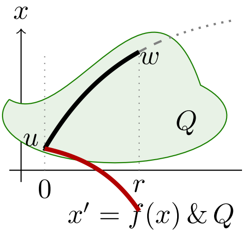


Differential Cut



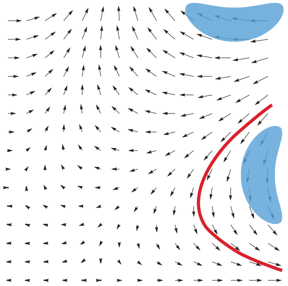


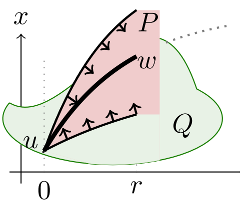
Differential Ghost



Differential Invariants for Differential Equations

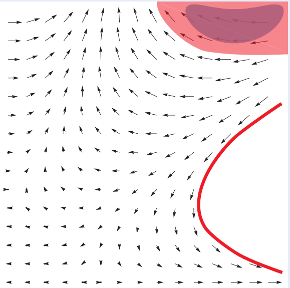
Differential Invariant

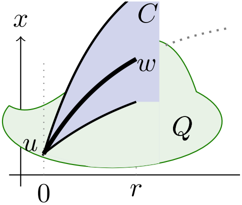




$x' = f(x) \,\&\, Q$

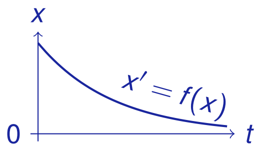
Differential Cut

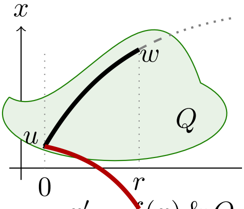




$x' = f(x) \,\&\, Q$

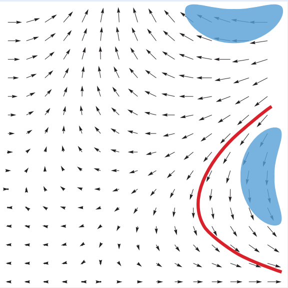
Differential Ghost



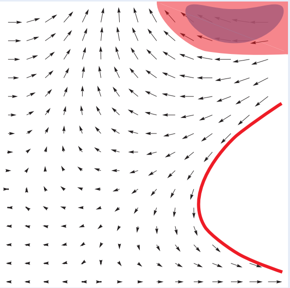


$x' = f(x) \,\&\, Q$

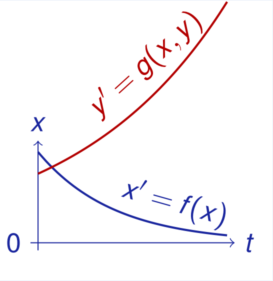
Differential Invariant

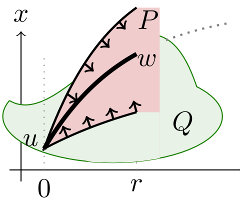


Differential Cut

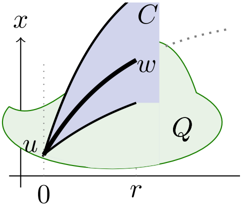


Differential Ghost

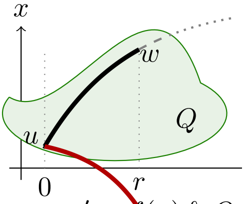




$x' = f(x) \,\&\, Q$

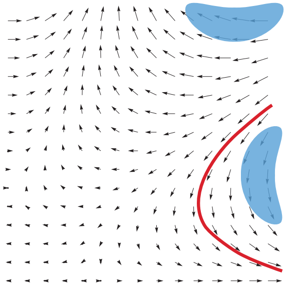


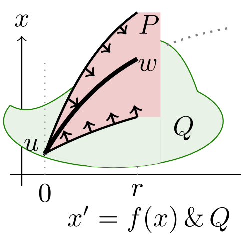
$x' = f(x) \,\&\, Q$



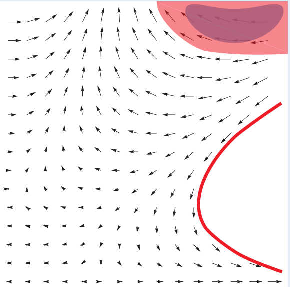
$x' = f(x) \,\&\, Q$

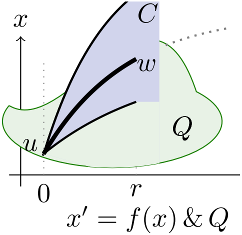
Differential Invariant



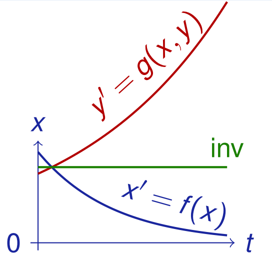


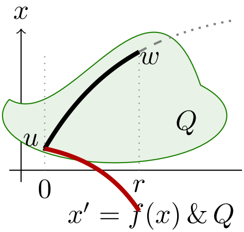
Differential Cut





Differential Ghost





Differential Invariant

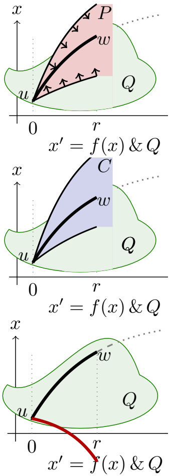
$$\frac{Q \rightarrow [x' := f(x)](P)'}{P \rightarrow [x' = f(x) \,\&\, Q]P}$$

Differential Cut

$$\frac{P \rightarrow [x' = f(x) \,\&\, Q]C \quad P \rightarrow [x' = f(x) \,\&\, Q \land C]P}{P \rightarrow [x' = f(x) \,\&\, Q]P}$$

Differential Ghost

$$\frac{P \leftrightarrow \exists y\, G \quad G \rightarrow [x' = f(x), y' = g(x,y) \,\&\, Q]G}{P \rightarrow [x' = f(x) \,\&\, Q]P}$$



deductive power added DI $\prec$ DI+DC $\prec$ DI+DC+DG

$$\omega[\![(e)']\!] = \sum_x \omega(x') \frac{\partial [\![e]\!]}{\partial x}(\omega)$$

Springer'10, LMCS'12, LICS'12, JAR'17, LICS'18, JACM'20

Differential Invariants for Differential Equations

Differential Invariant

$$\frac{Q \to [x' := f(x)](P)'}{P \to [x' = f(x) \,\&\, Q]P}$$

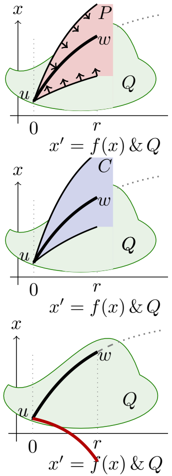
Differential Cut

$$\frac{P \to [x' = f(x) \,\&\, Q]C \quad P \to [x' = f(x) \,\&\, Q \land C]P}{P \to [x' = f(x) \,\&\, Q]P}$$

Differential Ghost

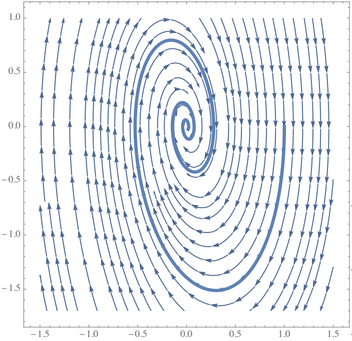
$$\frac{P \leftrightarrow \exists y\, G \quad G \to [x' = f(x), y' = g(x,y) \,\&\, Q]G}{P \to [x' = f(x) \,\&\, Q]P}$$

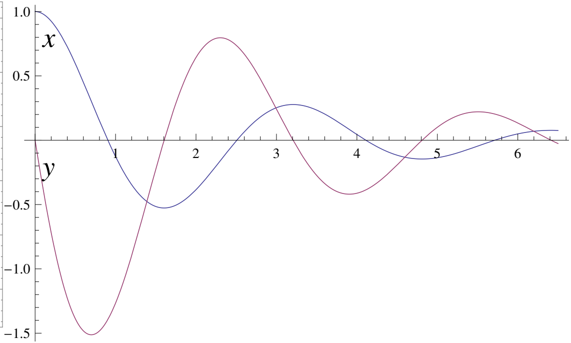
if $g(x,y) = a(x)y + b(x)$, so has long solution!



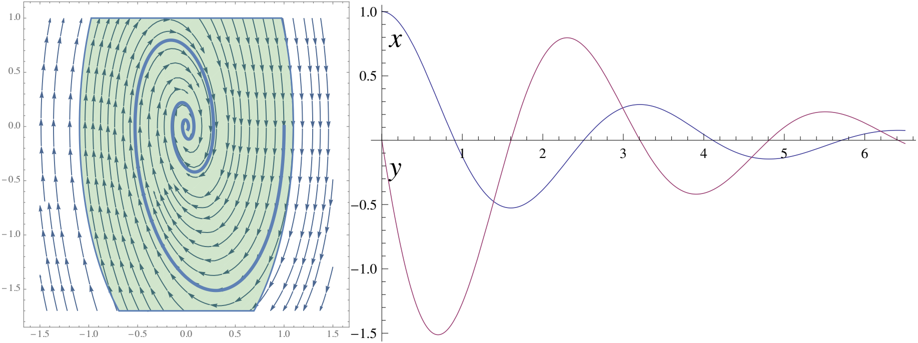
Springer'10, LMCS'12, LICS'12,JAR'17,LICS'18,JACM'20

$$\frac{}{\omega^2x^2+y^2\le c^2 \to [x'=y, y'=-\omega^2x-2d\omega y \,\&\, \omega\ge 0 \land d\ge 0]\,\omega^2x^2+y^2\le c^2}$$



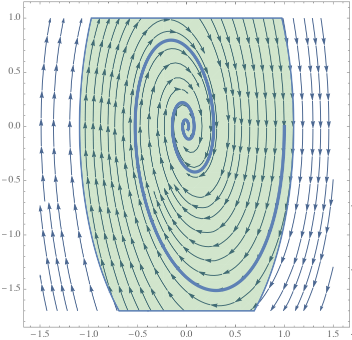


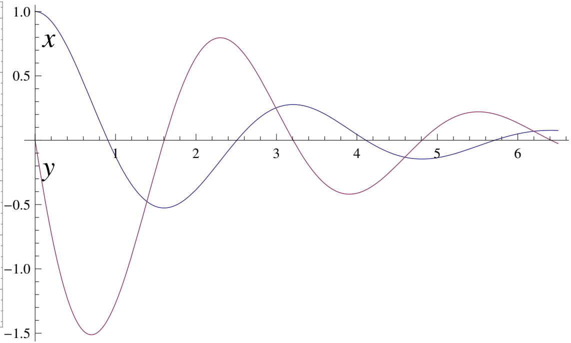
$$\omega^2x^2+y^2\leq c^2 \rightarrow [x' = y, y' = -\omega^2 x - 2d\omega y \,\&\, \omega\geq 0 \land d\geq 0]\, \omega^2x^2+y^2\leq c^2$$



damped oscillator

$$\frac{\omega \geq 0 \land d \geq 0 \rightarrow [x' := y][y' := -\omega^2 x - 2d\omega y]\, 2\omega^2 x x' + 2 y y' \leq 0}{\omega^2 x^2 + y^2 \leq c^2 \rightarrow [x' = y, y' = -\omega^2 x - 2d\omega y \,\&\, \omega \geq 0 \land d \geq 0]\, \omega^2 x^2 + y^2 \leq c^2}$$



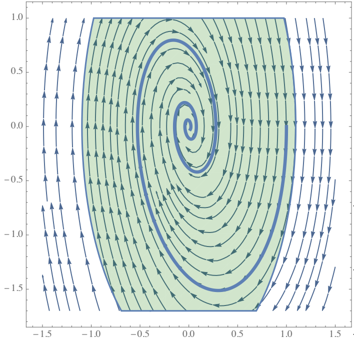


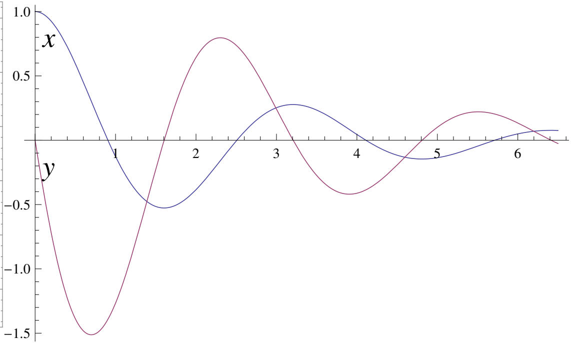
damped oscillator

$$\omega \geq 0 \land d \geq 0 \rightarrow 2\omega^2 x y + 2y(-\omega^2 x - 2d\omega y) \leq 0$$

$$\omega \geq 0 \land d \geq 0 \rightarrow [x' := y][y' := -\omega^2 x - 2d\omega y]\, 2\omega^2 x x' + 2y y' \leq 0$$

$$\omega^2 x^2 + y^2 \leq c^2 \rightarrow [x' = y, y' = -\omega^2 x - 2d\omega y \,\&\, \omega \geq 0 \land d \geq 0]\, \omega^2 x^2 + y^2 \leq c^2$$

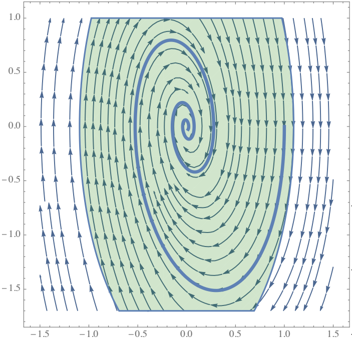


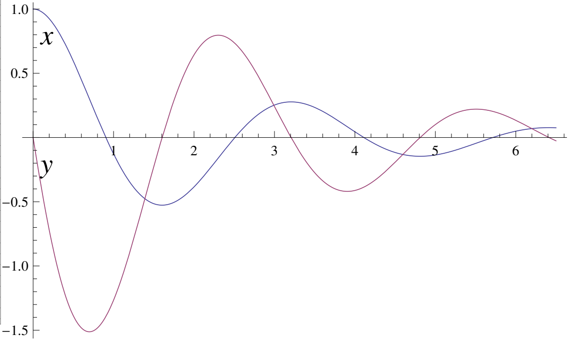


damped oscillator

Differential Invariants for Differential Equations

$$\frac{\frac{\frac{*}{\omega\geq 0 \wedge d\geq 0 \rightarrow 2\omega^2 xy + 2y(-\omega^2 x - 2d\omega y) \leq 0}}{\omega\geq 0 \wedge d\geq 0 \rightarrow [x':=y][y':=-\omega^2 x - 2d\omega y]\, 2\omega^2 x x' + 2yy' \leq 0}}{\omega^2 x^2 + y^2 \leq c^2 \rightarrow [x' = y, y' = -\omega^2 x - 2d\omega y \,\&\, \omega\geq 0 \wedge d\geq 0]\, \omega^2 x^2 + y^2 \leq c^2}}{}$$





damped oscillator

Differential Invariants for Differential Equations

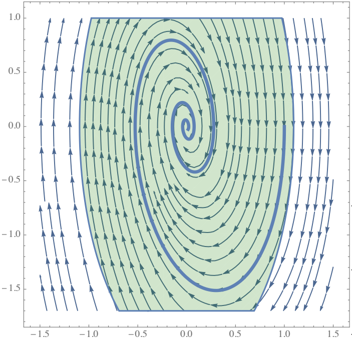
$$*$$

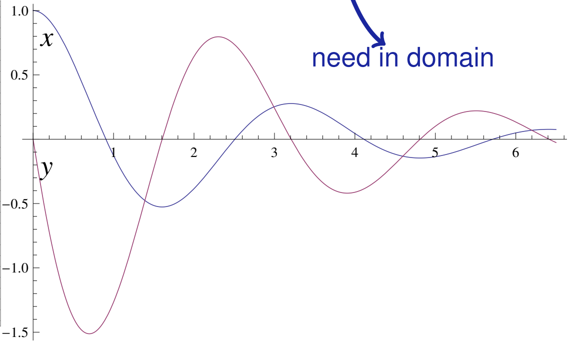
$$\omega\geq 0 \land {\color{red}d\geq 0} \rightarrow 2\omega^2 xy + 2y(-\omega^2 x - 2d\omega y) \leq 0$$

$$\omega\geq 0 \land {\color{red}d\geq 0} \rightarrow [x':=y][y':=-\omega^2 x - 2d\omega y]\, 2\omega^2 x x' + 2yy' \leq 0$$

$${\color{green}\omega^2x^2+y^2\leq c^2} \rightarrow [x' = y, y' = -\omega^2 x - 2d\omega y \,\&\, \omega\geq 0 \land {\color{red}d\geq 0}]\, {\color{green}\omega^2x^2+y^2\leq c^2}$$

need in domain

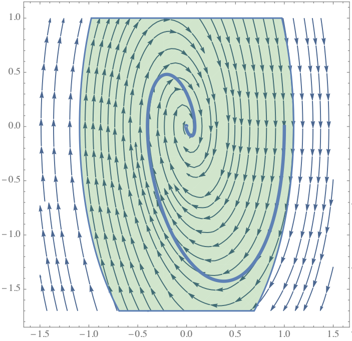


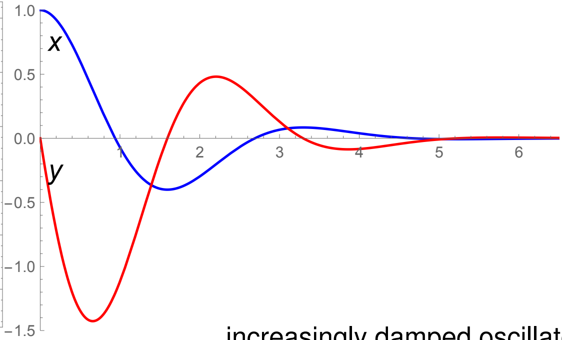


damped oscillator

$$\frac{}{\omega^2x^2+y^2\leq c^2 \rightarrow [x'=y, y'=-\omega^2x-2d\omega y, d'=7 \,\&\, \omega\geq 0]\, \omega^2x^2+y^2\leq c^2}$$

$$\omega^2x^2+y^2\le c^2 \rightarrow [x' = y, y' = -\omega^2x-2d\omega y, d'=7 \,\&\, \omega\ge 0]\, \omega^2x^2+y^2\le c^2$$





increasingly damped oscillator

$$\frac{\omega^2x^2+y^2\leq c^2 \rightarrow [x'=y, y'=-\omega^2x-2d\omega y, d'=7 \,\&\, \omega\geq 0 \land d\geq 0]\, \omega^2x^2+y^2\leq c^2}{\omega^2x^2+y^2\leq c^2 \rightarrow [x'=y, y'=-\omega^2x-2d\omega y, d'=7 \,\&\, \omega\geq 0]\, \omega^2x^2+y^2\leq c^2}$$

increasingly damped oscillator

$$\dfrac{\omega^2x^2+y^2\le c^2 \rightarrow [x'=y, y'=-\omega^2x-2d\omega y, d'=7 \,\&\, \omega\ge 0]\, \omega^2x^2+y^2\le c^2}{\omega^2x^2+y^2\le c^2 \rightarrow [x'=y, y'=-\omega^2x-2d\omega y, d'=7 \,\&\, \omega\ge 0 \land d\ge 0]\, \omega^2x^2+y^2\le c^2}$$

ask

$$\dfrac{}{d\ge 0 \rightarrow [x'=y, y'=-\omega^2x-2d\omega y, d'=7 \,\&\, \omega\ge 0]\, d\ge 0}$$

increasingly damped oscillator

$$\dfrac{\omega^2x^2+y^2\leq c^2 \rightarrow [x'=y, y'=-\omega^2x-2d\omega y, d'=7 \,\&\, \omega\geq 0\wedge d\geq 0]\, \omega^2x^2+y^2\leq c^2}{\omega^2x^2+y^2\leq c^2 \rightarrow [x'=y, y'=-\omega^2x-2d\omega y, d'=7 \,\&\, \omega\geq 0]\, \omega^2x^2+y^2\leq c^2}$$

$$\dfrac{\dfrac{}{\omega\geq 0 \rightarrow [d':=7]\, d'\geq 0}}{d\geq 0 \rightarrow [x'=y, y'=-\omega^2x-2d\omega y, d'=7 \,\&\, \omega\geq 0]\, d\geq 0}$$

increasingly damped oscillator

$$\dfrac{\omega^2x^2+y^2\le c^2 \rightarrow [x'=y, y'=-\omega^2x-2d\omega y, d'=7 \,\&\, \omega\ge0\land d\ge0]\, \omega^2x^2+y^2\le c^2}{\omega^2x^2+y^2\le c^2 \rightarrow [x'=y, y'=-\omega^2x-2d\omega y, d'=7 \,\&\, \omega\ge0]\, \omega^2x^2+y^2\le c^2}$$

$$\dfrac{\dfrac{\dfrac{}{\omega\ge0 \rightarrow 7\ge0}}{\omega\ge0 \rightarrow [d':=7]\, d'\ge0}}{d\ge0 \rightarrow [x'=y, y'=-\omega^2x-2d\omega y, d'=7 \,\&\, \omega\ge0]\, d\ge0}$$

increasingly damped oscillator

$$\dfrac{\dfrac{}{\omega^2x^2+y^2\le c^2 \rightarrow [x'=y, y'=-\omega^2x-2d\omega y, d'=7 \,\&\, \omega\ge 0 \land d\ge 0]\, \omega^2x^2+y^2\le c^2}}{\omega^2x^2+y^2\le c^2 \rightarrow [x'=y, y'=-\omega^2x-2d\omega y, d'=7 \,\&\, \omega\ge 0]\, \omega^2x^2+y^2\le c^2}$$

DC

$$\dfrac{\dfrac{\dfrac{*}{\omega\ge 0 \rightarrow 7\ge 0}}{\omega\ge 0 \rightarrow [d':=7]\, d'\ge 0}}{d\ge 0 \rightarrow [x'=y, y'=-\omega^2x-2d\omega y, d'=7 \,\&\, \omega\ge 0]\, d\ge 0}$$

increasingly damped oscillator

$$\omega\geq0\land d\geq0 \rightarrow [x':=y][y':=-\omega^2x-2d\omega y]\,2\omega^2xx'+2yy'\leq 0$$

$$\omega^2x^2+y^2\leq c^2 \rightarrow [x'=y,y'=-\omega^2x-2d\omega y,d'=7\,\&\,\omega\geq0\land d\geq0]\,\omega^2x^2+y^2\leq c^2$$

$$\omega^2x^2+y^2\leq c^2 \rightarrow [x'=y,y'=-\omega^2x-2d\omega y,d'=7\,\&\,\omega\geq0]\,\omega^2x^2+y^2\leq c^2$$

$$*$$

$$\omega\geq0 \rightarrow 7\geq0$$

$$\omega\geq0 \rightarrow [d':=7]\,d'\geq0$$

$$d\geq0 \rightarrow [x'=y,y'=-\omega^2x-2d\omega y,d'=7\,\&\,\omega\geq0]\,d\geq0$$

increasingly damped oscillator

$$\dfrac{\dfrac{\dfrac{\omega\geq 0\land d\geq 0 \rightarrow 2\omega^2 x y + 2y(-\omega^2 x - 2d\omega y) \leq 0}{\omega\geq 0\land d\geq 0 \rightarrow [x':=y][y':=-\omega^2 x - 2d\omega y]\,2\omega^2 x x' + 2y y' \leq 0}}{\omega^2 x^2+y^2\leq c^2 \rightarrow [x' = y, y' = -\omega^2 x-2d\omega y, d'=7\,\&\,\omega\geq 0\land d\geq 0]\,\omega^2 x^2+y^2\leq c^2}}{\omega^2 x^2+y^2\leq c^2 \rightarrow [x' = y, y' = -\omega^2 x-2d\omega y, d'=7\,\&\,\omega\geq 0]\,\omega^2 x^2+y^2\leq c^2}$$

$$\dfrac{\dfrac{\dfrac{*}{\omega\geq 0 \rightarrow 7\geq 0}}{\omega\geq 0 \rightarrow [d':=7]\,d'\geq 0}}{d\geq 0 \rightarrow [x' = y, y' = -\omega^2 x - 2d\omega y, d'=7\,\&\,\omega\geq 0]\,d\geq 0}$$

increasingly damped oscillator

$$\dfrac{\dfrac{\dfrac{*}{\omega\geq0\land d\geq0 \to 2\omega^2xy+2y(-\omega^2x-2d\omega y)\leq 0}}{\omega\geq0\land d\geq0 \to [x':=y][y':=-\omega^2x-2d\omega y]\,2\omega^2xx'+2yy'\leq 0}}{\omega^2x^2+y^2\leq c^2 \to [x'=y,y'=-\omega^2x-2d\omega y, d'=7\,\&\,\omega\geq0\land d\geq0]\,\omega^2x^2+y^2\leq c^2}}{\omega^2x^2+y^2\leq c^2 \to [x'=y,y'=-\omega^2x-2d\omega y, d'=7\,\&\,\omega\geq0]\,\omega^2x^2+y^2\leq c^2}$$

$$\dfrac{\dfrac{\dfrac{*}{\omega\geq0 \to 7\geq0}}{\omega\geq0 \to [d':=7]\,d'\geq0}}{d\geq0 \to [x'=y,y'=-\omega^2x-2d\omega y, d'=7\,\&\,\omega\geq0]\,d\geq0}$$

increasingly damped oscillator

$$*$$

$$\omega\geq 0 \land d\geq 0 \rightarrow 2\omega^2 xy + 2y(-\omega^2 x - 2d\omega y) \leq 0$$

$$\omega\geq 0 \land d\geq 0 \rightarrow [x':=y][y':=-\omega^2 x - 2d\omega y]\, 2\omega^2 x x' + 2yy' \leq 0$$

$$\omega^2 x^2 + y^2 \leq c^2 \rightarrow [x' = y, y' = -\omega^2 x - 2d\omega y, d'=7 \,\&\, \omega\geq 0 \land d\geq 0]\, \omega^2 x^2 + y^2 \leq c^2$$

$$\omega^2 x^2 + y^2 \leq c^2 \rightarrow [x' = y, y' = -\omega^2 x - 2d\omega y, d'=7 \,\&\, \omega\geq 0]\, \omega^2 x^2 + y^2 \leq c^2$$

init

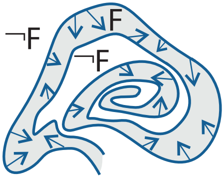
$$*$$

$$\omega\geq 0 \rightarrow 7\geq 0$$

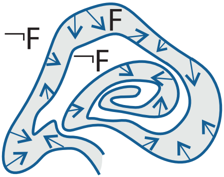
$$\omega\geq 0 \rightarrow [d':=7]\, d'\geq 0$$

$$d\geq 0 \rightarrow [x' = y, y' = -\omega^2 x - 2d\omega y, d'=7 \,\&\, \omega\geq 0]\, d\geq 0$$

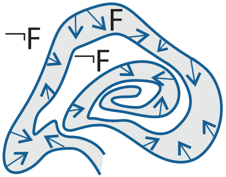
Could repeatedly diffcut in formulas to help the proof



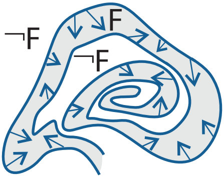
$$\frac{Q \rightarrow [x' := f(x)](F)'}{F \rightarrow [x' = f(x) \,\&\, Q]F}$$



$$\frac{\color{red}{F} \land Q \rightarrow [x' := f(x)](F)'}{F \rightarrow [x' = f(x) \,\&\, Q]F}$$



$$\frac{Q \rightarrow [x' := f(x)](F)'}{F \rightarrow [x' = f(x) \,\&\, Q]F}$$

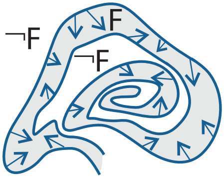


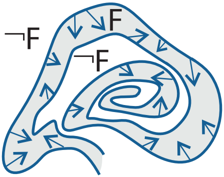
$$\frac{F \land Q \rightarrow [x' := f(x)](F)'}{F \rightarrow [x' = f(x) \,\&\, Q]F}$$

Example (Inductive hypothesis)

$$\frac{}{v^2 - 2v + 1 = 0 \rightarrow [v' = w, w' = -v]v^2 - 2v + 1 = 0}$$

Assuming Differential Invariance





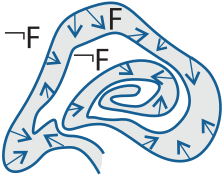
$$\frac{Q \rightarrow [x' := f(x)](F)'}{F \rightarrow [x' = f(x)\,\&\,Q]F}$$

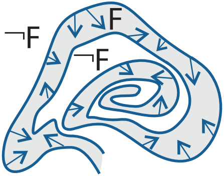
$$\frac{\color{red}{F} \land Q \rightarrow [x' := f(x)](F)'}{F \rightarrow [x' = f(x)\,\&\,Q]F}$$

Example (Inductive hypothesis)

$$\frac{\overline{\qquad\qquad\qquad}}{\frac{v^2 - 2v + 1 = 0 \rightarrow [\color{red}{v'}{:=}w][\color{red}{w'}{:=}{-v}]2v\color{red}{v'} - 2\color{red}{v'} = 0}{v^2 - 2v + 1 = 0 \rightarrow [v' = w, w' = -v]v^2 - 2v + 1 = 0}}$$

Assuming Differential Invariance





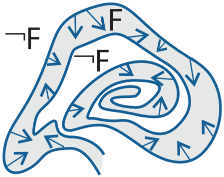
$$\frac{Q \rightarrow [x' := f(x)](F)'}{F \rightarrow [x' = f(x) \,\&\, Q]F}$$

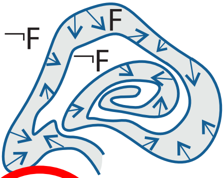
$$\frac{F \wedge Q \rightarrow [x' := f(x)](F)'}{F \rightarrow [x' = f(x) \,\&\, Q]F}$$

Example (Inductive hypothesis)

$$\frac{\frac{\overline{v^2 - 2v + 1 = 0 \rightarrow 2vw - 2w = 0}}{v^2 - 2v + 1 = 0 \rightarrow [v' := w][w' := -v]2vv' - 2v' = 0}}{v^2 - 2v + 1 = 0 \rightarrow [v' = w, w' = -v]v^2 - 2v + 1 = 0}$$

Assuming Differential Invariance

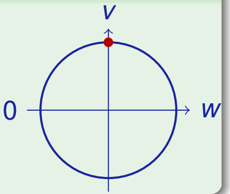




$$\frac{Q \rightarrow [x' := f(x)](F)'}{F \rightarrow [x' = f(x) \,\&\, Q]F} \qquad \frac{F \land Q \rightarrow [x' := f(x)](F)'}{F \rightarrow [x' = f(x) \,\&\, Q]F}$$

Example (Inductive hypothesis is unsound!)

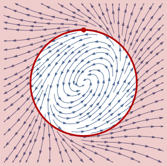
$$\cfrac{\cfrac{\cfrac{\text{(unsound)}}{v^2 - 2v + 1 = 0 \rightarrow 2vw - 2w = 0}}{v^2 - 2v + 1 = 0 \rightarrow [v' := w][w' := -v]2vv' - 2v' = 0}}{v^2 - 2v + 1 = 0 \rightarrow [v' = w, w' = -v]\, v^2 - 2v + 1 = 0}$$



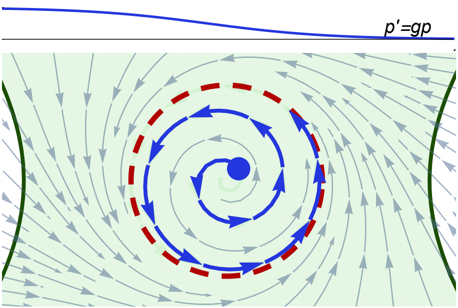
Induction for ODEs is subtle!

Darboux inequalities are DG

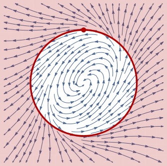
$$\frac{Q \rightarrow p^{\bullet} \geq gp}{p \succcurlyeq 0 \rightarrow [x' = f(x) \,\&\, Q]\, p \succcurlyeq 0} \quad (g \in \mathbb{R}[x])$$



$$\frac{(1-u^2-v^2)^{\bullet} \geq -\tfrac{1}{2}(u^2+v^2)(1-u^2-v^2)}{\ldots \rightarrow \left[\begin{array}{rl} u' = & -v + \frac{u}{4}(1-u^2-v^2) \\ v' = & u + \frac{v}{4}(1-u^2-v^2) \end{array}\right] \underbrace{1-u^2-v^2 > 0}}$$

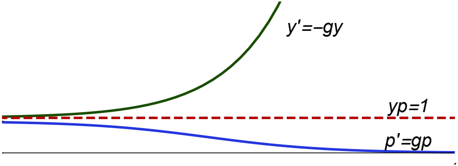


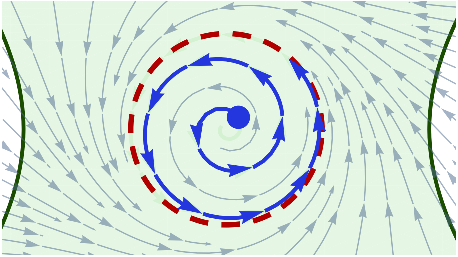
Definable $p^{\bullet}$ for Lie-derivative w.r.t. ODE



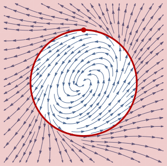
Darboux inequalities are DG

$$\frac{Q \to \dot{p} \geq gp}{p \succcurlyeq 0 \to [x' = f(x) \,\&\, Q]\, p \succcurlyeq 0} \quad (g \in \mathbb{R}[x])$$



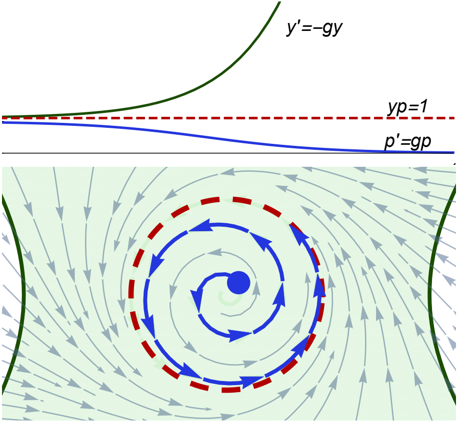


$$\frac{\dot{(1-u^2-v^2)} \geq -\frac{1}{2}(u^2+v^2)(1-u^2-v^2)}{\ldots \to \left[\begin{array}{rl} u' =& -v + \frac{u}{4}(1-u^2-v^2) \\ v' =& u + \frac{v}{4}(1-u^2-v^2) \\ y' =& \frac{1}{2}(u^2+v^2)y \end{array}\right] \underbrace{1-u^2-v^2 > 0}_{y(1-u^2-v^2)=1}}$$



Darboux inequalities are DG

$$\frac{Q \rightarrow p^{\bullet} \geq gp}{p \succcurlyeq 0 \rightarrow [x' = f(x) \,\&\, Q] p \succcurlyeq 0} \quad (g \in \mathbb{R}[x])$$



$$\frac{(1-u^2-v^2)^{\bullet} \geq -\tfrac{1}{2}(u^2+v^2)(1-u^2-v^2)}{\ldots \rightarrow \left[\begin{aligned} u' &= -v + \tfrac{u}{4}(1-u^2-v^2) \\ v' &= u + \tfrac{v}{4}(1-u^2-v^2) \\ y' &= \tfrac{1}{2}(u^2+v^2)y \\ z' &= -\tfrac{1}{4}(u^2+v^2)z \end{aligned}\right] \underbrace{1-u^2-v^2 > 0}_{y(1-u^2-v^2)=1}}$$

Example Proof Derived by Differential Ghosts

$$*$$

$$\mathbb{R} \quad Q \rightarrow (-gy)z^2 + y(2z(\tfrac{g}{2}z)) = 0$$

$$\text{dI} \quad yz^2 = 1 \rightarrow [x' = f(x), y' = -gy, z' = \tfrac{g}{2}z \,\&\, Q]\, yz^2 = 1$$

$$\text{M}[\cdot],\exists\text{R} \quad y > 0 \rightarrow \exists z\, [x' = f(x), y' = -gy, z' = \tfrac{g}{2}z \,\&\, Q]\, y > 0$$

$$\text{dG} \quad y > 0 \rightarrow [x' = f(x), y' = -gy \,\&\, Q]\, y > 0$$

$$*$$

$$Q \rightarrow p^{\bullet} \geq gp \qquad \mathbb{R}\; p^{\bullet} \geq gp, y > 0 \rightarrow p^{\bullet} y - gyp \geq 0$$

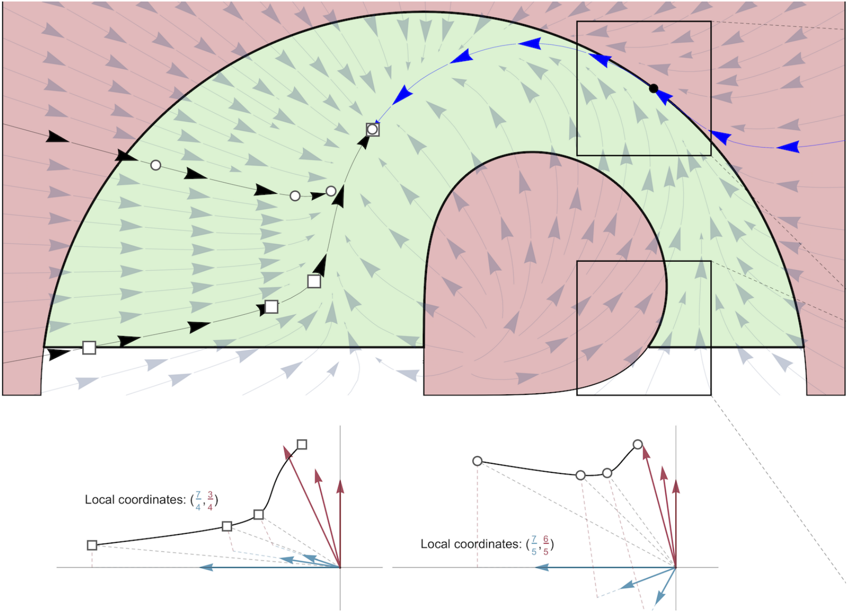
$$\text{cut} \quad Q, y > 0 \rightarrow p^{\bullet} y - gyp \geq 0$$

$$\text{dI} \quad p \succcurlyeq 0, y > 0 \rightarrow [x' = f(x), y' = -gy \,\&\, Q \land y > 0]\, py \succcurlyeq 0 \quad \triangleright$$

$$\text{dC} \quad p \succcurlyeq 0, y > 0 \rightarrow [x' = f(x), y' = -gy \,\&\, Q](y > 0 \land py \succcurlyeq 0)$$

$$\text{M}[\cdot],\exists\text{R} \quad p \succcurlyeq 0 \rightarrow \exists y\, [x' = f(x), y' = -gy \,\&\, Q]\, p \succcurlyeq 0$$

$$\text{dG} \quad p \succcurlyeq 0 \rightarrow [x' = f(x) \,\&\, Q]\, p \succcurlyeq 0$$



LICS'18,JACM'20

Differential Equation Axiomatization

Theorem (Algebraic Completeness) (LICS'18,JACM'20)

dL *calculus is a sound & complete axiomatization of algebraic invariants of polynomial differential equations. They are decidable by DI,DC,DG in* dL*.*

Theorem (Semialgebraic Completeness) (LICS'18,JACM'20)

dL *calculus with RI is a sound & complete axiomatization of semialgebraic invariants of differential equations. They are decidable in* dL*.*

Differential Equation Axiomatization

Theorem (Algebraic Completeness) (LICS'18,JACM'20)

dL *calculus is a sound & complete axiomatization of algebraic invariants of polynomial differential equations. They are decidable*

$$\text{DRI}\ [x' = f(x)\,\&\,Q]e = 0 \leftrightarrow \big(Q \rightarrow e'^{*} = 0\big) \qquad (Q \text{ open})$$

Theorem (Semialgebraic Completeness) (LICS'18,JACM'20)

dL *calculus with RI is a sound & complete axiomatization of semialgebraic invariants of differential equations. They are decidable*

$$\text{SAI}\ \forall x\,(P \rightarrow [x' = f(x)]P) \leftrightarrow \forall x\,\big(P \rightarrow P'^{*}\big) \land \forall x\,\big(\neg P \rightarrow (\neg P)'^{*-}\big)$$

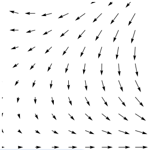
Definable e'^{*} is short for *all/significant* Lie derivative w.r.t. ODE
Definable e'^{*-} is w.r.t. backwards ODE $x' = -f(x)$. Also for P.

$$e'^{*} = 0 \equiv e{=}0 \land (e')'^{*}{=}0 \qquad (P \land Q)'^{*} \equiv P'^{*} \land Q'^{*}$$

$$e'^{*} \geq 0 \equiv e{\geq}0 \land (e{=}0 {\rightarrow} (e')'^{*}{\geq}0) \quad (P \lor Q)'^{*} \equiv P'^{*} \lor Q'^{*}$$

Differentials

Syntax $e ::= x \mid x' \mid c \mid e+k \mid e\cdot k \mid (e)'$



$\to \mathbb{R}$

Semantics $\omega[\![(e)']\!] = \sum_x \omega(x') \frac{\partial [\![e]\!]}{\partial x}(\omega)$

Axioms

$$(e+k)' = (e)' + (k)'$$

$$(e\cdot k)' = (e)'\cdot k + e\cdot (k)'$$

$$(c())' = 0 \qquad \text{for constants/numbers } c()$$

$$(x)' = x' \qquad \text{for variables } x \in \mathscr{V}$$

ODE

$$[\![x' = f(x)\,\&\,Q]\!] = \{(\varphi(0)|_{\{x'\}^\complement}, \varphi(r)) : \varphi \models x' = f(x) \land Q$$
$$\text{for some } \varphi : [0,r] \to \mathscr{S}, \text{ some } r \in \mathbb{R}\}$$

$$\varphi(z)(x') = \frac{d\varphi(t)(x)}{dt}(z) \qquad \ldots$$

JAR'17

Lemma (Differential lemma) (Differential value vs. Time-derivative)

If $\varphi \models x' = f(x) \land Q$ *for duration* $r>0$*, then for all* $0 \leq z \leq r$, $FV(e) \subseteq \{x\}$*:*

$$\varphi(z)[\![(e)']\!] = \frac{\mathrm{d}\varphi(t)[\![e]\!]}{\mathrm{d}t}(z)$$

Lemma (Differential assignment) (Effect on Differentials)

If $\varphi \models x' = f(x) \land Q$ *then* $\varphi \models P \leftrightarrow [x' := f(x)]P$

Lemma (Derivations) (Equations of Differentials)

$$\begin{aligned}
(e+k)' &= (e)' + (k)' \\
(e \cdot k)' &= (e)' \cdot k + e \cdot (k)' \\
(c())' &= 0 && \text{for constants/numbers } c() \\
(x)' &= x' && \text{for variables } x \in \mathscr{V}
\end{aligned}$$

Differential Substitution Lemmas

Lemma (Differential lemma) (Differential value vs. Time-derivative)

If $\varphi \models x' = f(x) \land Q$ *for duration* $r>0$*, then for all* $0 \le z \le r$, $FV(e) \subseteq \{x\}$:

Syntactic ▶ $$\varphi(z)[\![(e)']\!] = \frac{d\varphi(t)[\![e]\!]}{dt}(z)$$ ◀ Analytic

Lemma (Differential assignment) (Effect on Differentials)

If $\varphi \models x' = f(x) \land Q$ *then* $\varphi \models P \leftrightarrow [x' := f(x)]P$

Lemma (Derivations) (Equations of Differentials)

$$(e+k)' = (e)' + (k)'$$

$$(e \cdot k)' = (e)' \cdot k + e \cdot (k)'$$

$$(c())' = 0 \qquad \textit{for constants/numbers } c()$$

$$(x)' = x' \qquad \textit{for variables } x \in \mathcal{V}$$

Lemma (Differential lemma) (Differential value vs. Time-derivative)

$$DI \; \frac{[x' = f(x) \,\&\, Q](e)' = 0}{e = 0 \rightarrow [x' = f(x) \,\&\, Q]e = 0}$$

Lemma (Differential assignment) (Effect on Differentials)

$DE \; [x' = f(x) \,\&\, Q]P \leftrightarrow [x' = f(x) \,\&\, Q][x' := f(x)]P$

Lemma (Derivations) (Equations of Differentials)

$+'$	$(e+k)' = (e)' + (k)'$
$\cdot'$	$(e \cdot k)' = (e)' \cdot k + e \cdot (k)'$
c'	$(c())' = 0$
x'	$(x)' = x'$

Lemma (Differential lemma) (Differential value vs. Time-derivative)

If $\varphi \models x' = f(x) \land Q$ *for duration* $r>0$, *then for all* $0 \leq z \leq r$, $FV(e) \subseteq \{x\}$:

$$\varphi(z)[\![(e)']\!] = \frac{d\varphi(t)[\![e]\!]}{dt}(z)$$

Lemma (Differential assignment) (Effect on Differentials)

If $\varphi \models x' = f(x) \land Q$ *then* $\varphi \models P \leftrightarrow [x' := f(x)]P$

Axiomatics

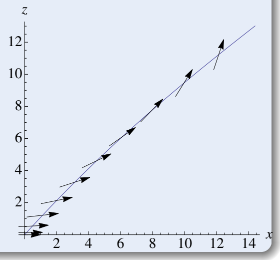
DE $[x' = f(x) \,\&\, Q]P \leftrightarrow [x' = f(x) \,\&\, Q][x' := f(x)]P$

$$\text{DI}\ \frac{[x' = f(x) \,\&\, Q](e)' = 0}{e = 0 \rightarrow [x' = f(x) \,\&\, Q]e = 0}$$

Example: Longitudinal Dynamics of an Airplane

Study (6th Order Longitudinal Flight Equations)

$u' = \frac{X}{m} - g\sin(\theta) - qw$	axial velocity
$w' = \frac{Z}{m} + g\cos(\theta) + qu$	vertical velocity
$x' = \cos(\theta)u + \sin(\theta)w$	range
$z' = -\sin(\theta)u + \cos(\theta)w$	altitude
$\theta' = q$	pitch angle
$q' = \frac{M}{I_{yy}}$	pitch rate



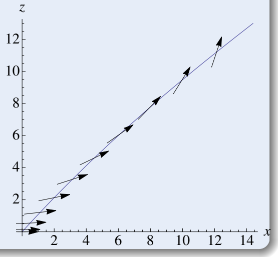
X : thrust along u Z : thrust along w M : thrust moment for w

g : gravity m : mass I_{yy} : inertia second diagonal

TACAS'14

Study (6th Order Longitudinal Flight Equations)

$u' = \frac{X}{m} - g\sin(\theta) - qw$	axial velocity
$w' = \frac{Z}{m} + g\cos(\theta) + qu$	vertical velocity
$x' = \cos(\theta)u + \sin(\theta)w$	range
$z' = -\sin(\theta)u + \cos(\theta)w$	altitude
$\theta' = q$	pitch angle
$q' = \frac{M}{I_{yy}}$	pitch rate

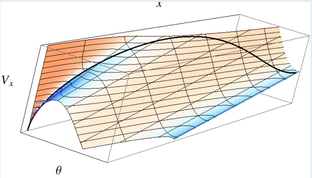


Result (DRI Automatically Generates Invariant Functions)

$$\frac{Mz}{I_{yy}} + g\theta + \left(\frac{X}{m} - qw\right)\cos(\theta) + \left(\frac{Z}{m} + qu\right)\sin(\theta)$$

$$\frac{Mx}{I_{yy}} - \left(\frac{Z}{m} + qu\right)\cos(\theta) + \left(\frac{X}{m} - qw\right)\sin(\theta)$$

$$-q^2 + \frac{2M\theta}{I_{yy}}$$



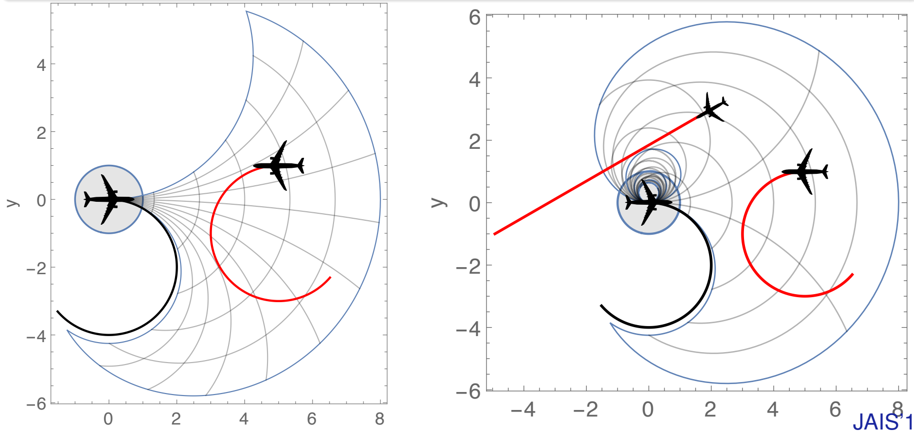
TACAS'14

Example: Dubins Dynamics of 2 Airplanes

Result (DRI Automatically Generates Invariants)

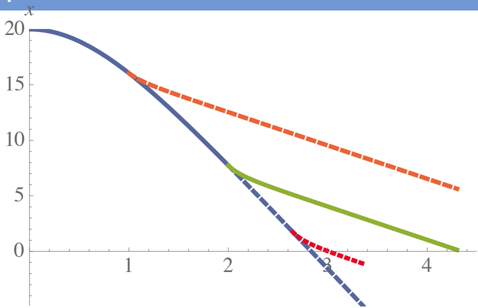
$$\omega_1 = 0 \land \omega_2 = 0 \rightarrow v_2 \sin\vartheta x = (v_2 \cos\vartheta - v_1) y > p(v_1 + v_2)$$

$$\omega_1 \neq 0 \lor \omega_2 \neq 0 \rightarrow -\omega_1 \omega_2 (x^2 + y^2) + 2 v_2 \omega_1 \sin\vartheta x + 2(v_1 \omega_2 - v_2 \omega_1 \cos\vartheta) y + 2 v_1 v_2 \cos\vartheta > 2 v_1 v_2 + 2p(v_2|\omega_1| + v_1|\omega_2|) + p^2|\omega_1 \omega_2|$$

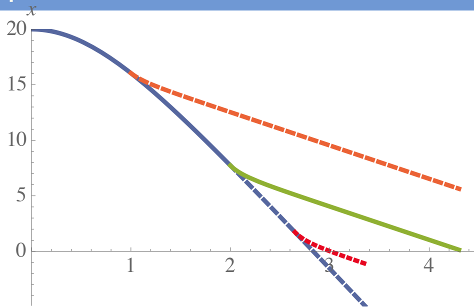


JAIS'14

Ex: Parachute Open or Keep Closed



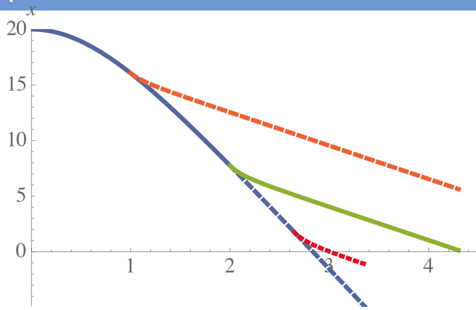
Ex: Parachute Open or Keep Closed



Example (Parachute)

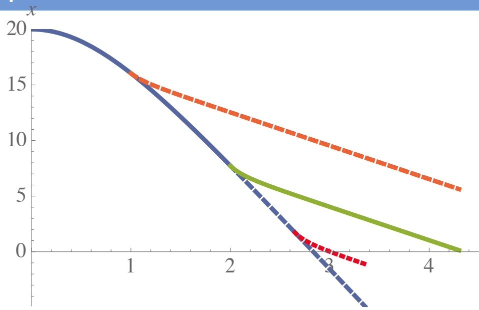
$$\Big((?(Q \land r = a) \cup r{:=}p); t{:=}0;$$
$$\{x' = v, v' = -g + rv^2, t' = 1 \,\&\, t \le T \land x \ge 0 \land v < 0\} \Big)^*$$

Ex: Parachute Open or Keep Closed



Example (Parachute)

$$\rightarrow [((?(Q \wedge r = a) \cup r{:=}p); t{:=}0;$$
$$\{x' = v, v' = -g + rv^2, t' = 1 \,\&\, t \leq T \wedge x \geq 0 \wedge v < 0\})^*]$$
$$(x = 0 \rightarrow v \geq m)$$



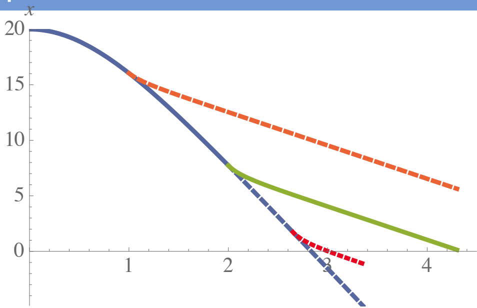
Example (Parachute)

$$\to \Big[\Big(\big(?(Q \land r = a) \cup r{:=}p\big); t{:=}0;$$
$$\{x' = v, v' = -g + rv^2, t' = 1 \,\&\, t \le T \land x \ge 0 \land v < 0\}\Big)^*\Big]$$
$$(x = 0 \to v \ge m)$$

Ex: Parachute Open or Keep Closed

$$Q \equiv v - gT > -\sqrt{g/p}$$

Conservatively bounded next velocity above parachute's limit velocity.



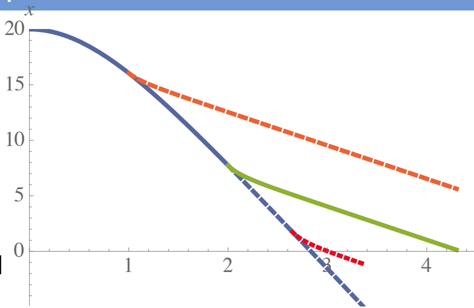
Example (Parachute)

$$m < -\sqrt{g/p} \rightarrow \Big[\big((?(Q \land r = a) \cup r := p); t := 0; \{x' = v, v' = -g + rv^2, t' = 1 \,\&\, t \le T \land x \ge 0 \land v < 0\}\big)^*\Big] (x = 0 \rightarrow v \ge m)$$

$$Q \equiv v - gT > -\sqrt{g/p}$$

Conservatively bounded next velocity above parachute's limit velocity. Limit by differential ghost:

$$y' = -\frac{p}{2}(v - \sqrt{g/p}) \quad y^2(\underbrace{v + \sqrt{g/p}}_{>0}) = 1$$



Example (Parachute)

$$m < -\sqrt{g/p} \rightarrow \Big[\Big(\big(?(Q \land r = a) \cup r := p\big); t := 0; \{x' = v, v' = -g + rv^2, t' = 1 \,\&\, t \leq T \land x \geq 0 \land v < 0\}\Big)^*\Big] (x = 0 \rightarrow v \geq m)$$

Ex: Parachute Open or Keep Closed

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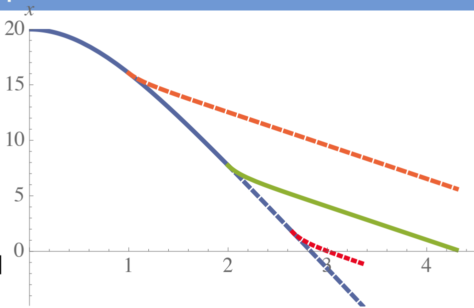


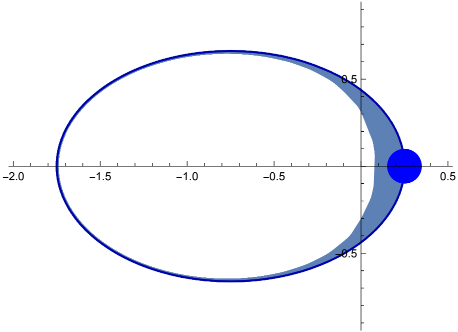
Conservatively bounded next velocity above parachute's limit velocity.
Limit by differential ghost:

$$y' = -\frac{p}{2}(v - \sqrt{g/p}) \quad y^2(\underbrace{v + \sqrt{g/p}}_{>0}) = 1$$

$v \geq \text{old}(v) - gt$ if closed

Example (Parachute)

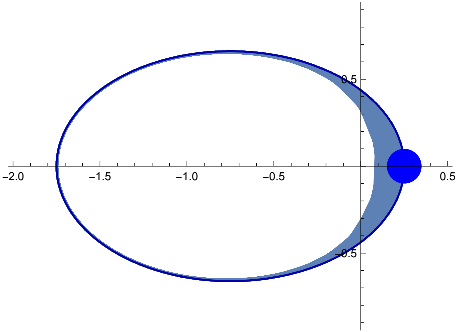
$$m < -\sqrt{g/p} \rightarrow \Big[\big((?(Q \land r = a) \cup r := p); t := 0;$$
$$\{x' = v, v' = -g + rv^2, t' = 1 \,\&\, t \leq T \land x \geq 0 \land v < 0\}\big)^*\Big]$$
$$(x = 0 \rightarrow v \geq m)$$



−2.0
−1.5
−1.0
−0.5
0.5
0.5
−0.5

Ex: Kepler Two Body Problem

- $-\frac{x}{\sqrt{x^2+y^2}}$ opposite direction
- $\frac{1}{x^2+y^2}$ inverse-square law

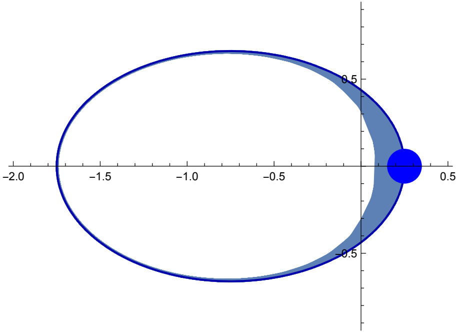


Example (Two Body Problem)

$$[x' = v, v' = -x/(x^2+y^2)^{3/2},$$
$$y' = w, w' = -y/(x^2+y^2)^{3/2}]$$

Ex: Kepler Two Body Problem

- $-\frac{x}{\sqrt{x^2+y^2}}$ opposite direction
- $\frac{1}{x^2+y^2}$ inverse-square law
- Energy preservation



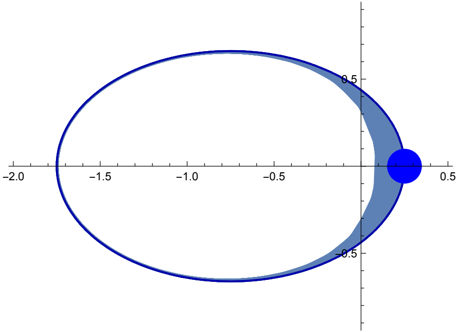
Example (Two Body Problem)

$$\frac{v^2+w^2}{2} - \frac{1}{\sqrt{x^2+y^2}} = E \rightarrow$$

$$[x' = v, v' = -x/(x^2+y^2)^{3/2},$$

$$y' = w, w' = -y/(x^2+y^2)^{3/2}]\ \frac{v^2+w^2}{2} - \frac{1}{\sqrt{x^2+y^2}} = E$$

- $-\frac{x}{\sqrt{x^2+y^2}}$ opposite direction
- $\frac{1}{x^2+y^2}$ inverse-square law
- Energy preservation
- Well-definedness



Example (Two Body Problem)

$$\frac{v^2+w^2}{2} - \frac{1}{\sqrt{x^2+y^2}} = E \rightarrow$$

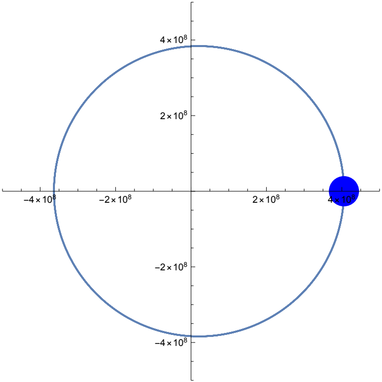
$$[x' = v, v' = -x/(x^2+y^2)^{3/2},$$

$$\& x \neq 0 \lor y \neq 0$$

$$y' = w, w' = -y/(x^2+y^2)^{3/2}] \frac{v^2+w^2}{2} - \frac{1}{\sqrt{x^2+y^2}} = E$$

Exercise: Moon Gravitates Around the Earth

- *G* Gravitational constant $6.67430 * 10^{-11}$
- *M* Mass of the Earth
- *m* Mass of the Moon



Example (Moon around Earth)

$$\cdots \rightarrow \big[x' = v, v' = -GMx/(x^2+y^2)^{3/2},$$
$$y' = w, w' = -GMy/(x^2+y^2)^{3/2} \,\&\, x \neq 0 \lor y \neq 0\big] \ldots$$

Differential Invariant

$$\frac{Q \rightarrow [x' := f(x)](P)'}{P \rightarrow [x' = f(x) \,\&\, Q]P}$$

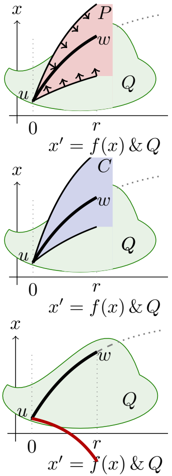
Differential Cut

$$\frac{P \rightarrow [x' = f(x) \,\&\, Q]C \quad P \rightarrow [x' = f(x) \,\&\, Q \land C]P}{P \rightarrow [x' = f(x) \,\&\, Q]P}$$

Differential Ghost

$$\frac{P \leftrightarrow \exists y\, G \quad G \rightarrow [x' = f(x), y' = g(x,y) \,\&\, Q]G}{P \rightarrow [x' = f(x) \,\&\, Q]P}$$

if $g(x,y) = a(x)y + b(x)$, so has long solution!



Springer'10, LMCS'12, LICS'12,JAR'17,LICS'18,JACM'20

Outline (Logic for Dynamical Systems)

Acknowledgments

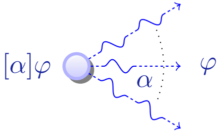
Logic of Autonomous Dynamical Systems, Karlsruhe Institute of Technology
Logical Systems lab, Carnegie Mellon University, Computer Science
Yong Kiam Tan, Brandon Bohrer, Nathan Fulton, Sarah Loos, Katherine Cordwell
Aditi Kabra, Jonathan Laurent, Noah Abou El Wafa
Stefan Mitsch, Khalil Ghorbal, Jean-Baptiste Jeannin, Andrew Sogokon

Unterstützt von / Supported by Alexander von Humboldt Stiftung / Foundation

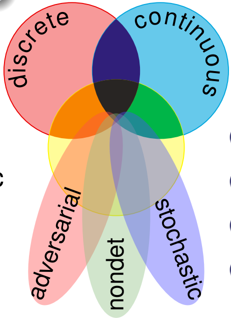
Logical foundations make a big difference for CPS, and vice versa

differential dynamic logic

$\text{dL} = \text{DL} + \text{HP}$



- Multi-dynamical systems
- Hybrid programs + dL logic
- Compositional proofs
- Decide invariant by dL

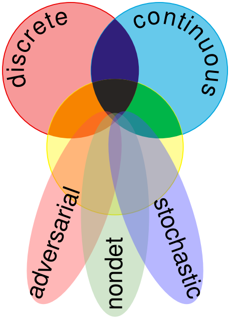


1. Analytic foundations
2. Practical proving
3. Significant applications
4. Bring sciences together

Programming CPS $=$ program cyber $+$ program physics $+$ mutual care

CPSs deserve proofs as safety evidence!

- Verified CPS implementations by ModelPlex FMSD’16
- Correct CPS execution PLDI’18
- CPS proof and tactic languages+libraries ITP’17
- Big CPS built from safe components STTT’18
- ODE invariance JACM’20
- ODE liveness FAC’21
- ODE stability TACAS’21
- Invariant generation FMSD’21
- Safe AI autonomy in CPS AAAI’18
- Refinement + system property proofs LICS’16
- CPS information flow LICS’18
- Hybrid games TOCL’15
- Constructive hybrid games IJCAR’20



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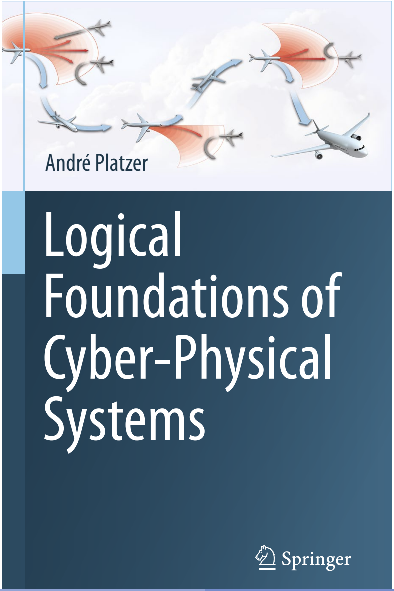

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