Playing Hybrid Games with KeYmaera

Jan-David Quesel[1] André Platzer[2]

1. University of Oldenburg, Department of Computing Science, Germany
2. Carnegie Mellon University, Computer Science Department, Pittsburgh, PA, USA

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Outline

1. Motivation
2. Differential Dynamic Game Logic (dDG$\mathcal{L}$)
3. Proof Calculus
4. Case Study
5. Conclusion

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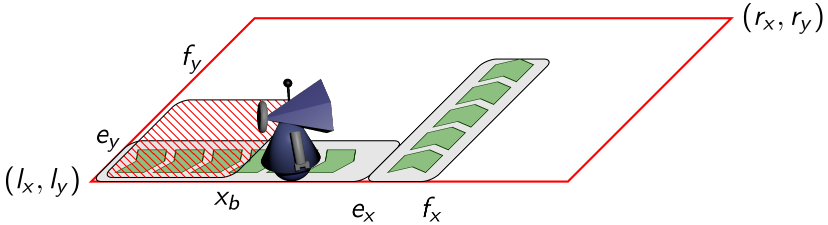
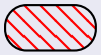
Automated Factory



Model

- (x, y): coordinates of the robot
- (v_x, v_y): velocities
- conveyor belts instantaneously increase the velocity of the robot

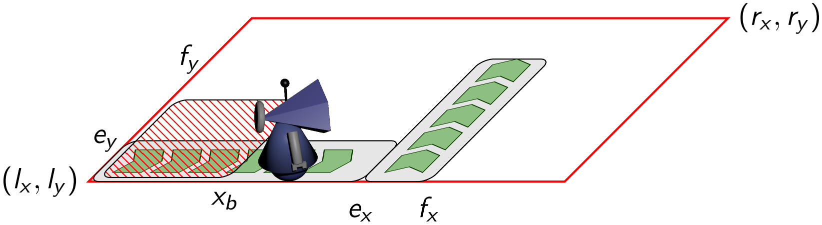
Primary objectives of the robot

- *Leave*  within ε time units.
- Do *not* leave .

Challenges

- Distributed, physical environment
- Possibly conflicting secondary objectives

Automated Factory

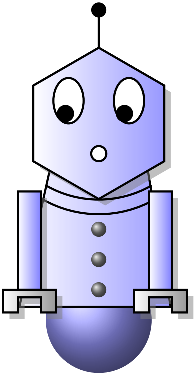


Is there a strategy for the robot to stay safe?

Modelling Language

Differential equations for robot movement

$$x' = v_x, \qquad v_x' = a_x,$$
$$y' = v_y, \qquad v_y' = a_y$$



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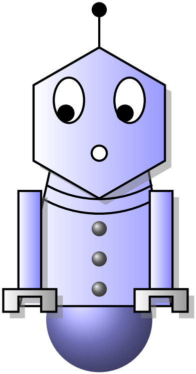
Guards/Constraints

$$l_x \leq x \leq r_x, \qquad v_x^2 \leq 2A(r_x - f_x)$$



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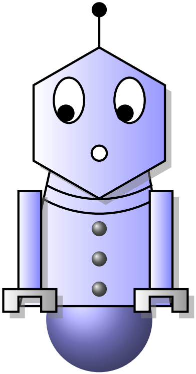
Discrete Assignments

$$a_x := -A, \qquad v_x := v_x + c_x, \qquad \left(s := \frac{v^2}{2b}\right)$$



Modelling Language

Hybrid Program	Effect
$\alpha;\ \beta$	sequential composition
$\alpha \cup \beta$	nondeterministic choice
α^*	nondeterministic repetition
$x := \theta$	discrete assignment (jump)
$x := *$	nondeterministic assignment
$(x_1' = \theta_1, \ldots, x_n' = \theta_n \& F)$	continuous evolution of x_i
$?F$	assert that formula F holds

Platzer, André
Differential dynamic logic for hybrid systems.
J. Autom. Reasoning **41**(2) (2008) 143–189

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Differential Dynamic Game Logic dDG$\mathcal{L}$

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Definition (Hybrid Game)

$$G ::= [\alpha] \mid \langle\alpha\rangle \mid (G_1 \cap G_2) \mid (G_1 \cup G_2) \mid (G_1 G_2) \mid (G)^{[*]} \mid (G)^{\langle*\rangle}$$

Falsifier vs. Verifier

Game Rules

Hybrid Game (informal) Rules

$[\alpha]$	Falsifier plays α
$(G_1 \cap G_2)$	Falsifier decides whether to play G_1 or G_2
$(G)^{[*]}$	Repeat G n times, where n is chosen in advance by Falsifier

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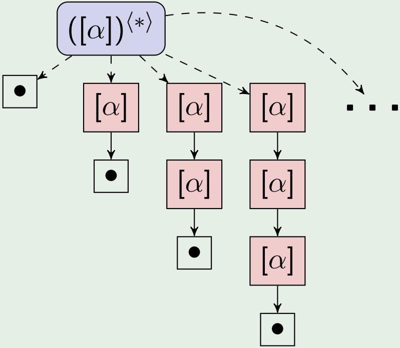
$(G_1 G_2)$	Play G_1 followed by G_2

Game Illustration

Example (Repetition with advance notice semantics)

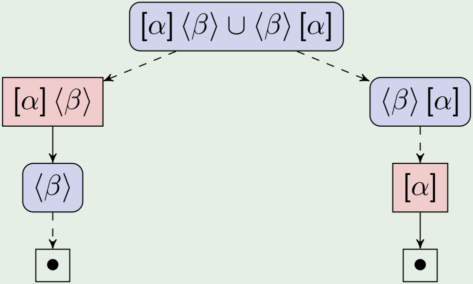


Observations

1. Countably infinite branching
2. Every path has finite depth

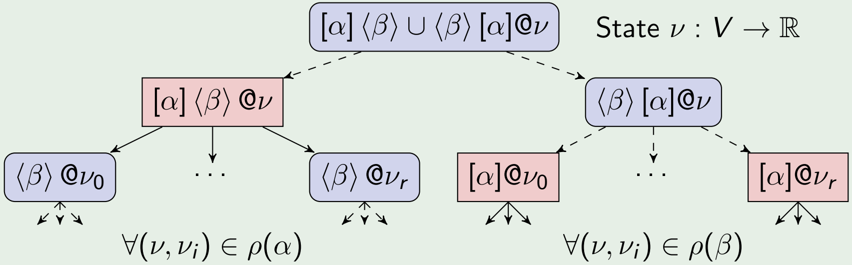
Game Illustration

Example (Explicit branching)



Game Illustration

Example (State)



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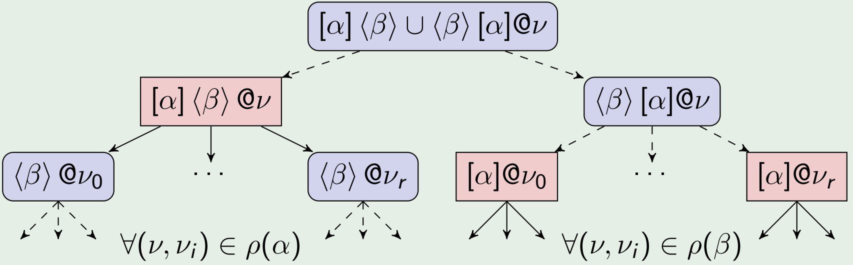
Definition (dDG$\mathcal{L}$ Formula)

$$\phi ::= \theta_1 \sim \theta_2 \mid \neg\phi \mid \phi \wedge \psi \mid \forall x\, \phi \mid G\ \phi$$

$\rightsquigarrow FOL_{\mathbb{R}}$ + Hybrid Games

Strategy, Play, and Winning

Example (Strategy)



Strategy, Play, and Winning

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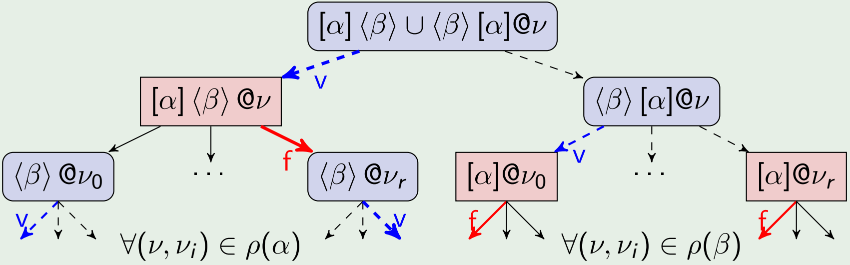

Strategy, Play, and Winning

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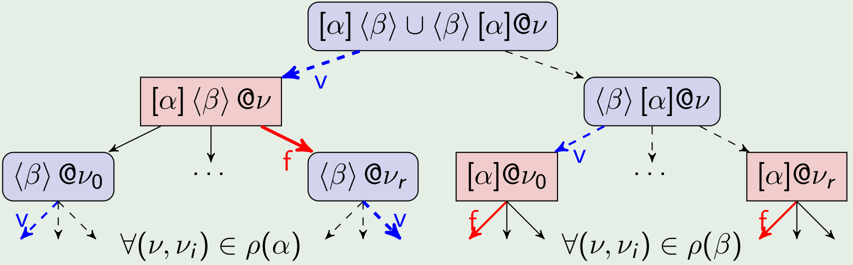

Definition (Strategy)

1. Strategy $s : \mathcal{G} \times Sta(V) \to (\mathcal{G} \cup \{\bullet, \bot, \top\}) \times Sta(V)$ maps game positions to followup positions.
2. s is compatible with G if $((g@\nu) \to s(g@\nu)) \in [\![G]\!]$ f.a. $g \in cl(G)$ and f.a. $\nu \in Sta(V)$.

$cl(G)$: closure under subgame

Strategy, Play, and Winning

Definition (Play)

$G \in \mathcal{G}$, $\nu \in Sta(V)$, two compatible strategies (**f** for Falsifier and **v** for Verifier), a play $p_{\mathbf{f},\mathbf{v}}(G@\nu)$ is defined by:

while $G \notin \{\bullet, \bot, \top\}$ do
Match form of G:

od
return $G@\nu$

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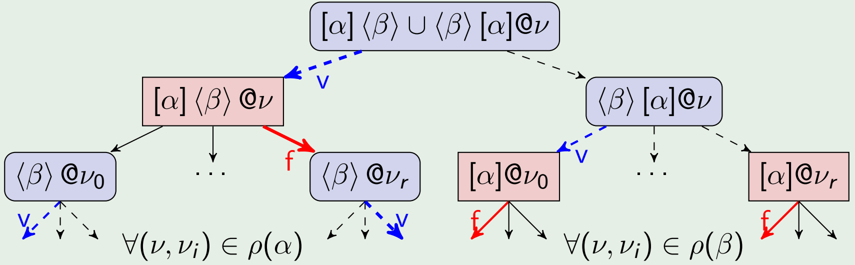
        Case $G_1 G_2 \Rightarrow$ do
            $G@\nu := p_{\mathbf{f},\mathbf{v}}(G_1@\nu)$ //*play* G_1
            If $G = \bullet$ then $G := G_2$ fi //*if* G_1 *terminated with* $\bullet$ *move to* G_2
        od
od
return $G@\nu$

Strategy, Play, and Winning

Example (Winning)



Strategy, Play, and Winning

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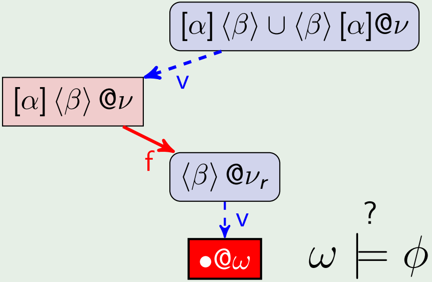
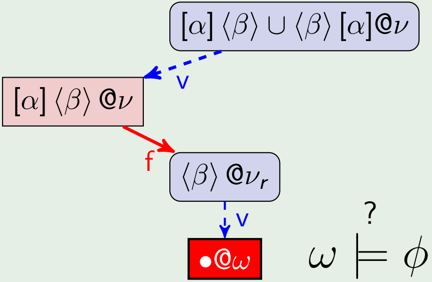


Strategy, Play, and Winning

Example (Winning)



Definition (Winning)

- Winning condition: dDG$\mathcal{L}$ formula ϕ
- Initial state ν
- G is won by Verifier iff G ends in a position $H@\omega$ where
  - $H = \bullet$ and $\omega \models \phi$
  - or $H = \top$.

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Extension

Theorem

dD$\mathcal{G}\mathcal{L}$ *is a conservative extension of* d$\mathcal{L}$*, i.e. for a* d$\mathcal{L}$ *formula* ϕ *holds:*

$$\models_{\mathsf{dDG}\mathcal{L}} \phi \textit{ iff } \models_{\mathsf{d}\mathcal{L}} \phi$$

Proof Calculus for d$\mathcal{L}$

10 propositional rules

(P1) $\dfrac{\vdash \phi}{\neg\phi \vdash}$

(P4) $\dfrac{\phi, \psi \vdash}{\phi \land \psi \vdash}$

(P7) $\dfrac{\phi \vdash \quad \psi \vdash}{\phi \lor \psi \vdash}$

(P2) $\dfrac{\phi \vdash}{\vdash \neg\phi}$

(P5) $\dfrac{\vdash \phi \quad \vdash \psi}{\vdash \phi \land \psi}$

(P8) $\dfrac{\vdash \phi, \psi}{\vdash \phi \lor \psi}$

(P3) $\dfrac{\phi \vdash \psi}{\vdash \phi \to \psi}$

(P6) $\dfrac{\vdash \phi \quad \psi \vdash}{\phi \to \psi \vdash}$

(P9) $\dfrac{}{\phi \vdash \phi}$

(P10) $\dfrac{\vdash \phi \quad \phi \vdash}{\vdash}$

Proof Calculus for d$\mathcal{L}$

13 dynamic rules

(D1) $\dfrac{\phi \land \psi}{\langle ?\phi \rangle \psi}$

(D2) $\dfrac{\phi \rightarrow \psi}{[?\phi]\psi}$

(D3) $\dfrac{\langle \alpha \rangle \phi \lor \langle \beta \rangle \phi}{\langle \alpha \cup \beta \rangle \phi}$

(D4) $\dfrac{[\alpha]\phi \land [\beta]\phi}{[\alpha \cup \beta]\phi}$

(D5) $\dfrac{\phi \lor \langle \alpha; \alpha^* \rangle \phi}{\langle \alpha^* \rangle \phi}$

(D6) $\dfrac{\phi \land [\alpha; \alpha^*]\phi}{[\alpha^*]\phi}$

(D7) $\dfrac{\langle\![\alpha]\!\rangle \langle\![\beta]\!\rangle \phi}{\langle\![\alpha; \beta]\!\rangle \phi}$

(D8) $\dfrac{\phi_x^\theta}{\langle\![x := \theta]\!\rangle \phi}$

(D9) $\dfrac{\exists t {\geq} 0\, (\chi t \land \langle x := y_x(t) \rangle \phi)}{\langle x' = \theta \,\&\, \chi \rangle \phi}$

(D10) $\dfrac{\forall t {\geq} 0\, (\chi t \rightarrow [x := y_x(t)]\phi)}{[x' = \theta \,\&\, \chi]\phi}$

(D11) $\dfrac{\phi \vdash \psi}{\langle\![\alpha]\!\rangle \phi \vdash \langle\![\alpha]\!\rangle \psi}$

(D12) $\dfrac{\phi \vdash [\alpha]\phi}{\phi \vdash [\alpha^*]\phi}$

(D13) $\dfrac{\phi(x) \vdash \langle \alpha \rangle \phi(x-1)}{\exists v\, \phi(v) \vdash \langle \alpha^* \rangle \exists v {\leq} 0\, \phi(v)}$

Proof Calculus for d$\mathcal{L}$

6 quantifier rules

$$\text{(F1)}\ \frac{\vdash \phi(s(X_1,\ldots,X_n))}{\vdash \forall x\, \phi(x)} \qquad \text{(F4)}\ \frac{\vdash \phi(X)}{\vdash \exists x\, \phi(x)}$$

$$\text{(F2)}\ \frac{\phi(s(X_1,\ldots,X_n)) \vdash}{\exists x\, \phi(x) \vdash} \qquad \text{(F5)}\ \frac{\phi(X) \vdash}{\forall x\, \phi(x) \vdash}$$

$$\text{(F3)}\ \frac{\vdash \mathrm{QE}(\forall X\, (\Phi(X) \rightarrow \Psi(X)))}{\Phi(s(X_1,\ldots,X_n)) \vdash \Psi(s(X_1,\ldots,X_n))} \qquad \text{(F6)}\ \frac{\vdash \mathrm{QE}(\exists X\, \bigwedge_i (\Phi_i \rightarrow \Psi_i))}{\Phi_1 \vdash \Psi_1 \quad \ldots \quad \Phi_n \vdash \Psi_n}$$

Proof Calculus for dDG$\mathcal{L}$

Calculus (dDG$\mathcal{L}$ specific rules)

(G1) $\dfrac{G_1\phi \vee G_2\phi}{(G_1 \cup G_2)\phi}$ (G2) $\dfrac{G_1\phi \wedge G_2\phi}{(G_1 \cap G_2)\phi}$ (G3) $\dfrac{G_1(G_2\phi)}{(G_1 G_2)\phi}$

(G4) $\dfrac{\vdash \forall^G(\phi \rightarrow \psi)}{G\phi \vdash G\psi}$

(G5) $\dfrac{\vdash \forall^G(\phi \rightarrow G\phi)}{\phi \vdash (G)^{[*]}\phi}$ (G6) $\dfrac{\vdash \forall^G \forall n > 0(\phi(n) \rightarrow G\ (\phi(n-1)))}{\exists n \phi(n) \vdash (G)^{\langle * \rangle} \exists n (n \leq 0 \wedge \phi(n))}$

$\forall^G$: universal closure over variables in G

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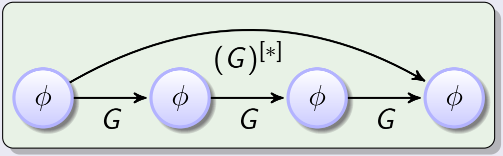

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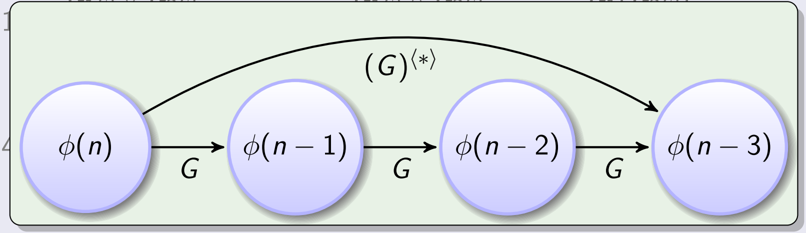
Proof Calculus for dDG$\mathcal{L}$

Calculus (dDG$\mathcal{L}$ specific rules)

(G1



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Sound but Incomplete

Theorem (Soundness)

The sequent calculus for dDG$\mathcal{L}$ *is sound.*

Theorem (Incompleteness)

The sequent calculus for dDG$\mathcal{L}$ *is incomplete.*

Proof Sketch (incompleteness)

1. x is a natural number iff

$$\langle y := 0; (y := y + 1)^* \rangle y = x$$

2. $FOL_{\mathbb{R}}$ + natural numbers: incompletness of the calculus follows by Gödel's incompletness theorem

Relative Completeness

Propositional Dynamic Logic (PDL)

- Game Logic: Game extension of PDL
- Game Logic is strictly more express than PDL:
  PDL cannot express the absence of an infinite g-branch
  ($\langle (g^d)^* \rangle$ *false*).

Parikh, R.:
The logic of games and its applications.
In: Annals of Discrete Mathematics. pp. 111–140. Elsevier (1985)

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  ($\langle (g^d)^* \rangle$ *false*).

d$\mathcal{L}$ encoding of $([\alpha])^{\langle * \rangle}$ *false*

$$\exists n \in \mathbb{N} : \forall Z : \exists 0 \leq i < n \in \mathbb{N} : \left[\vec{x} := Z^{(i)}; \alpha\right] \vec{x} \neq Z^{(i+1)}$$

where Z is interpreted as a sequence of real numbers.

Observation

Implicit quantification over states in games
$\leadsto$ completeness modulo d$\mathcal{L}$ unclear.

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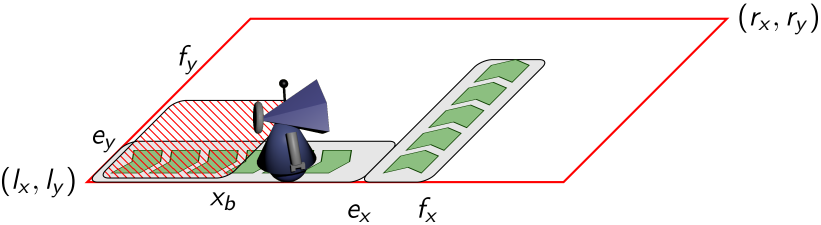

Model

- (x, y): coordinates of the robot
- (v_x, v_y): velocities
- conveyor belts instantaneously increase the velocity of the robot

Primary objectives of the robot

- *Leave* [hatched region] within ε time units.
- Do *not* leave [red box].

Challenges

- Distributed, physical environment
- Possibly conflicting secondary objectives

Robotic Factory Automation (*RF*)

Example (Environment vs. Robot)

$$\Big(\big[?true \cup (?(x < e_x \wedge y < e_y \wedge \mathrm{eff}_1 = 1);\ v_x := v_x + c_x;\ \mathrm{eff}_1 := 0)$$

$$\cup\, (?(e_x \le x \wedge y \le f_y \wedge \mathrm{eff}_2 = 1);\ v_y := v_y + c_y;\ \mathrm{eff}_2 := 0)\ \big]$$

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$$\langle\, a_x := *;\ ?(-A \leq a_x \leq A);$$

$$a_y := *;\ ?(-A \leq a_y \leq A);$$

$$t_s := 0 \,\rangle$$

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$$t_s := 0\ \rangle$$

$$[\, x' = v_x, y' = v_y, v_x' = a_x, v_y' = a_y, t' = 1, t_s' = 1 \& t_s \leq \varepsilon\ ]$$

$$\Big)^{[*]}$$

Robotic Factory Automation (RF)

Example (Environment vs. Robot)

$$\Big([\,?true \cup (?(x < e_x \wedge y < e_y \wedge \mathrm{eff}_1 = 1);\ v_x := v_x + c_x;\ \mathrm{eff}_1 := 0)$$

$$\cup\, (?(e_x \le x \wedge y \le f_y \wedge \mathrm{eff}_2 = 1);\ v_y := v_y + c_y;\ \mathrm{eff}_2 := 0)\ ]$$

$$\langle\, a_x := *;\ ?(-A \le a_x \le A);$$

$$a_y := *;\ ?(-A \le a_y \le A);$$

$$t_s := 0\ \rangle$$

$$(\,[x' = v_x, y' = v_y, v_x' = a_x, v_y' = a_y, t' = 1, t_s' = 1 \& t_s \le \varepsilon\ ]$$

$$\cup(\langle\, ?a_x v_x \le 0 \wedge a_y v_y \le 0;$$

$$\text{if } v_x = 0 \text{ then } a_x := 0 \text{ fi};$$

$$\text{if } v_y = 0 \text{ then } a_y := 0 \text{ fi}\ \rangle$$

$$[x' = v_x, y' = v_y, v_x' = a_x, v_y' = a_y, t' = 1, t_s' = 1$$

$$\& t_s \le \varepsilon \wedge a_x v_x \le 0 \wedge a_y v_y \le 0]))\Big)^{[*]}$$

Results

Proposition (Robot stays in □)

$$\models (x = y = 0 \land v_x = v_y = 0 \land \text{Controllability Assumptions}) \rightarrow (RF)(x \in [l_x, r_x] \land y \in [l_y, r_y])$$

Note: KeYmaera proof has 2471 proof steps on 742 branches (159 interactive steps)

Proposition (Stays in □ + leaves shaded region in time)

$RF|_x$: *RF projected to the x-axis*

$$\models (x = 0 \land v_x = 0 \land \text{Controllability Assumptions}) \rightarrow (RF|_x)(x \in [l_x, r_x] \land (t \geq \varepsilon \rightarrow (x \geq x_b)))$$

Note: KeYmaera proof has 375079 proof steps on 10641 branches (1673 interactive steps)

Outline

1 Motivation

2 Differential Dynamic Game Logic (dDG$\mathcal{L}$)

3 Proof Calculus

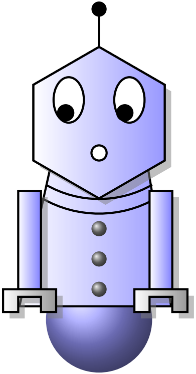
4 Case Study

5 Conclusion

Summary

We ...

- defined a logic for hybrid games (dDG$\mathcal{L}$).
- proved that dDG$\mathcal{L}$ is a conservative extension of d$\mathcal{L}$.
- presented a proof calculus for the logic.
- implemented the calculus in KeYmaera.
- showed a factory automation case study.
- proved the existence of a survival strategy for an robot in an hostile environment.



Outline

6 Related Work

7 Hybrid Games

8 Details

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7 Hybrid Games

8 Details

Structural Operational Semantics

Falsifier rules

(F1) $\dfrac{(\nu,\omega) \in \rho(\alpha)}{[\alpha]@\nu \to \bullet@\omega}$ (F2) $\dfrac{\rho(\alpha) = \emptyset}{[\alpha]@\nu \to \top@\nu}$ (F3) $\dfrac{G@\nu \to G'@\omega}{G \cap H@\nu \to G'@\omega}$

(F4) $\dfrac{G \cap H@\nu \to G'@\omega}{H \cap G@\nu \to G'@\omega}$ (F5) $\dfrac{n \in \mathbb{N}}{(G)^{[*]}@\nu \to G^n@\nu}$

Back

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Verifier rules

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Sequential rules

(S1) $\dfrac{G@\nu \to \bullet@\omega}{(G\ H)@\nu \to H@\omega}$ (S2) $\dfrac{G@\nu \to \bot@\omega}{(G\ H)@\nu \to \bot@\omega}$ (S3) $\dfrac{G@\nu \to \top@\omega}{(G\ H)@\nu \to \top@\omega}$

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Results (detailed)

Assumptions

$$x_b < \frac{1}{2}A\varepsilon^2 \wedge c_x > 0 \wedge (c_x + 4A\varepsilon)^2 \le 2A(r_x - f_x) \quad (1)$$

$$c_y > 0 \wedge c_y^2 \le 2A(r_y - l_y) \quad (2)$$

$$l_x = l_y = 0 \wedge r_x = r_y = 10 \wedge e_x = 2 \wedge e_y = 1 \wedge f_x = 3 \wedge f_y = 10 \wedge A = 2 \quad (3)$$

Proposition

$$\models (x = y = 0 \wedge v_x = v_y = 0 \wedge (1) \wedge (2) \wedge (3)) \rightarrow (RF)(x \in [l_x, r_x] \wedge y \in [l_y, r_y])$$

Proposition

$$\models (x = 0 \wedge v_y = 0 \wedge (1) \wedge (3)) \rightarrow (RF|_x)(x \in [l_x, r_x] \wedge (t \ge \varepsilon \rightarrow (x \ge x_b)))$$

RF projected to the x-axis (denoted $RF|_x$)

▸ Invariant ◂ Return

Invariant

Invariant

$$
\begin{aligned}
\mathrm{eff}_1 \in \{0,1\} \land x \geq l_x \land v_x \geq 0 \land (t \geq \varepsilon \rightarrow x \geq x_b) \\
\land (v_x + c_x \mathrm{eff}_1)^2 \leq 2A(r_x - x) \\
\land \Big(x < x_b \rightarrow t \leq \varepsilon \land \big(x_b - x \leq \frac{1}{2}A\varepsilon^2 - \frac{1}{2}At^2 \\
\land (\mathrm{eff}_1 = 1 \rightarrow v_x = At) \land (\mathrm{eff}_1 = 0 \rightarrow v_x = At + c_x) \\
\land\, r_x - x \geq \frac{(v_x + \mathrm{eff}_1 c_x)^2}{2A} + A(2\varepsilon - t)^2 + 2(2\varepsilon - t)(v_x + \mathrm{eff}_1 c_x) \big) \Big)
\end{aligned}
$$

Return