Refining Constructive Hybrid Games

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FSCD'20

Why Refine Constructive Hybrid Games?

Refining

$\alpha \leq \beta \quad \alpha \cong \beta \qquad \alpha \cup \beta \leq \text{if}(P)\ \alpha\ \text{else}\ \beta$

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Constructive Differential Game Logic (CdGL)

[P]a[safe]	<α>safe	Witness a safe behavior of α
{P}a{safe}	[α]safe	Observe α, universal proof of "safe"
	$\alpha \leq \beta$	Compute β strategy from α strategy

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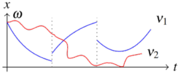
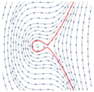
Hybrid





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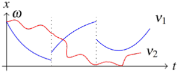
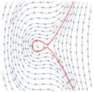


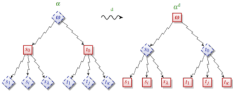


Games

<α>safe

[α]safe

α^d



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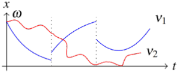

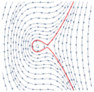

Games

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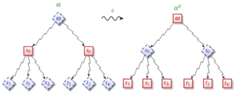
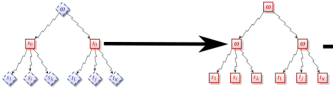


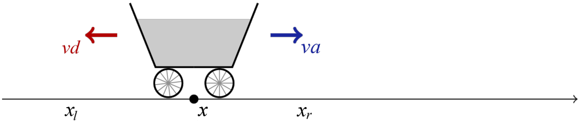
Enables



$\Gamma \vdash [\text{game}]P \quad \leq \quad \Gamma \vdash [\text{system}]P$

Strategies to Systems Synthesis

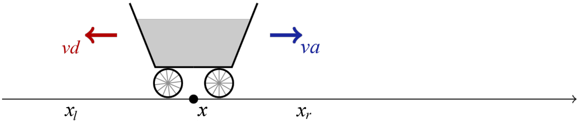
Constructive Hybrid Game: Push-pull



$$\text{safe} \equiv x_l < x_0 = x < x_r \to [\text{PP}](x = x_0)$$

$$\begin{aligned}\text{PP} \equiv \{&\{vd := -1 \cup vd := 1\};\\ &\{va := *; ?(-1 \le va \le 1)\}^d;\\ &\{x' = vd + va \,\&\, x_l \le x \le x_r\}\}^*\end{aligned}$$

Constructive Hybrid Game: Push-pull



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Either speed

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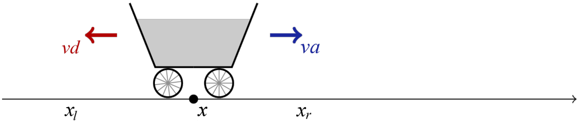

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Either speed

Limits

$$\text{PP} \equiv \{\{vd := -1 \cup vd := 1\};$$

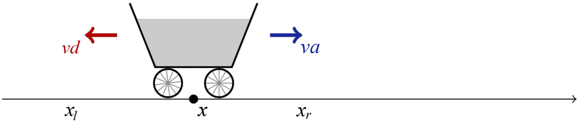
Speed

$$\{va := *; ?(-1 \leq va \leq 1)\}^d;$$

Switch

$$\{x' = vd + va \,\&\, x_l \leq x \leq x_r\}\}^*$$

Constructive Hybrid Game: Push-pull



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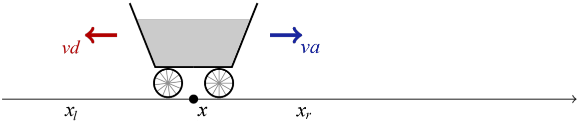
Physics

Constraint

Constructive Hybrid Game: Push-pull



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Speed

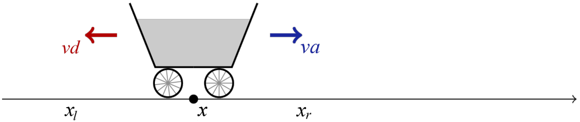
Switch

Physics

Constraint

Loop

Constructive Hybrid Game: Push-pull



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Speed

$$\{va := *; ?(-1 \le va \le 1)\}^d;$$

Switch

$$\{x' = vd + va \,\&\, x_l \le x \le x_r\}\}^*$$

Physics

Loop

Constraint

$$\alpha_{PP} = \{\{\{vd := -1; va := 1\}; \cup \{vd := 1; va := -1\}\}; x := *; x' := vd + va; ?x = x_0\}^*$$

Mirror

Types Give Constructive Semantics

$\ulcorner P \urcorner \quad : (\text{state} \Rightarrow \text{type} \quad)$

$\ulcorner \langle ?Q \rangle P \urcorner \, s \quad = \ulcorner Q \urcorner \, s * \ulcorner P \urcorner \, s$ — Prove test

$\ulcorner [?Q] P \urcorner \, s \quad = \ulcorner Q \urcorner \, s \Rightarrow \ulcorner P \urcorner \, s$ — Assume test

Types Give Constructive Semantics

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$$\ulcorner \langle x := * \rangle P \urcorner\ s = \Sigma v : \mathbb{R}.\ \ulcorner P \urcorner\ (\text{set } s\ x\ v)$$ — Choose x

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$$\ulcorner [x := *] P \urcorner\ s = \Pi v : \mathbb{R}.\ \ulcorner P \urcorner\ (\text{set } s\ x\ v)$$ — Receive x

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$$\ulcorner \langle \alpha \cup \beta \rangle P \urcorner \; s \quad = \ulcorner \langle \alpha \rangle P \urcorner \; s + \ulcorner \langle \beta \rangle P \urcorner \; s$$ — Choose branch

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$$\ulcorner P \urcorner \quad : (\text{state} \Rightarrow \text{type} \quad)$$

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\ulcorner \langle x := * \rangle P \urcorner\ s &= \Sigma v : \mathbb{R}.\ \ulcorner P \urcorner\ (\text{set}\ s\ x\ v) \\
\ulcorner \langle \alpha \cup \beta \rangle P \urcorner\ s &= \ulcorner \langle \alpha \rangle P \urcorner\ s + \ulcorner \langle \beta \rangle P \urcorner\ s \\
\ulcorner \langle \alpha^d \rangle P \urcorner\ s &= \ulcorner [\alpha] P \urcorner\ s
\end{aligned}$$

Prove test

Choose x

Choose branch

Switch

$$\begin{aligned}
\ulcorner [?Q] P \urcorner\ s &= \ulcorner Q \urcorner\ s \Rightarrow \ulcorner P \urcorner\ s \\
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\end{aligned}$$

Assume test

Receive x

Can't choose

Switch

Types Give Constructive Semantics

$$\ulcorner P \urcorner \quad : (\text{state} \Rightarrow \text{type} \quad)$$

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$$\ulcorner \langle \alpha^d \rangle P \urcorner \; s \quad = \ulcorner [\alpha] P \urcorner \; s$$ Switch

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$$\ulcorner [\alpha^d] P \urcorner \; s \quad = \ulcorner \langle \alpha \rangle P \urcorner \; s$$ Switch

$$\ulcorner \alpha \leq_{[]} \beta \urcorner \; s \quad = \left(\Pi P : (\text{state} \Rightarrow \text{type} \;). \left(\ulcorner [\alpha] P \urcorner \; s \Rightarrow \ulcorner [\beta] P \urcorner \; s\right)\right)$$

Types Give Constructive Semantics

$$\ulcorner P \urcorner : (\text{state} \Rightarrow \text{type}_{i+1})$$

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$$\ulcorner \langle \alpha^d \rangle P \urcorner\ s = \ulcorner [\alpha] P \urcorner\ s$$ — Switch

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$$\ulcorner [\alpha^d] P \urcorner\ s = \ulcorner \langle \alpha \rangle P \urcorner\ s$$ — Switch

$$\ulcorner \alpha \leq^i_{[]} \beta \urcorner\ s = \left(\Pi P : (\text{state} \Rightarrow \text{type}_i).\ \left(\ulcorner [\alpha] P \urcorner\ s \Rightarrow \ulcorner [\beta] P \urcorner\ s\right)\right)$$

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$$\text{R}[\cdot]\quad \frac{\Gamma \vdash [\alpha] P \qquad \Gamma \vdash \alpha \le^i_{[]} \beta}{\Gamma \vdash [\beta] P}\ 0 \qquad\qquad \text{R}\langle\cdot\rangle\quad \frac{\Gamma \vdash \langle \alpha \rangle P \qquad \Gamma \vdash \alpha \le^i_{\langle\rangle} \beta}{\Gamma \vdash \langle \beta \rangle P}\ 0$$

Refinements Subsume Game Algebra

$$\text{trans}\quad \frac{\Gamma \vdash \alpha \leq_{[]} \beta \quad \Gamma \vdash \beta \leq_{[]} \gamma}{\Gamma \vdash \alpha \leq_{[]} \gamma} \qquad \text{refl}\quad \Gamma \vdash \alpha \leq_{[]} \alpha$$

$$\cup\text{A}\quad \Gamma \vdash \{\alpha \cup \beta\} \cup \gamma \cong \alpha \cup \{\beta \cup \gamma\} \qquad \cup\text{c}\quad \Gamma \vdash \alpha \cup \beta \cong \beta \cup \alpha$$

$$;\text{d}_r\quad \Gamma \vdash \{\alpha \cup \beta\}; \gamma \cong \{\alpha; \gamma\} \cup \{\beta; \gamma\}$$

Refinements Resolve Strategic Choice

[∪]L1 $\quad \Gamma \vdash \alpha^d \leq_{[]} \{\alpha \cup \beta\}^d$ $\qquad$ [∪]L2 $\quad \Gamma \vdash \beta^d \leq_{[]} \{\alpha \cup \beta\}^d$

[:∗] $\quad \Gamma \vdash \{x := f\}^d \leq_{[]} \{x := *\}^d$

$$;\text{G} \quad \frac{\cdot \vdash \alpha_1 \leq_{[]} \alpha_2 \quad \cdot \vdash \beta_1 \leq_{[]} \beta_2}{\cdot \vdash \alpha_1;\beta_1 \leq_{[]} \alpha_2;\beta_2}$$

Refinements Resolve Strategic Choice

[∪]L1 $\Gamma \vdash \alpha^d \leq_{[]} \{\alpha \cup \beta\}^d$ [∪]L2 $\Gamma \vdash \beta^d \leq_{[]} \{\alpha \cup \beta\}^d$

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;G $$\frac{\cdot \vdash \alpha_1 \leq_{[]} \alpha_2 \quad \cdot \vdash \beta_1 \leq_{[]} \beta_2}{\cdot \vdash \alpha_1; \beta_1 \leq_{[]} \alpha_2; \beta_2}$$

;S $$\frac{\Gamma \vdash \boldsymbol{\alpha}_1 \leq_{[]} \alpha_2 \quad \Gamma \vdash [\boldsymbol{\alpha}_1]\beta_1 \leq_{[]} \beta_2}{\Gamma \vdash \boldsymbol{\alpha}_1; \beta_1 \leq_{[]} \alpha_2; \beta_2}$$ [1]

[1] $\boldsymbol{\alpha}_1$ is a hybrid system

ODEs are Solved or Abstracted

Assignment

ODE

$$\text{solve}\quad \frac{\Gamma \vdash t = 0 \wedge d \geq 0 \quad \Gamma \vdash [t := *; ?0 \leq t \leq d; x := sol]Q}{\Gamma \vdash \{t := d; x := sol; t' := 1; x' := f\} \leq_{[]} \{t' = 1, x' = f \,\&\, Q\}^d}\ ^1$$

$$\text{DC}\quad \frac{\Gamma \vdash [x' = f \,\&\, P]Q}{\Gamma \vdash \{x' = f \,\&\, P\} \cong \{x' = f \,\&\, P \wedge Q\}}$$

$$\text{DW}\quad \Gamma \vdash \{x := *; x' := f; ?Q\} \leq_{[]} \{x' = f \,\&\, Q\}$$

[1] *sol* solves ODE, $\{t, t', x, x'\}$ not free in d

Game Proofs are Reified as Systems

$$(\text{Proof of } [\alpha]P \text{ or } \langle\alpha\rangle P) \leadsto \text{System}$$

First IH $\boldsymbol{\alpha}$

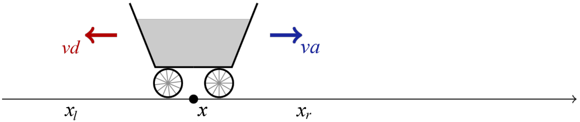
$$\langle{:=}\rangle\mathrm{I}\ \frac{\Gamma(x_0), x = f_x^{x_0} \vdash P}{\Gamma(x) \vdash \langle x := f\rangle P} \leadsto x := f; \boldsymbol{\alpha} \qquad \langle{:*}\rangle\mathrm{I}\ \frac{\Gamma(x_0), x = f_x^{x_0} \vdash P}{\Gamma(x) \vdash \langle x := *\rangle P} \leadsto x := f; \boldsymbol{\alpha}$$

$$\mathrm{dw}\ \frac{\Gamma(x_0), Q \vdash P}{\Gamma(x) \vdash [x' = f \,\&\, Q]P} \leadsto x := *; x' := f; ?Q; \boldsymbol{\alpha}$$

$$\mathrm{dc}\ \frac{\Gamma \vdash [x' = f \,\&\, Q]R \quad \Gamma \vdash [x' = f \,\&\, Q \land R]P}{\Gamma \vdash [x' = f \,\&\, Q]P} \leadsto \boldsymbol{\beta}$$

Second IH $\boldsymbol{\beta}$

Cart Proof Reifies Strategy



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$$\begin{aligned} \alpha_{PP} = \Big\{ & \{\{vd := -1; va := 1\} \\ & \cup \{vd := 1; va := -1\}\}; \\ & x := *; x' := vd + va; ?x = x_0 \Big\}^* \end{aligned}$$

Let $\mathcal{A}$ be standard mirroring strategy for PP, then $\mathcal{A} \rightsquigarrow \alpha_{PP}$

Theory

Let $\mathcal{A}$ be a proof of $(\Gamma \vdash [\alpha]P)$ and let $\mathcal{A} \rightsquigarrow \boldsymbol{\alpha}$. [1]

Theorem (Systemhood)

$\boldsymbol{\alpha}$ is a system, i.e., it does not contain dualities.

Theorem (Reification transfer)

$\Gamma \vdash [\boldsymbol{\alpha}]P$ is provable.

Theorem (Reification refinement)

$\Gamma \vdash \boldsymbol{\alpha} \leq_{[]} \alpha$ is provable.

[1] Recursively assume Γ free of duals β^d

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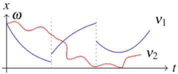
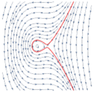
Hybrid



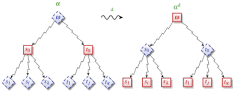


Games

$\langle\alpha\rangle$safe

$[\alpha]$safe

α^d



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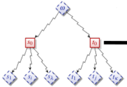

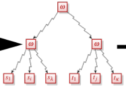

$\Gamma \vdash [\text{game}]P \quad \leq \quad \Gamma \vdash [\text{system}]P$

Strategies to Systems

Synthesis