Differential Game Logic

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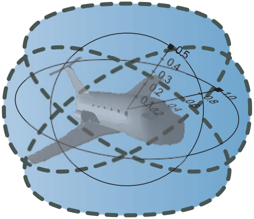
ACM TOCL 2015



Outline

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Cyber-Physical Systems Analysis: Aircraft Example

Which control decisions are safe for aircraft collision avoidance?

Cyber-Physical Systems

CPSs combine cyber capabilities with physical capabilities
to solve problems that neither part could solve alone.

Can you trust a computer to control physics?

Can you trust a computer to control physics?

1. Depends on how it has been programmed
2. And on what will happen if it malfunctions

Rationale

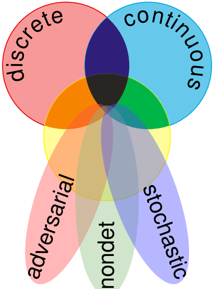
1. Safety guarantees require analytic foundations.
2. A common foundational core helps all application domains.
3. Foundations revolutionized digital computer science & our society.
4. Need even stronger foundations when software reaches out into our physical world.

CPSs deserve proofs as safety evidence!

CPSs are Multi-Dynamical Systems

CPS Dynamics

CPS are characterized by multiple facets of dynamical systems.



CPS Compositions

CPS combines multiple simple dynamical effects.

Descriptive simplification

Tame Parts

Exploiting compositionality tames CPS complexity.

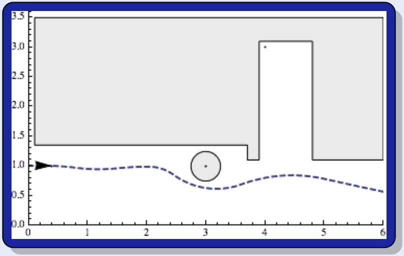
Analytic simplification

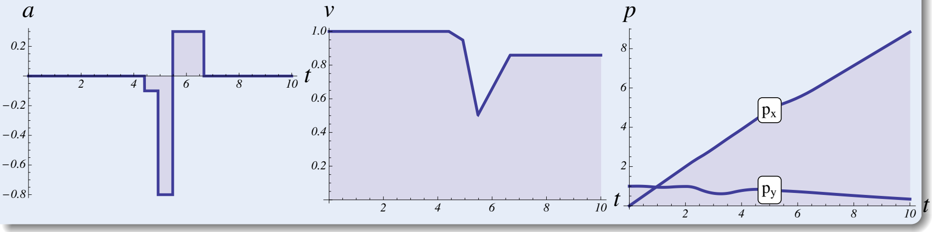
CPS Analysis: Robot Control

Challenge (Hybrid Systems)

Fixed rule describing state evolution with both

- Discrete dynamics (control decisions)
- Continuous dynamics (differential equations)





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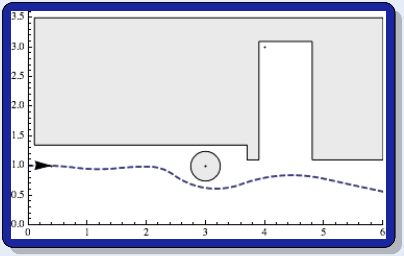
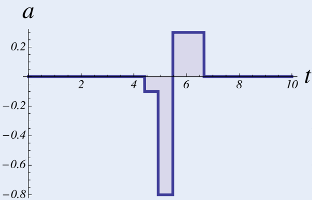
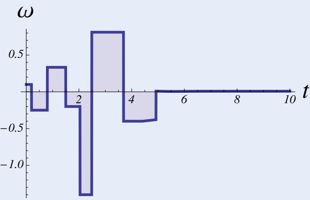
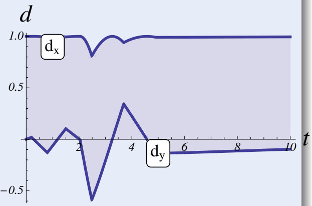
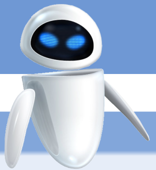








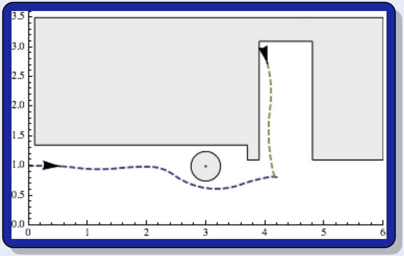
CPS Analysis: Robot Control

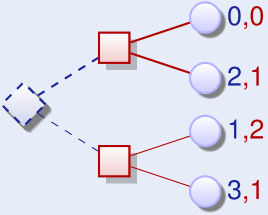


Challenge (Games)

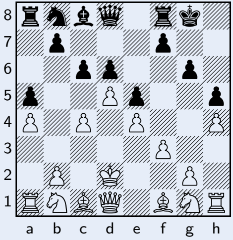
Game rules describing play evolution with both

- Angelic choices (player ◇ Angel)
- Demonic choices (player □ Demon)

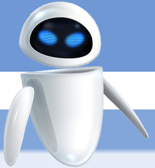




◇\□	Tr	Pl
Trash	1,2	0,0
Plant	0,0	2,1



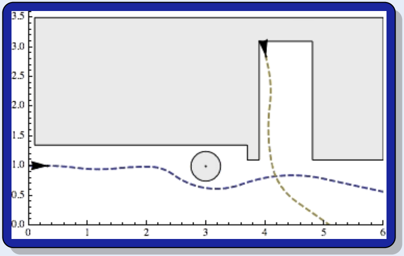
CPS Analysis: Robot Control

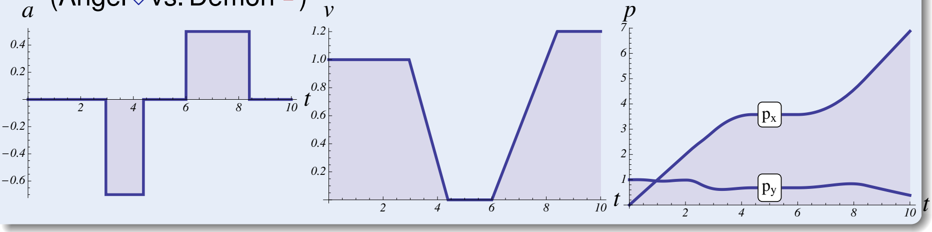


Challenge (Hybrid Games)

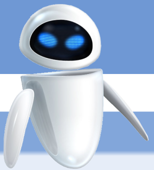
Game rules describing play evolution with

- Discrete dynamics (control decisions)
- Continuous dynamics (differential equations)
- Adversarial dynamics (Angel ◇ vs. Demon □)





CPS Analysis: Robot Control



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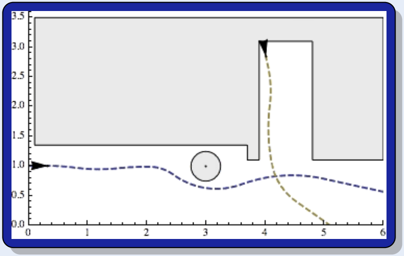

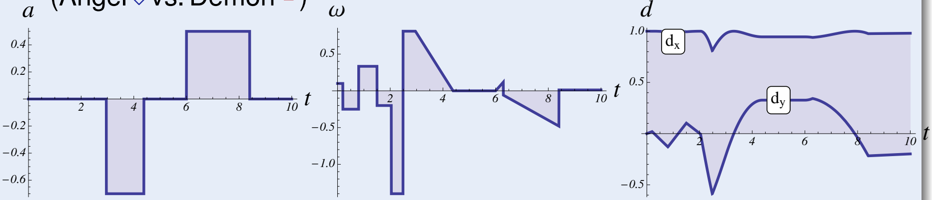

CPS Analysis: RoboCup Soccer

Challenge (Hybrid Games)

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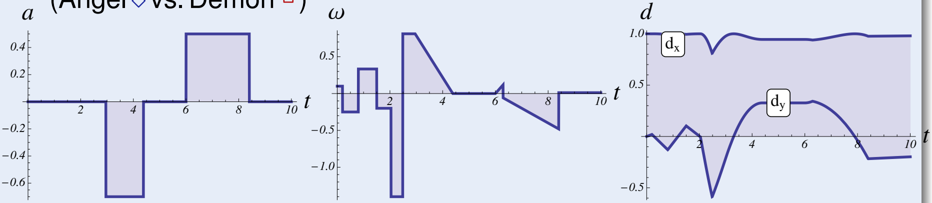

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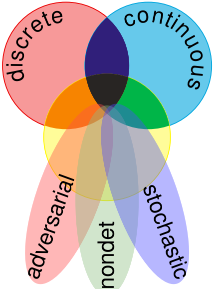

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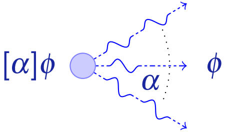
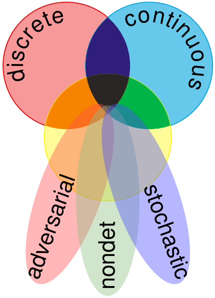
Dynamic Logics for Dynamical Systems

differential dynamic logic

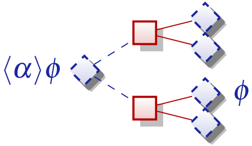
dL = DL + HP





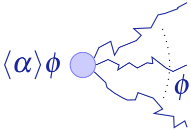
differential game logic

dGL = GL + HG



stochastic differential DL

SdL = DL + SHP



quantified differential DL

QdL = FOL + DL + QHP

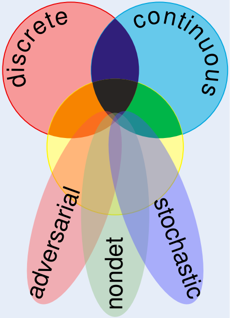
JAR'08,CADE'11,LMCS'12,LICS'12,LICS'12

TOCL'15,JAR'17,TOCL'17,JACM'20

Dynamic Logics for Dynamical Systems

Dynamic Logics

- DL has been introduced for programs Pratt'76,Harel,Kozen
- Its real calling are dynamical systems
- DL excels at providing simple+elegant logical foundations for dynamical systems
- CPSs are multi-dynamical systems
- DL for CPS are multi-dynamical



JAR'08,CADE'11,LMCS'12,LICS'12,LICS'12

TOCL'15,JAR'17,TOCL'17,JACM'20

Contributions

Logical foundations for hybrid games

1. Compositional programming language for hybrid games
2. Compositional logic and proof calculus for winning strategy existence
3. Hybrid games determined
4. Winning region computations terminate after $\geq \omega_1^{\mathrm{CK}}$ iterations
5. Separate truth (∃ winning strategy) vs. proof (winning certificate) vs. proof search (automatic construction)
6. Sound & relatively complete
7. Expressiveness
8. Fragments successful in applications
9. Generalizations in logic enable more applications

TOCL'15

Outline

Differential Game Logic: Syntax

Definition (Hybrid game α)

$$x := f(x) \mid ?Q \mid x' = f(x) \mid \alpha \cup \beta \mid \alpha;\beta \mid \alpha^* \mid \alpha^d$$

Definition (dGL Formula P)

$$p(e_1,\dots,e_n) \mid e \geq \tilde{e} \mid \neg P \mid P \wedge Q \mid \forall x\, P \mid \exists x\, P \mid \langle\alpha\rangle P \mid [\alpha]P$$

TOCL'15

Differential Game Logic: Syntax

Discrete Assign

Test Game

Differential Equation

Choice Game

Seq. Game

Repeat Game

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All Reals

Some Reals

TOCL'15

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TOCL'15

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All Reals | Some Reals | Angel Wins

TOCL'15

Differential Game Logic: Syntax

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All Reals
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Demon Wins

TOCL'15

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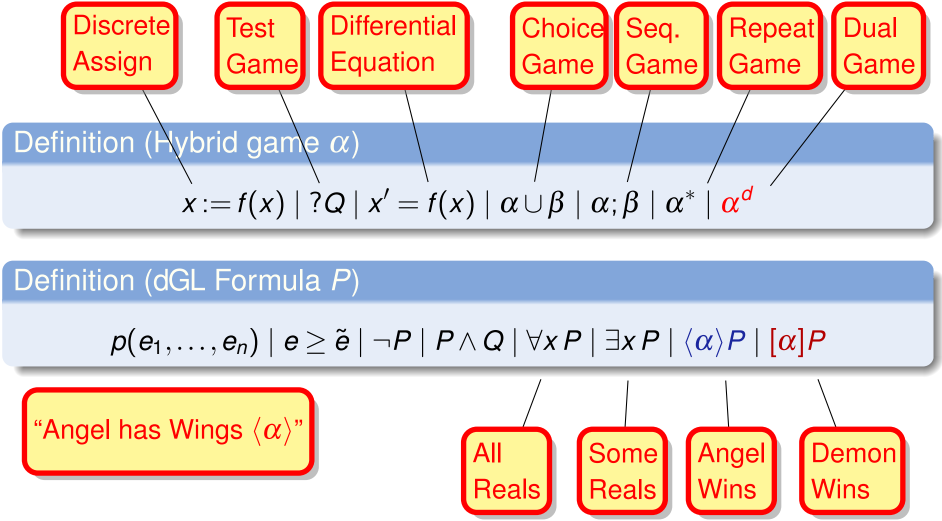

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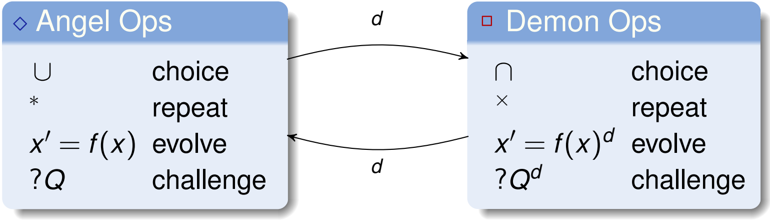
"Angel has Wings $\langle\alpha\rangle$"

All Reals | Some Reals | Angel Wins | Demon Wins

TOCL'15

Game Operators



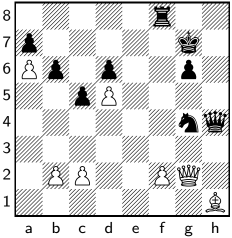
◇ Angel Ops

$\cup$	choice
$*$	repeat
$x' = f(x)$	evolve
$?Q$	challenge

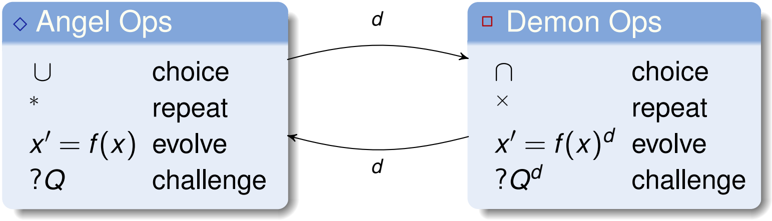
◻ Demon Ops

$\cap$	choice
$\times$	repeat
$x' = f(x)^d$	evolve
$?Q^d$	challenge

Duality operator ^{d} passes control between players



Game Operators



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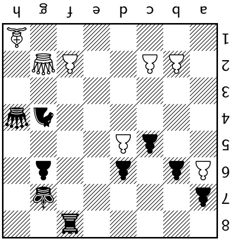
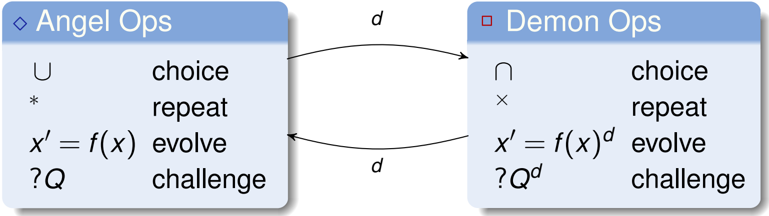


Definable Game Operators



◇ Angel Ops

$\cup$	choice
$*$	repeat
$x' = f(x)$	evolve
$?Q$	challenge

d

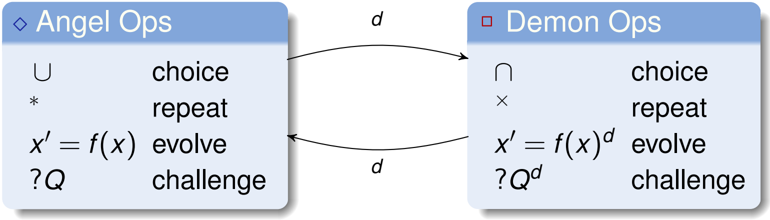
d

◻ Demon Ops

$\cap$	choice
$\times$	repeat
$x' = f(x)^d$	evolve
$?Q^d$	challenge

$$
\begin{aligned}
\mathtt{if}(Q)\,\alpha\,\mathtt{else}\,\beta &\equiv \\
\mathtt{while}(Q)\,\alpha &\equiv \\
\alpha \cap \beta &\equiv \\
\alpha^{\times} &\equiv \\
(x' = f(x) \,\&\, Q)^d &\quad x' = f(x) \,\&\, Q \\
(x := f(x))^d &\quad x := f(x) \\
?Q^d &\quad ?Q
\end{aligned}
$$

Definable Game Operators



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$?Q$	challenge

□ Demon Ops

$\cap$	choice
$\times$	repeat
$x' = f(x)^d$	evolve
$?Q^d$	challenge

$$\mathtt{if}(Q)\,\alpha\,\mathtt{else}\,\beta \equiv (?Q;\alpha) \cup (?\neg Q;\beta)$$

$$\mathtt{while}(Q)\,\alpha \equiv (?Q;\alpha)^*;?\neg Q$$

$$\alpha \cap \beta \equiv (\alpha^d \cup \beta^d)^d$$

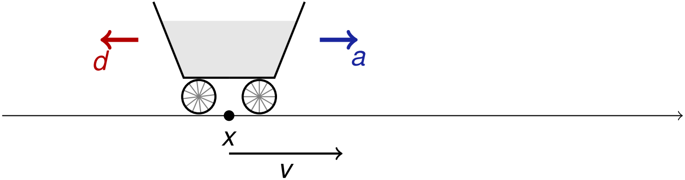
$$\alpha^\times \equiv ((\alpha^d)^*)^d$$

$$(x' = f(x) \,\&\, Q)^d \not\equiv x' = f(x) \,\&\, Q$$

$$(x := f(x))^d \equiv x := f(x)$$

$$?Q^d \not\equiv ?Q$$

Example: Push-around Cart



$$v \geq 1 \rightarrow$$

$$\Big[\big((d := 1 \cup d := -1)^d ; (a := 1 \cup a := -1) ; \{x' = v, v' = a + d\}\big)^*\Big]\, v \geq 0$$

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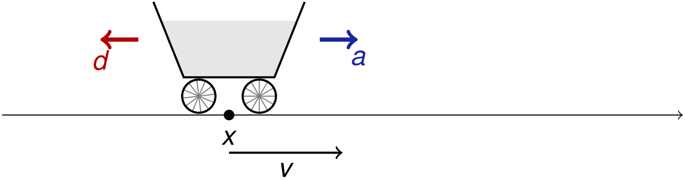

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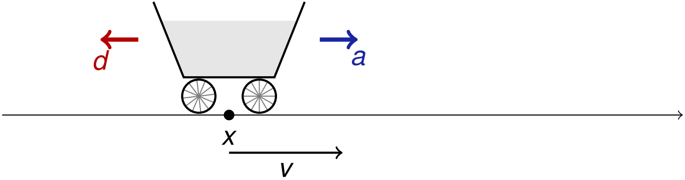

$$\vDash v \geq 1 \rightarrow$$

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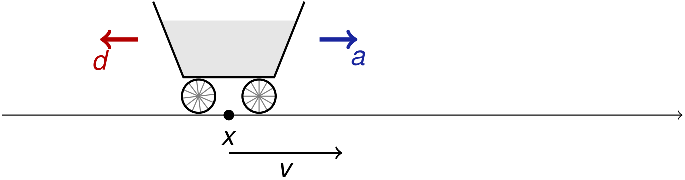

$\vDash v \geq 1 \rightarrow$ *d* before *a* can compensate

$$[((d := 1 \cap d := -1); (a := 1 \cup a := -1); \{x' = v, v' = a + d\})^*]\, v \geq 0$$

$x \geq 0 \land v \geq 0 \rightarrow$

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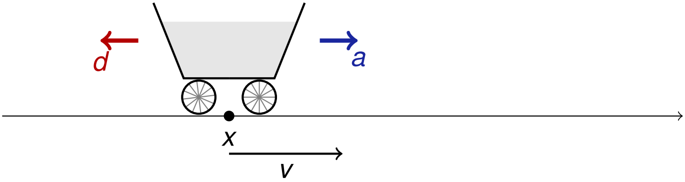

$\vDash v \geq 1 \to$ *d* before *a* can compensate

$$\left[\left((d:=1 \cap d:=-1);(a:=1 \cup a:=-1);\{x'=v,v'=a+d\}\right)^*\right] v \geq 0$$

$\vDash x \geq 0 \land v \geq 0 \to$

$$\left[\left((d:=1 \cap d:=-1);(a:=1 \cup a:=-1);\{x'=v,v'=a+d\}\right)^*\right] x \geq 0$$

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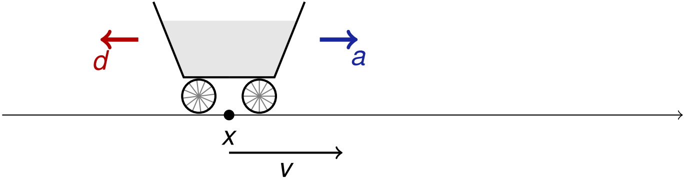

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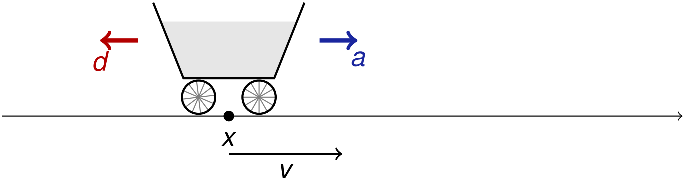

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$$[((d:=1 \cap d:=-1);(a:=1 \cup a:=-1);\{x'=v, v'=a+d\})^*]\, v \geq 0$$

$\vDash x \geq 0 \rightarrow$ boring by skip

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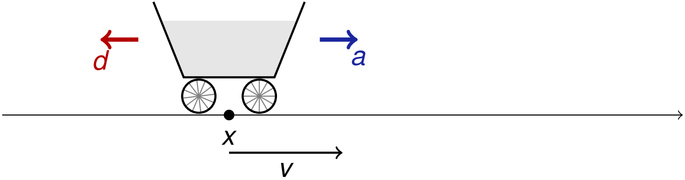

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$$\Big\langle\big((d := 1 \cap d := -1); (a := 1 \cup a := -1); \{x' = v, v' = a + d\}\big)^*\Big\rangle x \geq 0$$

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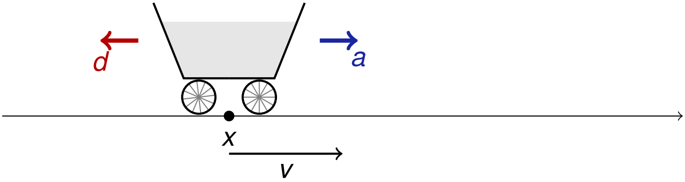

$\vDash v \geq 1 \rightarrow$ *d* before *a* can compensate

$$\big[\big((d:=1 \cap d:=-1);(a:=1 \cup a:=-1);\{x'=v, v'=a+d\}\big)^*\big]\,v \geq 0$$

$\not\vDash$ counterstrategy $d:=-1$

$$\big\langle\big((d:=1 \cap d:=-1);(a:=1 \cup a:=-1);\{x'=v, v'=a+d\}\big)^*\big\rangle\,x \geq 0$$

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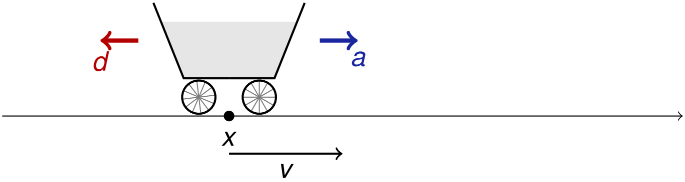

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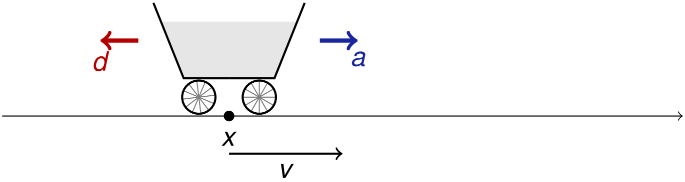


$\vDash v \geq 1 \rightarrow$ *d* before *a* can compensate

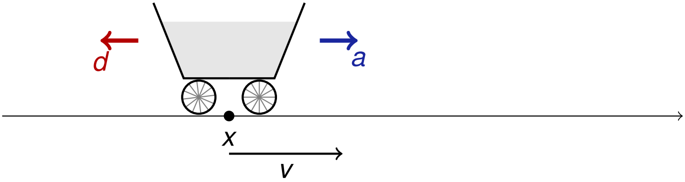
$$[((d := 1 \cap d := -1);(a := 1 \cup a := -1);\{x' = v, v' = a + d\})^*]\, v \geq 0$$

$\nvDash$ counterstrategy $d := -1$

$$\langle((d := 1 \cap d := -1);(a := 1 \cup a := -1);\{x' = v, v' = a + d\})^*\rangle\, x \geq 0$$

$\vDash \langle((d := 1 \cap d := -1);\ (a := 2 \cup a := -2);\ \{x' = v, v' = a + d\})^*\rangle\, x \geq 0$

Example: Push-around Cart



$\vDash v \geq 1 \rightarrow$ d before a can compensate

$$\Big[\big((d:=1 \cap d:=-1);(a:=1 \cup a:=-1);\{x'=v, v'=a+d\}\big)^*\Big]\, v \geq 0$$

$\nvDash$ counterstrategy $d:=-1$

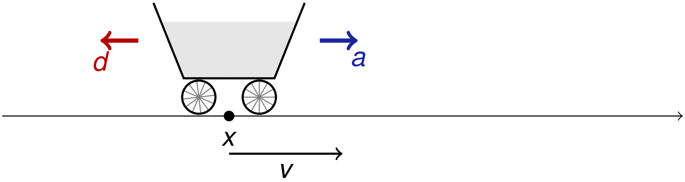
$$\Big\langle\big((d:=1 \cap d:=-1);(a:=1 \cup a:=-1);\{x'=v, v'=a+d\}\big)^*\Big\rangle\, x \geq 0$$

$$\vDash \Big\langle\big((d:=1 \cap d:=-1);\ (a:=2 \cup a:=-2);\ \{x'=v, v'=a+d\}\big)^*\Big\rangle\, x \geq 0$$

$$\Big\langle\big((d:=2 \cap d:=-2);\ (a:=2 \cup a:=-2);$$

$$t:=0;\ \{x'=v, v'=a+d, t'=1 \,\&\, t \leq 1\}\big)^*\Big\rangle\, x^2 \geq 100$$

Example: Push-around Cart



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$\vDash \langle((d:=2 \cap d:=-2);\ (a:=2 \cup a:=-2);$ $a:=d$ then $a:=2\,\mathrm{sign}\,v$

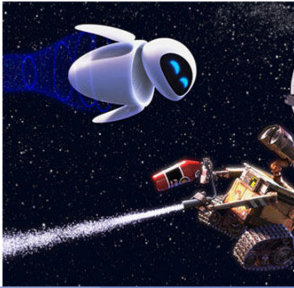
$$t:=0;\ \{x'=v, v'=a+d, t'=1 \,\&\, t \leq 1\})^*\rangle\, x^2 \geq 100$$

Example: EVE and WALL·E

$$
\begin{aligned}
&(w-e)^2 \le 1 \land v = f \to \\
&\Big\langle \Big((u := 1 \cap u := -1); \\
&\quad (g := 1 \cup g := -1); \\
&\quad t := 0; \\
&\quad (w' = v, v' = u, e' = f, f' = g, t' = 1 \,\&\, t \le 1)^d \\
&\Big)^{\times} \Big\rangle\, (w-e)^2 \le 1
\end{aligned}
$$

EVE at *e* plays Angel's part controlling *g*

WALL·E at *w* plays Demon's part controlling *u*



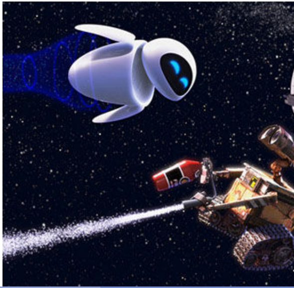
Example: EVE and WALL·E and the World

$$
\begin{aligned}
&(w-e)^2 \leq 1 \land v = f \to \\
&\big\langle \big((u := 1 \cap u := -1); \\
&\quad (g := 1 \cup g := -1); \\
&\quad t := 0; \\
&\quad (w' = v, v' = u, e' = f, f' = g, t' = 1 \,\&\, t \leq 1)^d \\
&\big)^\times \big\rangle\, (w-e)^2 \leq 1
\end{aligned}
$$

EVE at e plays Angel's part controlling g

WALL·E at w plays Demon's part controlling u

EVE assigned environment's time to WALL·E



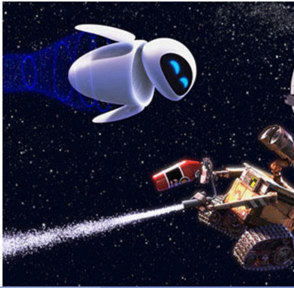
Example: WALL·E and EVE

$$
\begin{aligned}
&(w-e)^2 \le 1 \land v = f \to \\
&[((u:=1 \cap u:=-1); \\
&\quad (g:=1 \cup g:=-1); \\
&\quad t:=0; \\
&\quad (w'=v, v'=u, e'=f, f'=g, t'=1 \,\&\, t \le 1) \\
&)^{\times}]\ (w-e)^2 > 1
\end{aligned}
$$

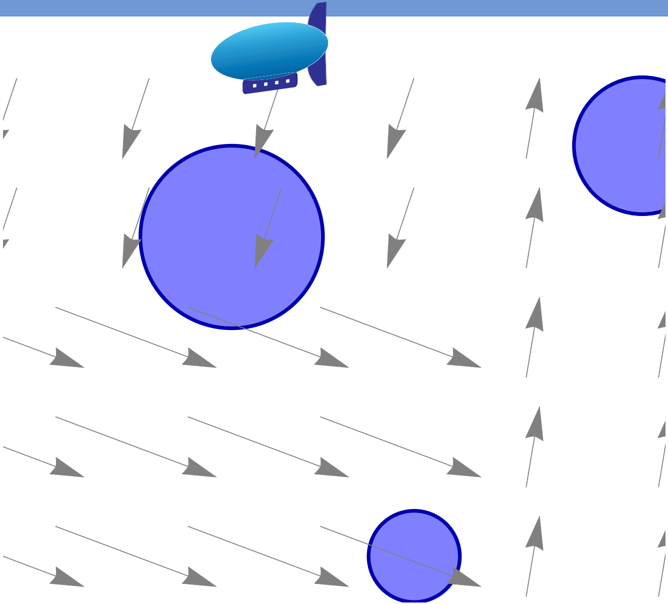
WALL·E at w plays Demon's part controlling u

EVE at e plays Angel's part controlling g

WALL·E assigned environment's time to EVE



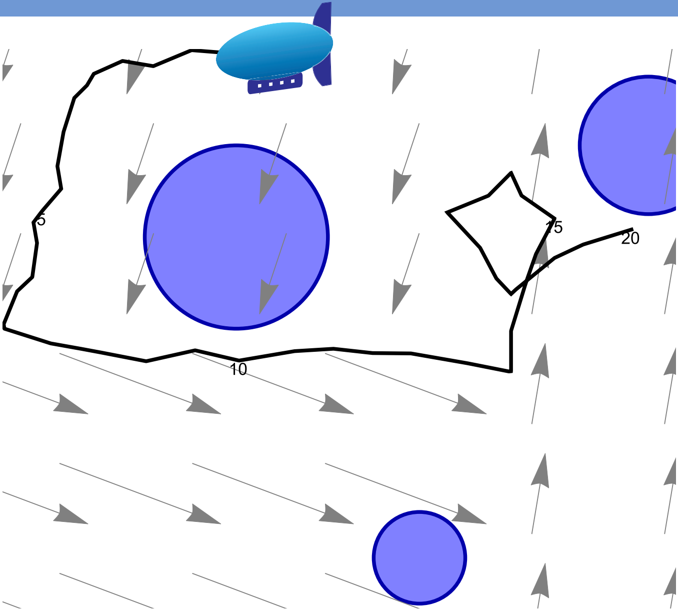
Zeppelin Obstacle Parcours



avoid obstacles
changing wind
local turbulence
$x' = f(x, y, z)$

TOCL'17

Zeppelin Obstacle Parcours



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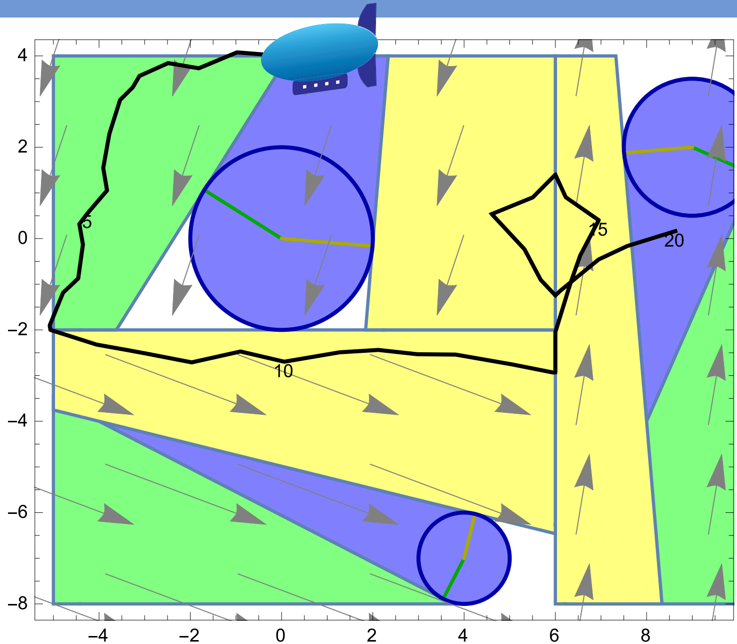
TOCL'17

Zeppelin Obstacle Parcours



avoid obstacles
changing wind
local turbulence
$x' = f(x, y, z)$

TOCL'17

Differential Game Logic: Denotational Semantics

Definition (Hybrid game α) $[\![\cdot]\!] : \mathrm{HG} \to (\wp(\mathscr{S}) \to \wp(\mathscr{S}))$

$$\begin{aligned}
\varsigma_{x:=f(x)}(X) &= \{s \in \mathscr{S} : s_x^{[\![f(x)]\!]_s} \in X\} \\
\varsigma_{x'=f(x)}(X) &= \{\varphi(0) \in \mathscr{S} : \varphi(r) \in X, \frac{\mathrm{d}\,\varphi(t)(x)}{\mathrm{d}t}(\zeta) = [\![f(x)]\!]_{\varphi(\zeta)} \text{ for all } \zeta\} \\
\varsigma_{?Q}(X) &= [\![Q]\!] \cap X \\
\varsigma_{\alpha\cup\beta}(X) &= \varsigma_\alpha(X) \cup \varsigma_\beta(X) \\
\varsigma_{\alpha;\beta}(X) &= \varsigma_\alpha(\varsigma_\beta(X)) \\
\varsigma_{\alpha^*}(X) &= \bigcap\{Z \subseteq \mathscr{S} : X \cup \varsigma_\alpha(Z) \subseteq Z\} \\
\varsigma_{\alpha^d}(X) &= (\varsigma_\alpha(X^\complement))^\complement
\end{aligned}$$

Definition (dGL Formula P) $[\![\cdot]\!] : \mathrm{Fml} \to \wp(\mathscr{S})$

$$\begin{aligned}
[\![e \geq \tilde{e}]\!] &= \{s \in \mathscr{S} : [\![e]\!]_s \geq [\![\tilde{e}]\!]_s\} \\
[\![\neg P]\!] &= ([\![P]\!])^\complement \\
[\![P \wedge Q]\!] &= [\![P]\!] \cap [\![Q]\!] \\
[\![\langle\alpha\rangle P]\!] &= \varsigma_\alpha([\![P]\!]) \\
[\![[\alpha]P]\!] &= \delta_\alpha([\![P]\!])
\end{aligned}$$

Differential Game Logic: Denotational Semantics

Definition (Hybrid game α: denotational semantics)

$$\varsigma_{x:=f(x)}(X) \; =$$

Differential Game Logic: Denotational Semantics

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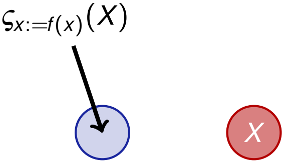

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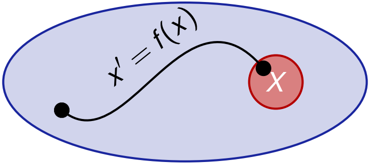

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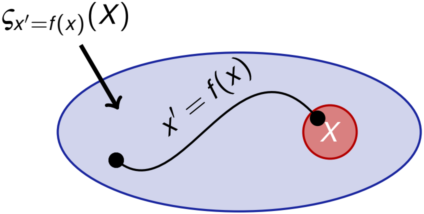

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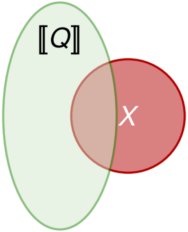
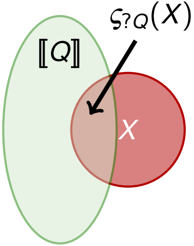


Differential Game Logic: Denotational Semantics

Definition (Hybrid game α: denotational semantics)

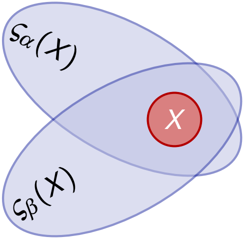
$$\varsigma_{?Q}(X) = [\![Q]\!] \cap X$$



Differential Game Logic: Denotational Semantics

Definition (Hybrid game α: denotational semantics)

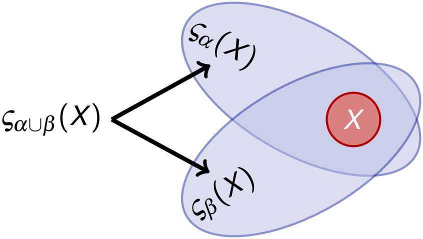
$\varsigma_{\alpha\cup\beta}(X) =$



Differential Game Logic: Denotational Semantics

Definition (Hybrid game α: denotational semantics)

$$\varsigma_{\alpha\cup\beta}(X) = \varsigma_\alpha(X) \cup \varsigma_\beta(X)$$



Differential Game Logic: Denotational Semantics

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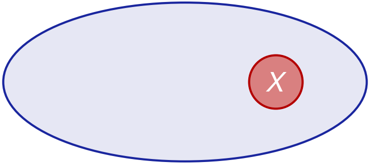

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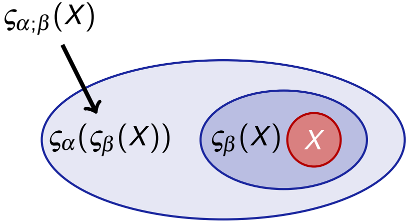

Differential Game Logic: Denotational Semantics

Definition (Hybrid game α: denotational semantics)

$$\varsigma_{\alpha^*}(X) =$$

Differential Game Logic: Denotational Semantics

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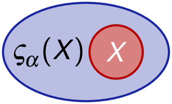

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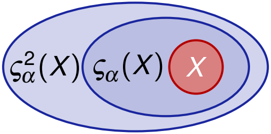

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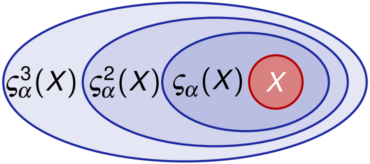

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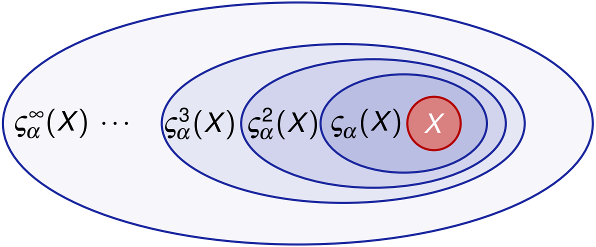

Differential Game Logic: Denotational Semantics

Definition (Hybrid game α: denotational semantics)

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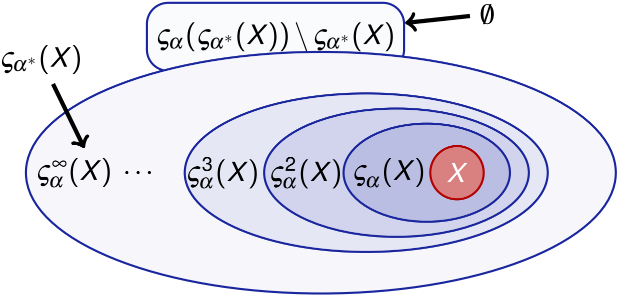
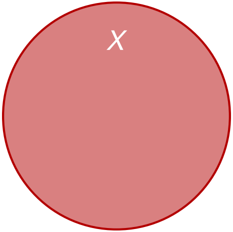


Differential Game Logic: Denotational Semantics

Definition (Hybrid game α: denotational semantics)

$\varsigma_{\alpha^d}(X) =$



Differential Game Logic: Denotational Semantics

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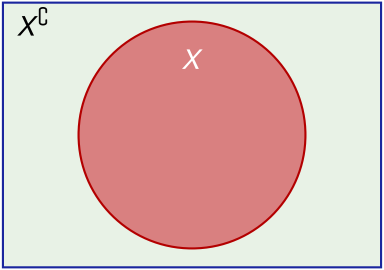

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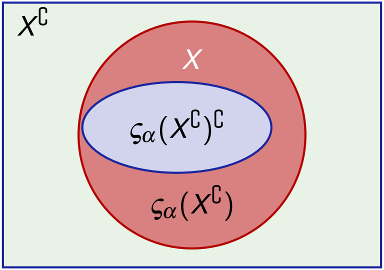

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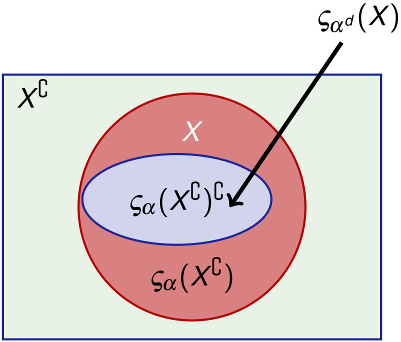
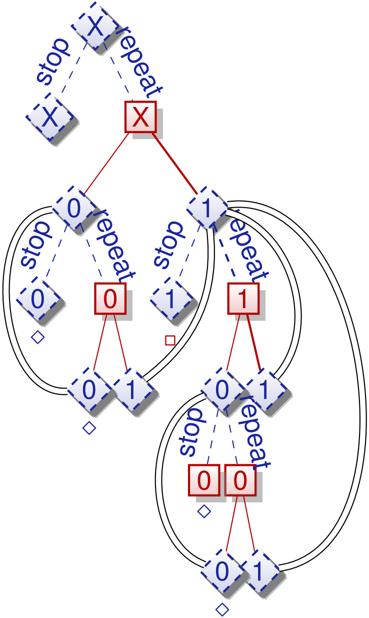


Filibusters

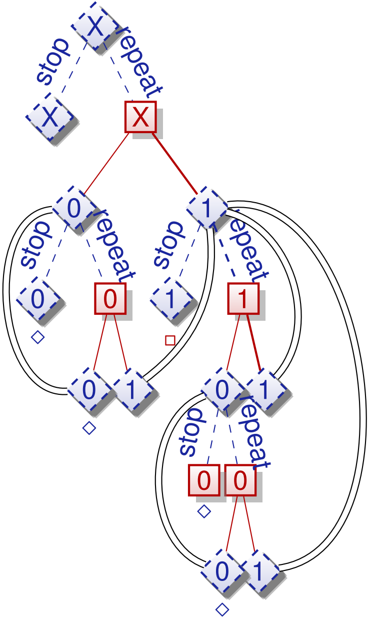
$$\langle (x := 0 \cap x := 1)^* \rangle x = 0$$



Filibusters & The Significance of Finitude

$$\langle (x := 0 \cap x := 1)^* \rangle x = 0$$

$\overset{\text{wfd}}{\leadsto}$ false unless $x = 0$



Filibusters & The Significance of Finitude

$\langle (x' = 1^d; x := 0)^* \rangle x = 0$

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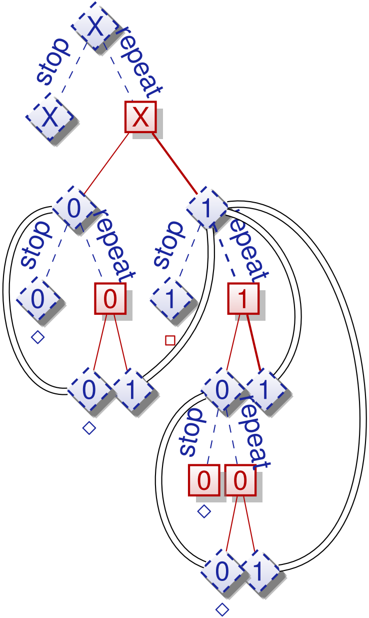

Filibusters & The Significance of Finitude

$\overset{<\infty}{\rightsquigarrow}$ true

$\langle (x' = 1^d ; x := 0)^* \rangle x = 0$

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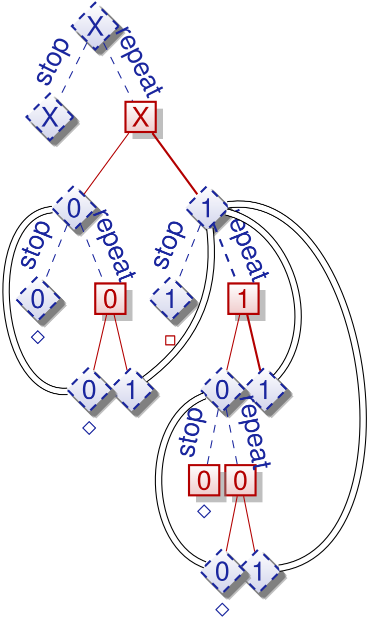

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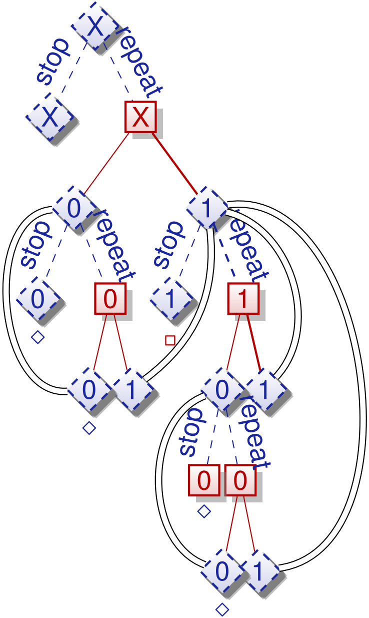
Well-defined games
can't be postponed forever



Consistency & Determinacy

Theorem (Consistency & determinacy)

Hybrid games are consistent and determined, i.e. $\vDash \neg\langle\alpha\rangle\neg\phi \leftrightarrow [\alpha]\phi$.

Corollary (Determinacy: At least one player wins)

$\vDash \neg\langle\alpha\rangle\neg\phi \rightarrow [\alpha]\phi$, *thus* $\vDash \langle\alpha\rangle\neg\phi \vee [\alpha]\phi$.

Corollary (Consistency: At most one player wins)

$\vDash [\alpha]\phi \rightarrow \neg\langle\alpha\rangle\neg\phi$, *thus* $\vDash \neg([\alpha]\phi \wedge \langle\alpha\rangle\neg\phi)$

Winning Region Fixpoint Iterations

Definition (Hybrid game α)

$$\varsigma_{\alpha^*}(X) = \bigcap\{Z \subseteq \mathscr{S} : X \cup \varsigma_\alpha(Z) \subseteq Z\}$$

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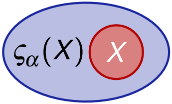

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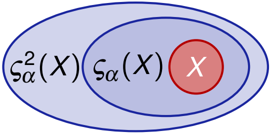

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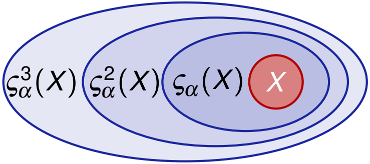
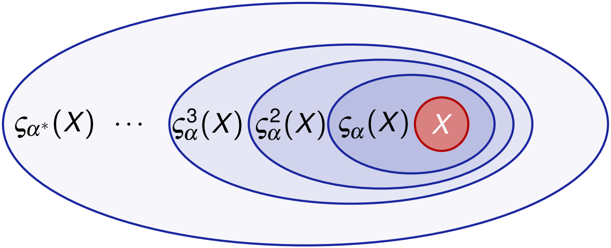


Winning Region Fixpoint Iterations

Definition (Hybrid game α)

$$\varsigma_{\alpha^*}(X) = \bigcap\{Z \subseteq \mathscr{S} : X \cup \varsigma_\alpha(Z) \subseteq Z\} = \varsigma_\alpha^\infty(X) \qquad \text{(Knaster-Tarski)}$$



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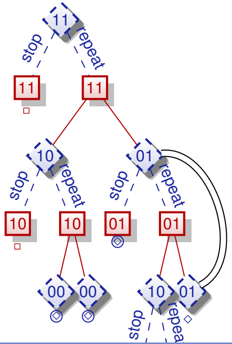
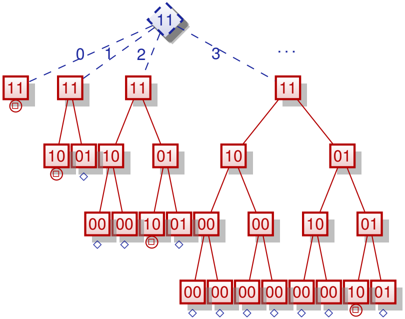
Alternative (Advance notice semantics)

$$\varsigma_{\alpha^*}(X) \overset{?}{=} \bigcup\nolimits_{n<\omega} \varsigma_{\alpha^n}(X) \qquad \text{where } \alpha^{n+1} \equiv \alpha^n;\alpha \quad \alpha^0 \equiv ?true$$





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$$\varsigma_{\alpha^*}(X) \overset{?}{=} \bigcup_{n<\omega} \varsigma_\alpha^n(X)$$

$$\varsigma_\alpha^0(X) \overset{\text{def}}{=} X$$

$$\varsigma_\alpha^{\kappa+1}(X) \overset{\text{def}}{=} X \cup \varsigma_\alpha(\varsigma_\alpha^\kappa(X))$$

Example

$$\langle (x:=1; x'=1^d \cup x:=x-1)^* \rangle\, (0 \le x < 1)$$

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$$\langle (x := 1; x' = 1^d \cup x := x - 1)^* \rangle (0 \le x < 1) \qquad \varsigma_\alpha^n([0,1)) = [0,n) \ne \mathbb{R}$$

Winning Region Fixpoint Iterations

Definition (Hybrid game α)

$$\varsigma_{\alpha^*}(X) = \bigcap\{Z \subseteq \mathscr{S} : X \cup \varsigma_\alpha(Z) \subseteq Z\} = \varsigma_\alpha^\infty(X)$$ (Knaster-Tarski)

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$$\varsigma_\alpha^{\kappa+1}(X) \overset{\text{def}}{=} X \cup \varsigma_\alpha(\varsigma_\alpha^\kappa(X))$$

$$\varsigma_\alpha^\lambda(X) \overset{\text{def}}{=} \bigcup_{\kappa<\lambda} \varsigma_\alpha^\kappa(X)$$ $\lambda \neq 0$ a limit ordinal

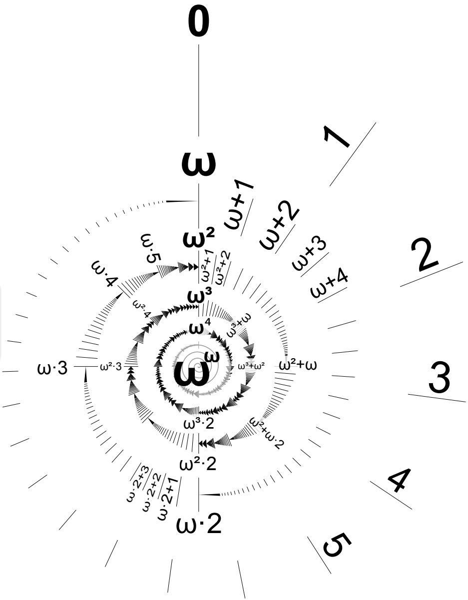
Example

$\langle(x:=1;x'=1^d \cup x:=x-1)^*\rangle(0 \leq x < 1)$ $\qquad \varsigma_\alpha^n([0,1)) = [0,n) \neq \mathbb{R}$

Strategic Closure Ordinal $\geq \omega_1^{CK}$

Theorem

Hybrid game closure ordinal $\geq \omega_1^{CK}$

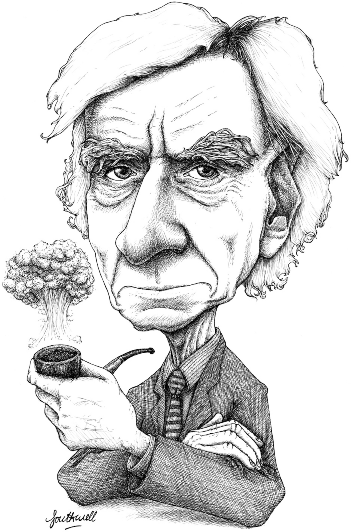


The Power of Implicit Definitions

Implicit Definitions

The advantages of implicit definition over construction are roughly those of theft over honest toil.

— Bertrand Russell



Outline

1. CPS Game Motivation
2. Differential Game Logic
   - Syntax
   - Example: Push-around Cart
   - Example: Robot Dance
   - Differential Hybrid Games
   - Denotational Semantics
   - Determinacy
   - Strategic Closure Ordinals
3. Axiomatization
   - Axiomatics
   - Example: Robot Soccer
   - Soundness and Completeness
   - Separating Axioms
4. Expressiveness
5. Differential Hybrid Games
6. Summary

Differential Game Logic: Axiomatization

$[\cdot]$ $[\alpha]P \leftrightarrow$

$\langle := \rangle$ $\langle x := f(x)\rangle p(x) \leftrightarrow$

$\langle ' \rangle$ $\langle x' = f(x)\rangle P \leftrightarrow$

$\langle ? \rangle$ $\langle ?Q\rangle P \leftrightarrow$

$\langle \cup \rangle$ $\langle \alpha \cup \beta\rangle P \leftrightarrow$

$\langle ; \rangle$ $\langle \alpha ; \beta\rangle P \leftrightarrow$

$\langle ^* \rangle$ $\langle \alpha^*\rangle P \leftrightarrow$

$\langle ^d \rangle$ $\langle \alpha^d\rangle P \leftrightarrow$

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Differential Game Logic: Axiomatization

$[\cdot]$ $[\alpha]P \leftrightarrow \neg\langle\alpha\rangle\neg P$

$\langle:=\rangle$ $\langle x := f(x)\rangle p(x) \leftrightarrow p(f(x))$

$\langle'\rangle$ $\langle x' = f(x)\rangle P \leftrightarrow \exists t{\geq}0\, \langle x := y(t)\rangle P$

$\langle?\rangle$ $\langle ?Q\rangle P \leftrightarrow (Q \wedge P)$

$\langle\cup\rangle$ $\langle \alpha\cup\beta\rangle P \leftrightarrow \langle\alpha\rangle P \vee \langle\beta\rangle P$

$\langle;\rangle$ $\langle \alpha;\beta\rangle P \leftrightarrow \langle\alpha\rangle\langle\beta\rangle P$

$\langle^*\rangle$ $\langle \alpha^*\rangle P \leftrightarrow P \vee \langle\alpha\rangle\langle\alpha^*\rangle P$

$\langle^d\rangle$ $\langle \alpha^d\rangle P \leftrightarrow \neg\langle\alpha\rangle\neg P$

M $\dfrac{P \rightarrow Q}{\langle\alpha\rangle P \rightarrow \langle\alpha\rangle Q}$

FP $\dfrac{P \vee \langle\alpha\rangle Q \rightarrow Q}{\langle\alpha^*\rangle P \rightarrow Q}$

MP $\dfrac{P \quad P \rightarrow Q}{Q}$

$\forall$ $\dfrac{p \rightarrow Q}{p \rightarrow \forall x\, Q}$ $\quad (x \notin \mathrm{FV}(p))$

US $\dfrac{\varphi}{\varphi_{p(\cdot)}^{\psi(\cdot)}}$

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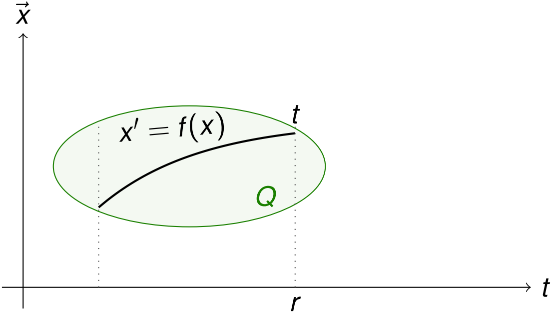
Defining Evolution Domain Constraints

$x' = f(x) \,\&\, Q$ $\qquad$ $x' = f(x); ?(Q)$



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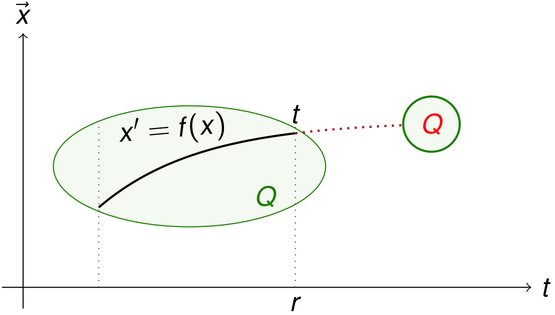
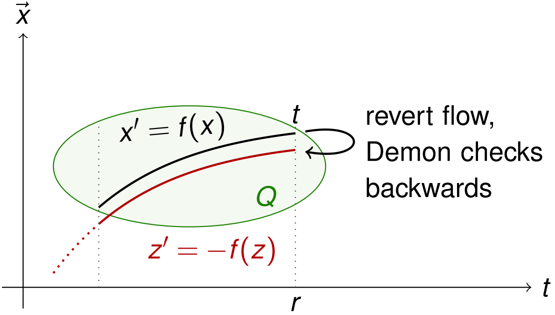


Defining Evolution Domain Constraints

$x' = f(x) \,\&\, Q$ $\qquad\qquad$ $x' = f(x); (z := x; z' = -f(z))^d; ?(Q(z))$



revert flow,
Demon checks Q
backwards

Defining Evolution Domain Constraints

$x' = f(x) \,\&\, Q$

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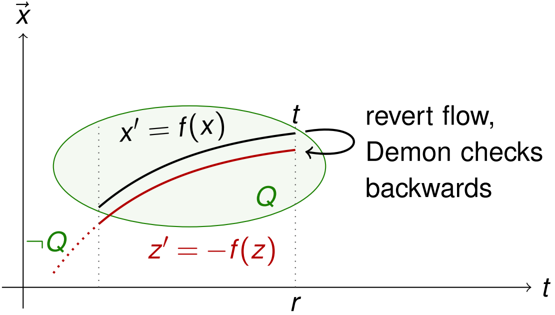

revert flow,
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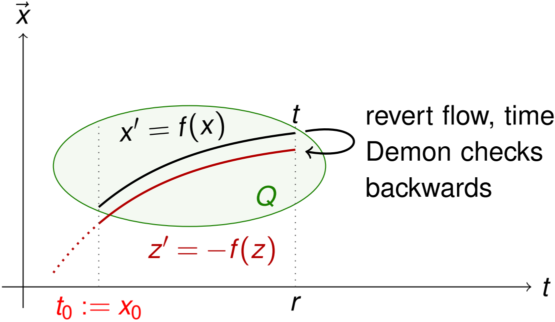
Defining Evolution Domain Constraints

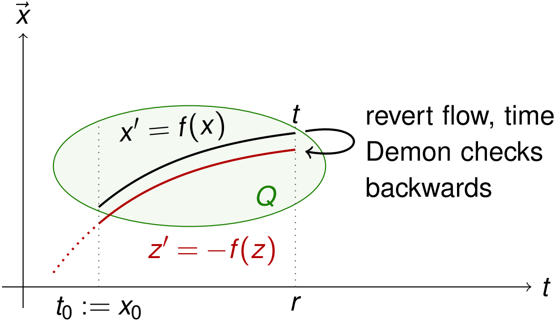
$$x' = f(x) \,\&\, Q \equiv t_0 := x_0;\, x' = f(x);\, (z := x;\, z' = -f(z))^d;\, ?(z_0 \geq t_0 \rightarrow Q(z))$$



revert flow, time x_0;
Demon checks Q
backwards

"There and Back Again" Game

$$x' = f(x) \,\&\, Q \equiv t_0 := x_0; x' = f(x); (z := x; z' = -f(z))^d; ?(z_0 \geq t_0 \to Q(z))$$



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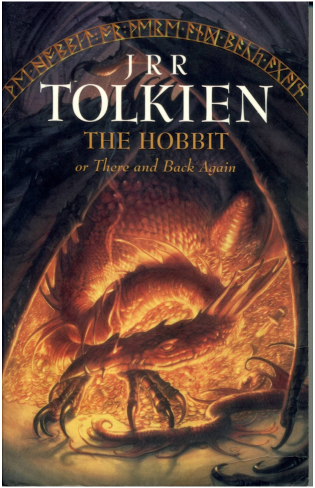
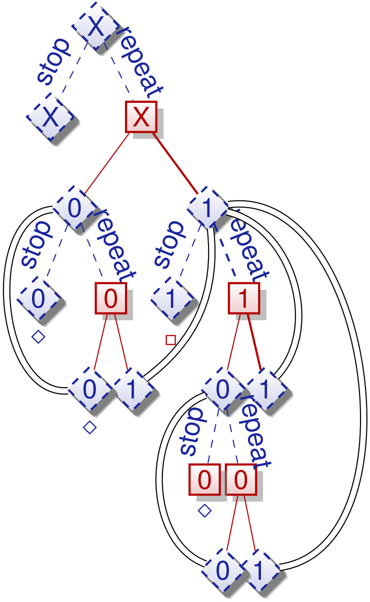


Lemma

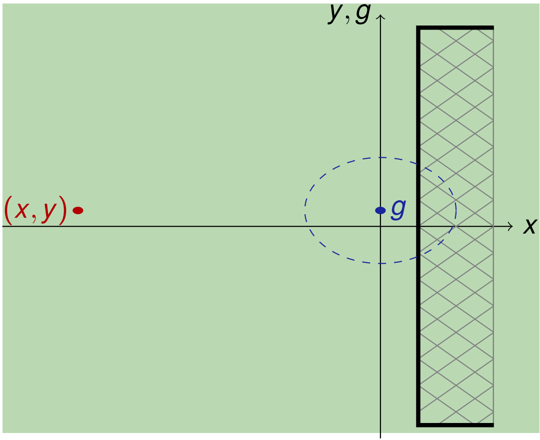
Evolution domains definable by games

Example Proof: Dual Filibuster

$$\begin{array}{ll}
 & \qquad * \\ \hline
\mathbb{R} & x=0 \to 0=0 \lor 1=0 \\ \hline
\langle:=\rangle & x=0 \to \langle x:=0\rangle x=0 \lor \langle x:=1\rangle x=0 \\ \hline
\langle\cup\rangle & x=0 \to \langle x:=0 \cup x:=1\rangle x=0 \\ \hline
\langle^{d}\rangle & x=0 \to \neg\langle x:=0 \cap x:=1\rangle \neg x=0 \\ \hline
[\cdot] & x=0 \to [x:=0 \cap x:=1]x=0 \\ \hline
\text{ind} & x=0 \to [(x:=0 \cap x:=1)^{*}]x=0 \\ \hline
\langle^{d}\rangle & x=0 \to \langle (x:=0 \cup x:=1)^{\times}\rangle x=0
\end{array}$$

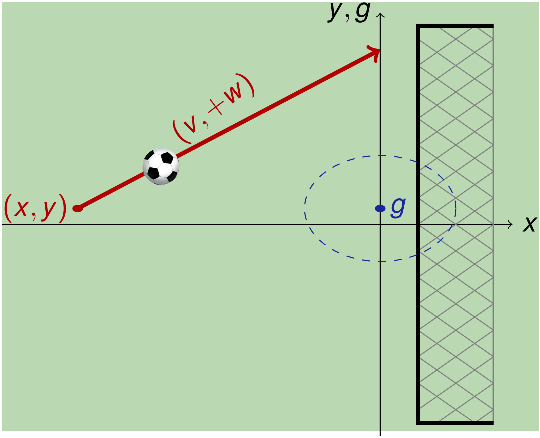


Example: Goalie in Robot Soccer



$$x < 0 \land v > 0 \land y = g \to \Big\langle \big(w := +w \cap w := -w\big); \Big(\big(u := +u \cup u := -u\big); \{x' = v, y' = w, g' = u\}\Big)^* \Big\rangle\, x^2 + (y-g)^2 \le 1$$

Example: Goalie in Robot Soccer



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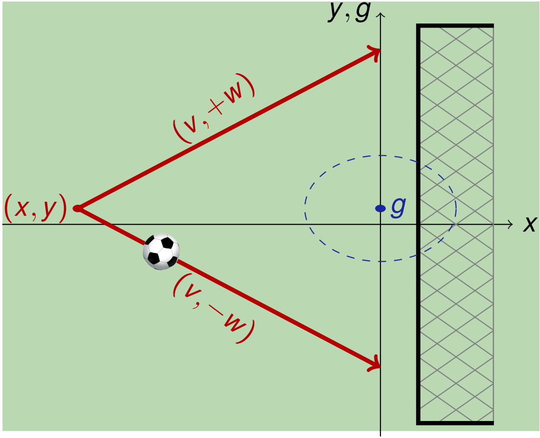

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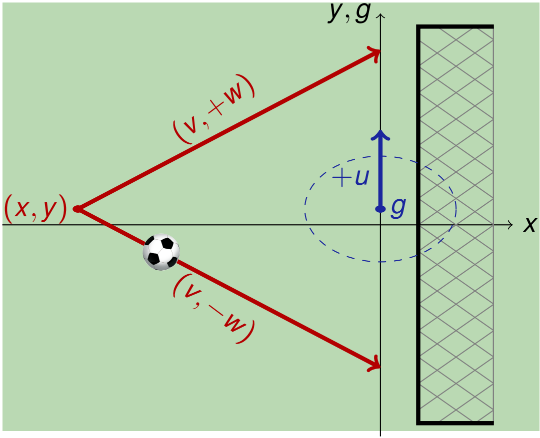

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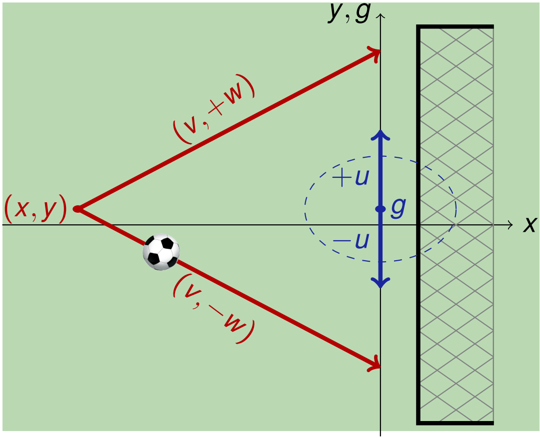

$$x < 0 \land v > 0 \land y = g \rightarrow$$
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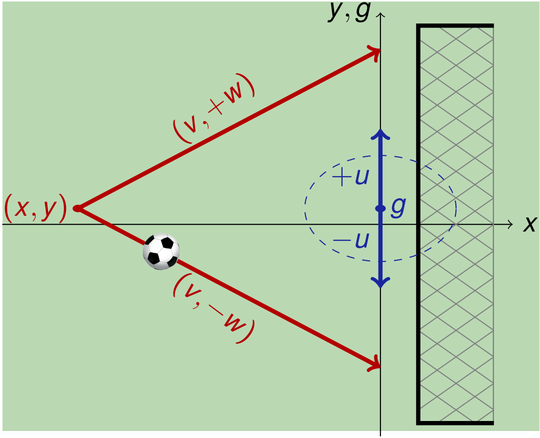
$$\Big(\big(u := +u \cup u := -u\big); \{x' = v, y' = w, g' = u\}\Big)^* \Big\rangle\, x^2 + (y-g)^2 \leq 1$$

Example: Goalie in Robot Soccer



$$\left(\frac{x}{v}\right)^2 (u-w)^2 \leq 1 \land$$

$$x < 0 \land v > 0 \land y = g \rightarrow$$

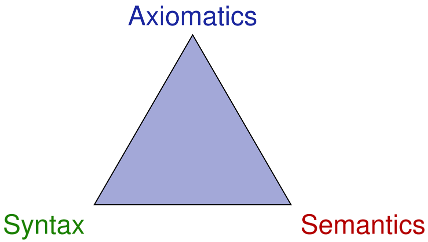
$$\langle (w := +w \cap w := -w);$$

$$((u := +u \cup u := -u); \{x' = v, y' = w, g' = u\})^* \rangle\, x^2 + (y-g)^2 \leq 1$$

Soundness

Theorem (Soundness)

dGL *proof calculus is sound i.e. all provable formulas are valid*



Soundness

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Proof.

$$\langle\cup\rangle \quad \langle\alpha \cup \beta\rangle P \leftrightarrow \langle\alpha\rangle P \vee \langle\beta\rangle P$$

$$\langle;\rangle \quad \langle\alpha;\beta\rangle P \leftrightarrow \langle\alpha\rangle\langle\beta\rangle P$$

$$[\cdot] \quad [\alpha]P \leftrightarrow \neg\langle\alpha\rangle\neg P$$

$$\text{M} \quad \frac{P \rightarrow Q}{\langle\alpha\rangle P \rightarrow \langle\alpha\rangle Q}$$

□

Soundness

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Proof.

$\langle\cup\rangle$ $[\![\langle\alpha\cup\beta\rangle P]\!] = \varsigma_{\alpha\cup\beta}([\![P]\!]) = \varsigma_\alpha([\![P]\!]) \cup \varsigma_\beta([\![P]\!]) = [\![\langle\alpha\rangle P]\!] \cup [\![\langle\beta\rangle P]\!] = [\![\langle\alpha\rangle P \vee \langle\beta\rangle P]\!]$

$\langle\cup\rangle$ $\langle\alpha\cup\beta\rangle P \leftrightarrow \langle\alpha\rangle P \vee \langle\beta\rangle P$

$\langle;\rangle$ $[\![\langle\alpha;\beta\rangle P]\!] = \varsigma_{\alpha;\beta}([\![P]\!]) = \varsigma_\alpha(\varsigma_\beta([\![P]\!])) = \varsigma_\alpha([\![\langle\beta\rangle P]\!]) = [\![\langle\alpha\rangle\langle\beta\rangle P]\!]$

$\langle;\rangle$ $\langle\alpha;\beta\rangle P \leftrightarrow \langle\alpha\rangle\langle\beta\rangle P$

$[\cdot]$ is sound by determinacy

$[\cdot]$ $[\alpha]P \leftrightarrow \neg\langle\alpha\rangle\neg P$

M Assume the premise $P \rightarrow Q$ is valid, i.e. $[\![P]\!] \subseteq [\![Q]\!]$.
Then the conclusion $\langle\alpha\rangle P \rightarrow \langle\alpha\rangle Q$ is valid, i.e.
$[\![\langle\alpha\rangle P]\!] = \varsigma_\alpha([\![P]\!]) \subseteq \varsigma_\alpha([\![Q]\!]) = [\![\langle\alpha\rangle Q]\!]$ by monotonicity.

$$\text{M} \quad \frac{P \rightarrow Q}{\langle\alpha\rangle P \rightarrow \langle\alpha\rangle Q}$$

□

Soundness & Completeness

Theorem (Completeness)

dGL *calculus is a sound & complete axiomatization of hybrid games relative to any (differentially) expressive logic L.*

$$\vDash \varphi \quad \text{iff} \quad Taut_L \vdash \varphi$$

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Soundness & Completeness: Consequences

Corollary (Constructive)

Constructive and Moschovakis-coding-free. (Minimal: $x' = f(x)$, $\exists$, $[\alpha^*]$)

Remark (Coquand & Huet) (Inf.Comput'88)

Modal analogue for $\langle\alpha^*\rangle$ of characterizations in Calculus of Constructions

Corollary (Meyer & Halpern) (J.ACM'82)

$F \rightarrow \langle\alpha\rangle G$ semidecidable for uninterpreted programs.

Corollary (Schmitt) (Inf.Control.'84)

$[\alpha]$-free semidecidable for uninterpreted programs.

Corollary

Uninterpreted game logic with even ^{d} in $\langle\alpha\rangle$ is semidecidable.

Soundness & Completeness: Consequences

Corollary

Harel'77 convergence rule unnecessary for hybrid games, hybrid systems, discrete programs.

Corollary (Characterization of hybrid game challenges)

- $[\alpha^*]G$*: Succinct invariants* *discrete* Π^0_2
- $[x' = f(x)]G$ *and* $\langle x' = f(x)\rangle G$*: Succinct differential (in)variants* Δ^1_1
- $\exists x\, G$*: Complexity depends on Herbrand disjunctions:* *discrete* Π^1_1
  ✓ *uninterpreted* ✓ *reals* × $\exists x\, [\alpha^*]G$ Π^1_1*-complete for discrete* α

Corollary (Hybrid version of Parikh's result) (FOCS'83)

**-free* dGL *complete relative to* dL*, relative to continuous, or to discrete*
^{d}*-free* dGL *complete relative to* dL*, relative to continuous, or to discrete*

Soundness & Completeness: Consequences

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  ✓ *uninterpreted* ✓ *reals* × $\exists x\, [\alpha^*]G$ Π^1_1*-complete for discrete* α

set is Π^0_n iff it's $\{x : \forall y_1\, \exists y_2 \forall y_3 \dots y_n\, \varphi(x, y_1, \dots, y_n)\}$ for a decidable φ
set is Σ^0_n iff it's $\{x : \exists y_1\, \forall y_2 \exists y_3 \dots y_n\, \varphi(x, y_1, \dots, y_n)\}$ for a decidable φ
set is Π^1_1 iff it's $\{x : \forall f\, \exists y\, \varphi(x, y, f)\}$ for a decidable φ and functions f
set is Σ^1_1 iff it's $\{x : \exists f\, \forall y\, \varphi(x, y, f)\}$ for a decidable φ and functions f
$\Delta^i_n = \Sigma^i_n \cap \Pi^i_n$

Soundness & Completeness: Consequences

Corollary (ODE Completeness) (+LICS'12)

dGL *complete relative to ODE for hybrid games with finite-rank Borel winning regions.*

Corollary (Continuous Completeness)

dGL *complete relative to* $L_{\mu D}$, *continuous modal* μ, *over* $\mathbb{R}$

Corollary (Discrete Completeness) (+LICS'12)

dGL + *Euler axiom complete relative to discrete* L_μ *over* $\mathbb{R}$

Soundness & Completeness: Consequences

$$\langle(\underbrace{\underbrace{x:=1;x'=1^d}_{\beta}\cup\underbrace{x:=x-1}_{\gamma}}_{\alpha})^*\rangle 0\leq x<1$$

▸ Fixpoint style proof technique

$$\begin{array}{lr}
 & * \\ \hline
\mathbb{R} & \forall x\,(0{\leq}x{<}1\vee\forall t{\geq}0\,p(1+t)\vee p(x-1)\rightarrow p(x))\rightarrow(\mathit{true}\rightarrow p(x)) \\ \hline
\langle{:=}\rangle & \forall x\,(0{\leq}x{<}1\vee\langle x:=1\rangle\neg\exists t{\geq}0\,\langle x:=x+t\rangle\neg p(x)\vee p(x-1)\rightarrow p(x))\rightarrow(\mathit{true}\rightarrow p(x)) \\ \hline
\langle{'}\rangle & \forall x\,(0{\leq}x{<}1\vee\langle x:=1\rangle\neg\langle x'=1\rangle\neg p(x)\vee p(x-1)\rightarrow p(x))\rightarrow(\mathit{true}\rightarrow p(x)) \\ \hline
\langle;\rangle,\langle^d\rangle & \forall x\,(0{\leq}x{<}1\vee\langle\beta\rangle p(x)\vee\langle\gamma\rangle p(x)\rightarrow p(x))\rightarrow(\mathit{true}\rightarrow p(x)) \\ \hline
\langle\cup\rangle & \forall x\,(0{\leq}x{<}1\vee\langle\beta\cup\gamma\rangle p(x)\rightarrow p(x))\rightarrow(\mathit{true}\rightarrow p(x)) \\ \hline
\mathrm{US} & \forall x\,(0{\leq}x{<}1\vee\langle\alpha\rangle\langle\alpha^*\rangle 0{\leq}x{<}1\rightarrow\langle\alpha^*\rangle 0{\leq}x{<}1)\rightarrow(\mathit{true}\rightarrow\langle\alpha^*\rangle 0{\leq}x{<}1) \\ \hline
\langle{*}\rangle & \mathit{true}\rightarrow\langle\alpha^*\rangle 0{\leq}x{<}1
\end{array}$$

More Axioms

$\mathrm{K}\quad [\alpha](P \to Q) \to ([\alpha]P \to [\alpha]Q)$ $\qquad \mathrm{M}_{[\cdot]}\ \dfrac{P \to Q}{[\alpha]P \to [\alpha]Q}$

$\overleftarrow{\mathrm{M}}\quad \langle\alpha\rangle(P \lor Q) \to \langle\alpha\rangle P \lor \langle\alpha\rangle Q$ $\qquad \mathrm{M}\quad \langle\alpha\rangle P \lor \langle\alpha\rangle Q \to \langle\alpha\rangle(P \lor Q)$

$\mathrm{I}\quad [\alpha^*](P \to [\alpha]P) \to (P \to [\alpha^*]P)$ $\qquad \forall\mathrm{I}\quad (P \to [\alpha]P) \to (P \to [\alpha^*]P)$

$\mathrm{B}\quad \langle\alpha\rangle\exists x\, P \to \exists x\, \langle\alpha\rangle P$ $\qquad (x \notin \alpha)\ \overleftarrow{\mathrm{B}}\quad \exists x\, \langle\alpha\rangle P \to \langle\alpha\rangle\exists x\, P$

$\mathrm{G}\quad \dfrac{P}{[\alpha]P}$ $\qquad \mathrm{M}_{[\cdot]}\ \dfrac{P \to Q}{[\alpha]P \to [\alpha]Q}$

$\mathrm{R}\quad \dfrac{P_1 \land P_2 \to Q}{[\alpha]P_1 \land [\alpha]P_2 \to [\alpha]Q}$ $\qquad \mathrm{M}_{[\cdot]}\ \dfrac{P_1 \land P_2 \to Q}{[\alpha](P_1 \land P_2) \to [\alpha]Q}$

$\mathrm{FA}\ \langle\alpha^*\rangle P \to P \lor \langle\alpha^*\rangle(\neg P \land \langle\alpha\rangle P)$ $\qquad \overleftarrow{[*]}\ [\alpha^*]P \leftrightarrow P \land [\alpha^*][\alpha]P$

More Axioms ???

~~K~~	$[\alpha](P \to Q) \to ([\alpha]P \to [\alpha]Q)$	$\text{M}_{[\cdot]}$	$\dfrac{P \to Q}{[\alpha]P \to [\alpha]Q}$
~~$\overleftarrow{\text{M}}$~~	$\langle\alpha\rangle(P \vee Q) \to \langle\alpha\rangle P \vee \langle\alpha\rangle Q$	M	$\langle\alpha\rangle P \vee \langle\alpha\rangle Q \to \langle\alpha\rangle(P \vee Q)$
~~I~~	$[\alpha^*](P \to [\alpha]P) \to (P \to [\alpha^*]P)$	$\forall$I	$(P\to[\alpha]P) \to (P\to[\alpha^*]P)$
~~B~~	$\langle\alpha\rangle\exists x\, P \to \exists x\, \langle\alpha\rangle P$	$(x\notin\alpha)$ $\overleftarrow{\text{B}}$	$\exists x\, \langle\alpha\rangle P \to \langle\alpha\rangle\exists x\, P$
~~G~~	$\dfrac{P}{[\alpha]P}$	$\text{M}_{[\cdot]}$	$\dfrac{P \to Q}{[\alpha]P \to [\alpha]Q}$
~~R~~	$\dfrac{P_1 \wedge P_2 \to Q}{[\alpha]P_1 \wedge [\alpha]P_2 \to [\alpha]Q}$	$\text{M}_{[\cdot]}$	$\dfrac{P_1 \wedge P_2 \to Q}{[\alpha](P_1 \wedge P_2) \to [\alpha]Q}$
~~FA~~	$\langle\alpha^*\rangle P \to P \vee \langle\alpha^*\rangle(\neg P \wedge \langle\alpha\rangle P)$	~~$\overleftarrow{[*]}$~~	$[\alpha^*]P \leftrightarrow P \wedge [\alpha^*][\alpha]P$

Separating Axioms

Theorem (Axiomatic separation: hybrid systems vs. hybrid games)

Axiomatic separation is exactly K, I, C, B, V, G. dGL *is a subregular, sub-Barcan, monotonic modal logic without loop induction axioms.*

$\cancel{K}$	$[\alpha](P \rightarrow Q) \rightarrow ([\alpha]P \rightarrow [\alpha]Q)$	$M_{[\cdot]}$	$\dfrac{P \rightarrow Q}{[\alpha]P \rightarrow [\alpha]Q}$
$\cancel{\overleftarrow{M}}$	$\langle\alpha\rangle(P \vee Q) \rightarrow \langle\alpha\rangle P \vee \langle\alpha\rangle Q$	M	$\langle\alpha\rangle P \vee \langle\alpha\rangle Q \rightarrow \langle\alpha\rangle(P \vee Q)$
$\cancel{I}$	$[\alpha^*](P \rightarrow [\alpha]P) \rightarrow (P \rightarrow [\alpha^*]P)$	∀I	$(P \rightarrow [\alpha]P) \rightarrow (P \rightarrow [\alpha^*]P)$
$\cancel{B}$	$\langle\alpha\rangle\exists x\, P \rightarrow \exists x\, \langle\alpha\rangle P$ $(x \notin \alpha)$	$\overleftarrow{B}$	$\exists x\, \langle\alpha\rangle P \rightarrow \langle\alpha\rangle\exists x\, P$
$\cancel{G}$	$\dfrac{P}{[\alpha]P}$	$M_{[\cdot]}$	$\dfrac{P \rightarrow Q}{[\alpha]P \rightarrow [\alpha]Q}$
$\cancel{R}$	$\dfrac{P_1 \wedge P_2 \rightarrow Q}{[\alpha]P_1 \wedge [\alpha]P_2 \rightarrow [\alpha]Q}$	$M_{[\cdot]}$	$\dfrac{P_1 \wedge P_2 \rightarrow Q}{[\alpha](P_1 \wedge P_2) \rightarrow [\alpha]Q}$
$\cancel{FA}$	$\langle\alpha^*\rangle P \rightarrow P \vee \langle\alpha^*\rangle(\neg P \wedge \langle\alpha\rangle P)$	$\cancel{\overleftarrow{[*]}}$	$[\alpha^*]P \leftrightarrow P \wedge [\alpha^*][\alpha]P$

Outline

Expressiveness

Theorem (Expressive Power: hybrid systems $<$ hybrid games)

dGL *for hybrid games strictly more expressive than* dL *for hybrid games:*

$$\mathrm{dL} < \mathrm{dGL}$$

Expressiveness

Theorem (Expressive Power: hybrid systems $<$ hybrid games)

dGL *for hybrid games strictly more expressive than* dL *for hybrid games:*

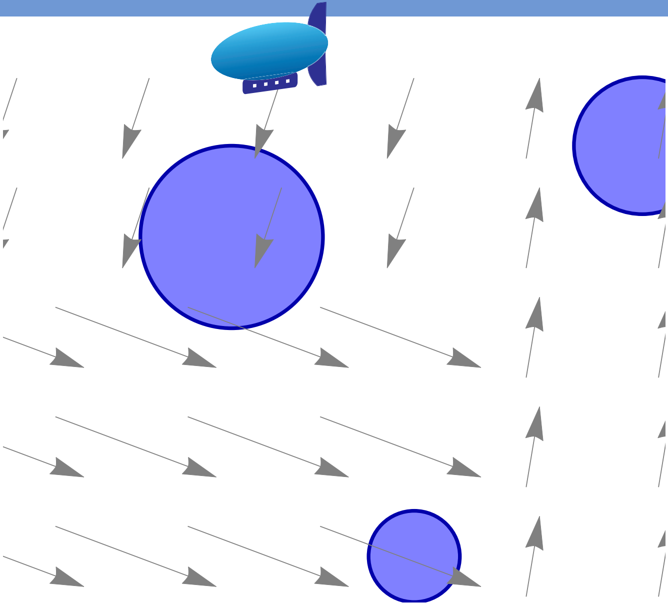
$$\text{dL} < \text{dGL}$$

First-order adm. $\mathbb{R}$

Inductive adm. $\mathbb{R}$

Outline

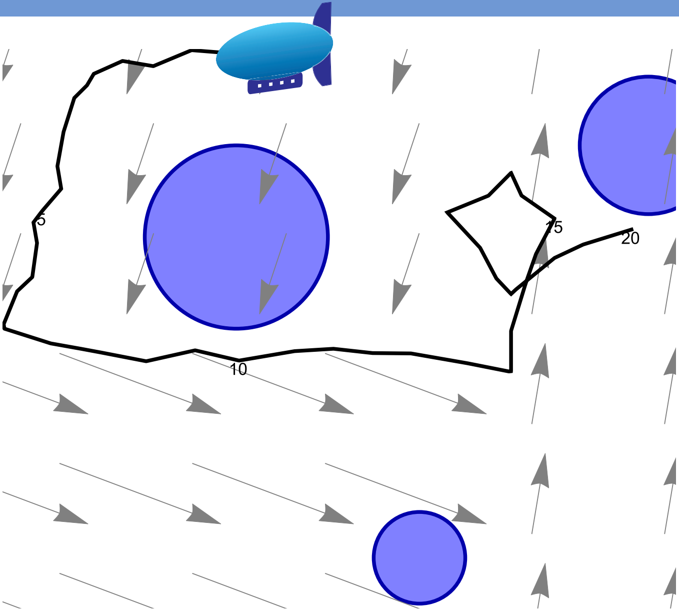
Zeppelin Obstacle Parcours



avoid obstacles
changing wind
local turbulence
$x' = f(x, y, z)$

TOCL'17

Zeppelin Obstacle Parcours



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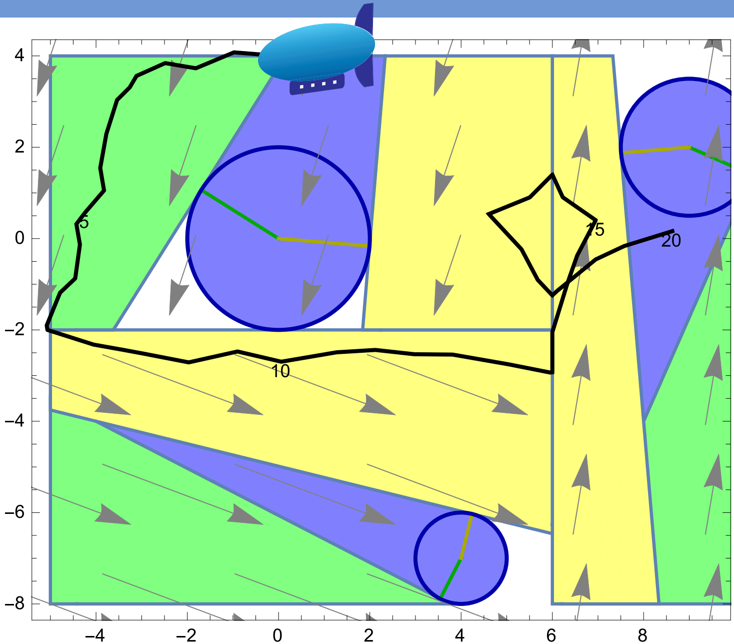
Zeppelin Obstacle Parcours



avoid obstacles
changing wind
local turbulence
$x' = f(x, y, z)$

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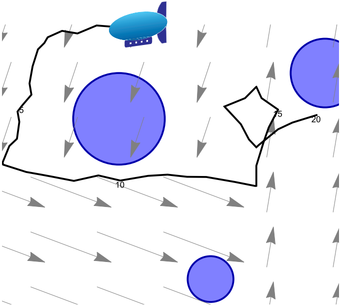
Zeppelin Obstacle Parcours

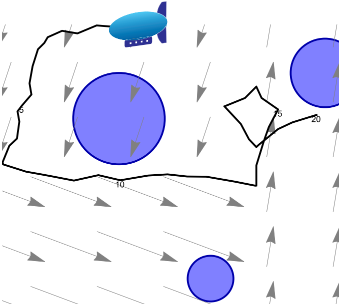
$$c > 0 \land \|x-o\|^2 \geq c^2 \to \\ \Big[\Big(v:=*;o:=*;c:=*;?C; \\ \{x' = v+py+rz \,\&^d\, y\in B \,\&\, z\in B\} \\ \Big)^*\Big] \, \|x-o\|^2 \geq c^2$$



- ✓ airship at $x \in \mathbb{R}^2$
- ✓ propeller p controlled in any direction $y \in B$, i.e. $y_1^2 + y_2^2 \leq 1$
- × sporadically changing homogeneous wind field $v \in \mathbb{R}^2$
- × sporadically changing obstacle $o \in \mathbb{R}^2$ of size c subject to C
- × continuously local turbulence of magnitude r in any direction $z \in B$

Zeppelin Obstacle Parcours

$$\begin{aligned}
&c > 0 \land \|x - o\|^2 \geq c^2 \to \\
&\quad \Big[\big(v := *; o := *; c := *; ?C; \\
&\qquad \{x' = v + py + rz \&^d y \in B \& z \in B\} \\
&\quad \big)^*\Big] \, \|x - o\|^2 \geq c^2
\end{aligned}$$



- $r > p$
- $p > \|v\| + r$
- $\|v\| + r > p > r$

✓ airship at $x \in \mathbb{R}^2$

✓ propeller p controlled in any direction $y \in B$, i.e. $y_1^2 + y_2^2 \leq 1$

× sporadically changing homogeneous wind field $v \in \mathbb{R}^2$

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Zeppelin Obstacle Parcours

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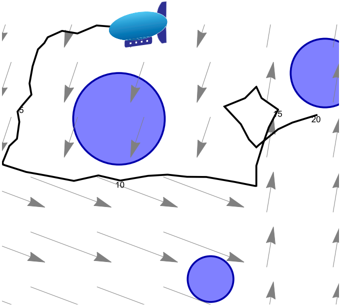
$$\Big[\Big(v := *; o := *; c := *; ?C;$$
$$\{x' = v + py + rz\&^{d} y \in B\& z \in B\}$$
$$\Big)^*\Big] \|x - o\|^2 \geq c^2$$

- × $r > p$ hopeless
- ● $p > \|v\| + r$
- ● $\|v\| + r > p > r$



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Zeppelin Obstacle Parcours

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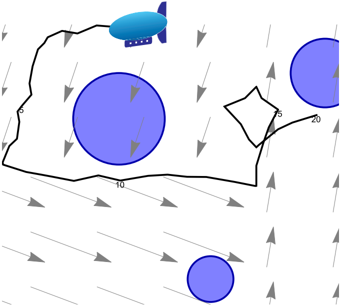
- × $r > p$ hopeless
- ✓ $p > \|v\| + r$ super-powered
- $\|v\| + r > p > r$



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Zeppelin Obstacle Parcours

$$
\begin{aligned}
& c > 0 \land \|x - o\|^2 \geq c^2 \rightarrow \\
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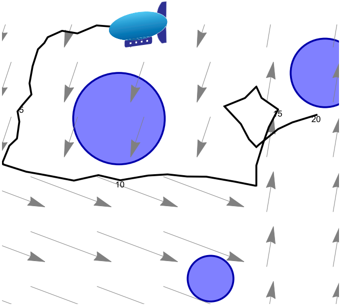
& \qquad \{x' = v + py + rz \,\&^d\, y \in B \,\&\, z \in B\} \\
& \quad \Big)^*\Big] \, \|x - o\|^2 \geq c^2
\end{aligned}
$$

- × $r > p$ hopeless
- ✓ $p > \|v\| + r$ super-powered
- ? $\|v\| + r > p > r$ our challenge



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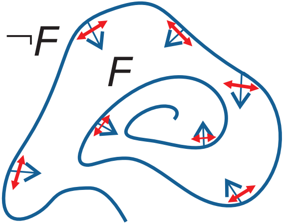
Differential Game Invariants

Theorem (Differential Game Invariants)

$$\text{DGI}\quad \frac{\exists y \in Y \forall z \in Z\, [x' := f(x,y,z)](F)'}{F \rightarrow [x' = f(x,y,z) \&^d y \in Y \& z \in Z]F}$$

Theorem (Differential Game Refinement)

$$\frac{\forall u \in U \exists y \in Y \forall z \in Z \exists v \in V \forall x\, (f(x,y,z) = g(x,u,v))}{[x' = g(x,u,v) \&^d u \in U \& v \in V]F \rightarrow [x' = f(x,y,z) \&^d y \in Y \& z \in Z]F}$$



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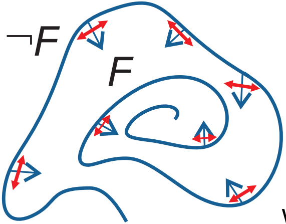
Differential Game Invariants

Theorem (Differential Game Invariants)

$$\text{DGI}\quad \frac{\exists y\in Y\,\forall z\in Z\,[x':=f(x,y,z)](F)'}{F\rightarrow[x'=f(x,y,z)\,\&^d\,y\in Y\,\&\,z\in Z]F}$$

Theorem (Differential Game Refinement)

$$\frac{\forall u\in U\,\exists y\in Y\,\forall z\in Z\,\exists v\in V\,\forall x\,(f(x,y,z)=g(x,u,v))}{[x'=g(x,u,v)\,\&^d\,u\in U\,\&\,v\in V]F\rightarrow[x'=f(x,y,z)\,\&^d\,y\in Y\,\&\,z\in Z]F}$$



$$\text{DGI}\;\frac{\text{[:=]}\;\dfrac{\dfrac{*}{\exists y{\in}I\,\forall z{\in}I\,0\leq 3x^2(-1+2y+z)}}{\exists y{\in}I\,\forall z{\in}I\,[x':=-1+2y+z]0\leq 3x^2x'}}{1\leq x^3\rightarrow[x'=-1+2y+z\,\&^d\,y\in I\,\&\,z\in I]1\leq x^3}$$

where $y\in I\overset{\text{def}}{\equiv}-1\leq y\leq 1$

TOCL'17

Outline

1. CPS Game Motivation
2. Differential Game Logic
   - Syntax
   - Example: Push-around Cart
   - Example: Robot Dance
   - Differential Hybrid Games
   - Denotational Semantics
   - Determinacy
   - Strategic Closure Ordinals
3. Axiomatization
   - Axiomatics
   - Example: Robot Soccer
   - Soundness and Completeness
   - Separating Axioms
4. Expressiveness
5. Differential Hybrid Games
6. **Summary**

Further Challenges

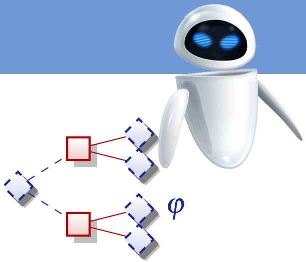
Several extensions …

1. Draws
2. Cooperative games with coalitions
3. Rewards
4. Payoffs other than ± 1

…are all expressible already. Direct syntactic support?

1. Compositional concurrent hybrid games
2. Imperfect information hybrid games
3. Constructive dGL to retain winning strategies as proof terms IJCAR'20
4. Differential games + hybrid games TOCL'17
5. Application in airborne collision avoidance games TECS
6. Structure proof language for hybrid games+systems TECS'21

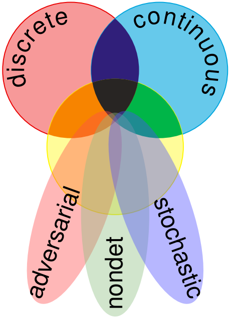
Differential Game Logic

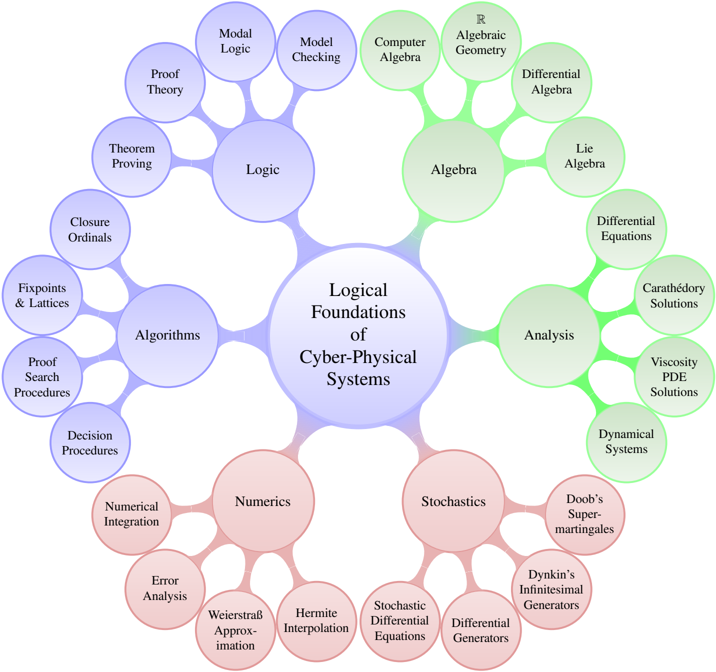


differential game logic

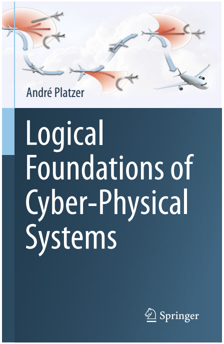
$\mathrm{dGL} = \mathrm{GL} + \mathrm{HG} = \mathrm{dL} + {}^{d}$

- Logic for hybrid games
- Compositional PL + logic
- Discrete + continuous + adversarial
- Winning region iteration $\geq\omega_1^{\mathrm{CK}}$
- Sound & rel. complete axiomatization
- Hybrid games $>$ hybrid systems
- ^{d} radical challenge yet smooth extension
- Stochastic $\approx$ adversarial





- Logical Foundations of Cyber-Physical Systems
  - Logic
    - Modal Logic
    - Model Checking
    - Proof Theory
    - Theorem Proving
  - Algebra
    - Computer Algebra
    - ℝ Algebraic Geometry
    - Differential Algebra
    - Lie Algebra
  - Analysis
    - Differential Equations
    - Carathéodory Solutions
    - Viscosity PDE Solutions
    - Dynamical Systems
  - Stochastics
    - Doob's Super-martingales
    - Dynkin's Infinitesimal Generators
    - Differential Generators
    - Stochastic Differential Equations
  - Numerics
    - Hermite Interpolation
    - Weierstraß Approx-imation
    - Error Analysis
    - Numerical Integration
  - Algorithms
    - Closure Ordinals
    - Fixpoints & Lattices
    - Proof Search Procedures
    - Decision Procedures



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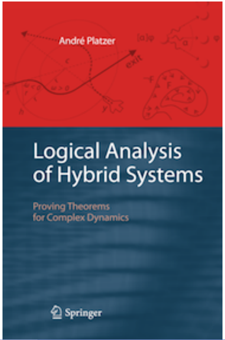
Logical Foundations of Cyber-Physical Systems

Springer



André Platzer

Logical Analysis of Hybrid Systems

Proving Theorems for Complex Dynamics

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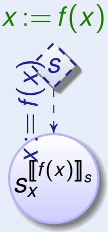
Outline

7 Operational Semantics

Differential Game Logic: Operational Semantics

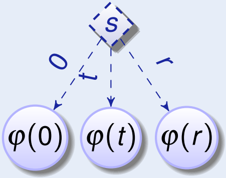
Definition (Hybrid game α: operational semantics)



Differential Game Logic: Operational Semantics

Definition (Hybrid game α: operational semantics)

$x' = f(x) \& Q$



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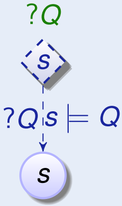

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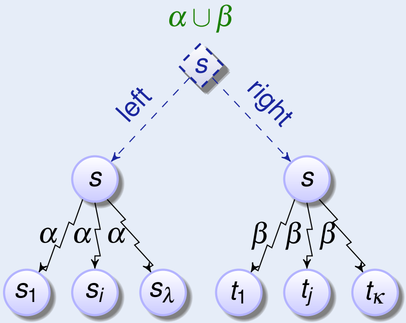

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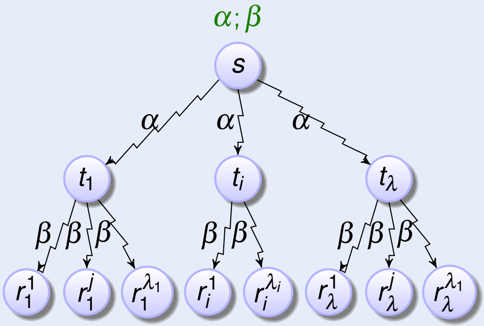

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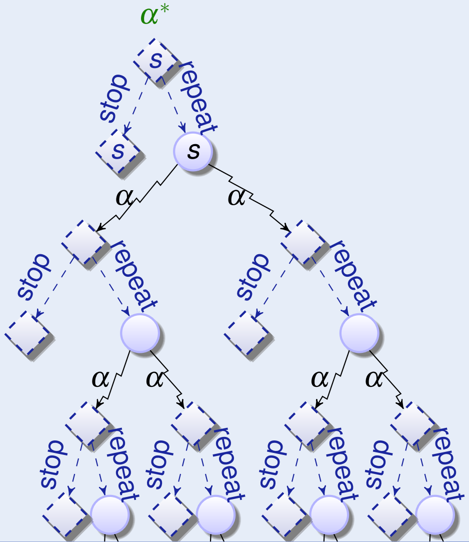

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Definition (Hybrid game α: operational semantics)