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Pegasus: a framework for sound continuous invariant generation

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Introduction

What this talk is about

Theorem proving in cyber-physical systems (CPS).

Why? **Fully rigorous proofs of correctness**.

Important for **safety-critical** embedded systems.

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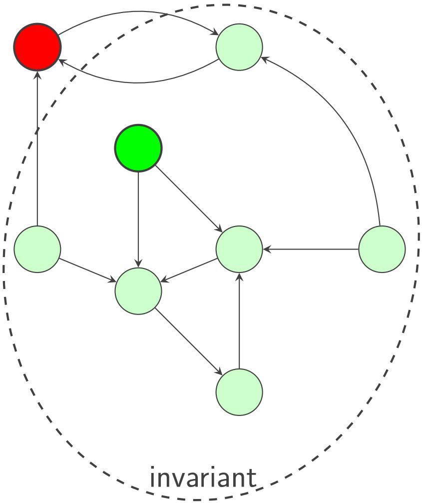
Important for **safety-critical** embedded systems.

Problem: Theorem proving in CPS is **not fully automatic**.

Safety verification relies on finding the right **invariants**.

Invariants in verification



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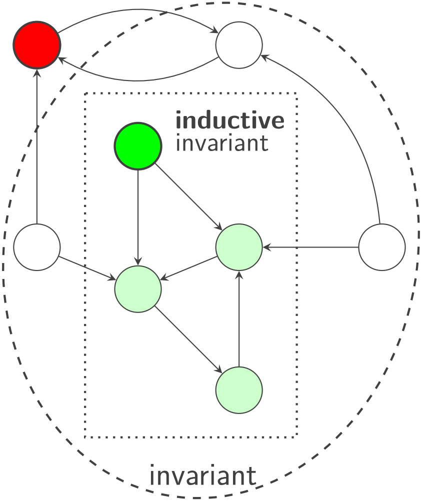


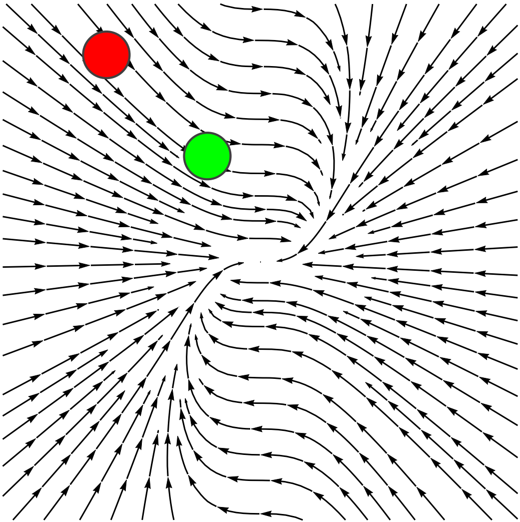
Continuous invariants

ODEs:

$\vec{x}' = f(\vec{x})$

$\vec{x} \in \mathbb{R}^n$

$\text{Init} \subseteq \mathbb{R}^n$



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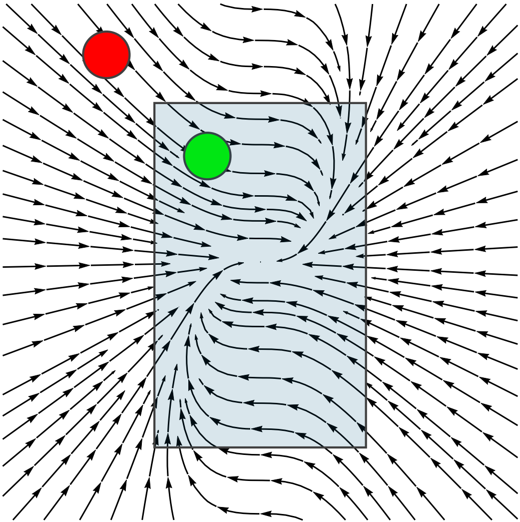

Checking continuous invariants

Checking whether a formula defines a continuous (inductive) invariant is **decidable** (Liu, Zhan & Zhao, EMSOFT 2011).

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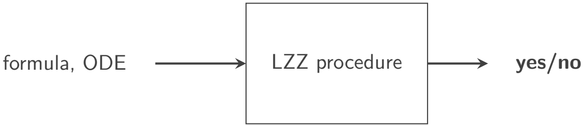

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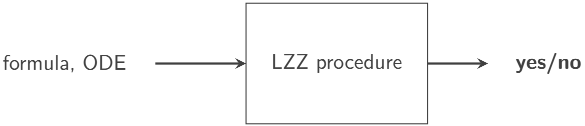

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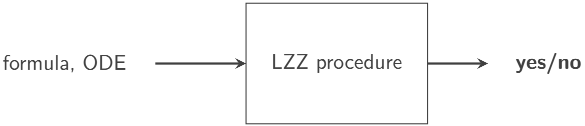

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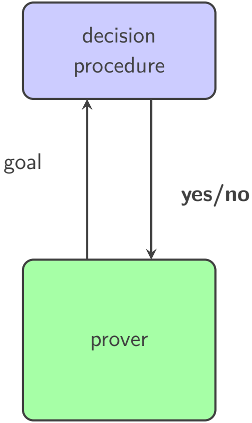
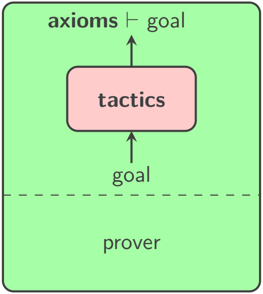
Handling decidable problems

Design choices in proof assistants



Using external oracles



Formal proof using tactics

Handling invariants

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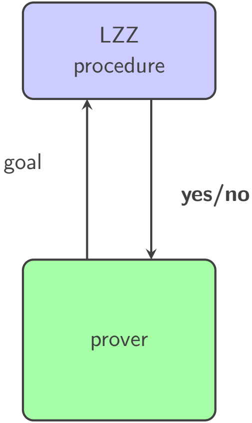
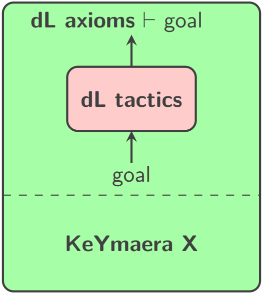


“PVS-style”



LCF-style

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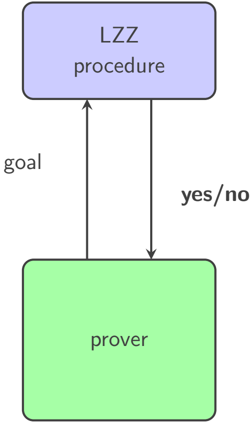
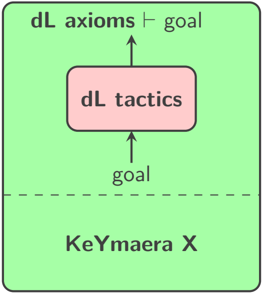


"PVS-style"

Less soundness-critical code



LCF-style

Generating continuous invariants

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Excellent progress made this decade on the **invariant checking problem**.

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Practical bottleneck for proof automation.

Generating continuous invariants

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In theory, we can search for invariants using **template formulas**:

$$a_0 + a_1 x + a_2 y + a_3 x^2 + a_4 xy + a_5 y^2 < 0 \wedge b_0 + b_1 x + b_2 y \geq 0$$

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More practical alternatives are needed.

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More practical methods for invariant generation exist.

These are

- more specialized,
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Challenge:

- build a **system for navigating this spectrum**,
- use it to improve proof automation in KeYmaera X.

Continuous invariant generator

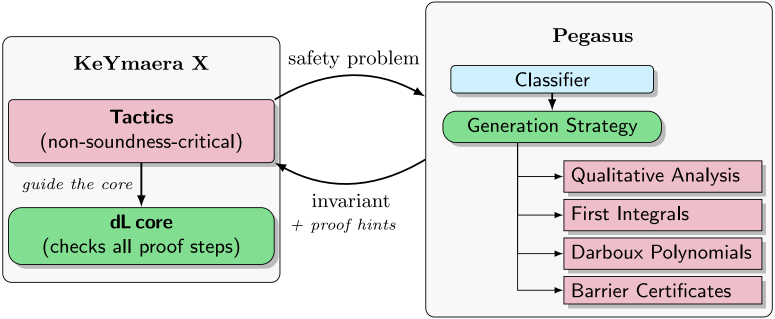
Pegasus is an automatic continuous invariant generator.

http://pegasus.keymaeraX.org

As of version 1.0, Pegasus (implemented in Wolfram Language) has

- a simple continuous safety verification problem classifier,
- implementation of invariant generation methods,
- a strategy for combining invariant generation methods,
- proof hints for KeYmaera X.

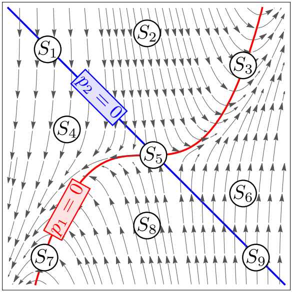
Sound integration architecture

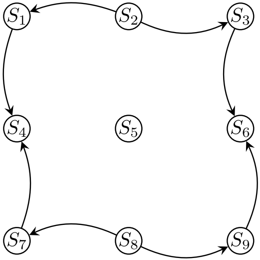


Discrete abstraction

Partition $\mathbb{R}^n$ into discrete states $S_1, \ldots, S_k$ defined by some predicates.

Compute the discrete transition relation.



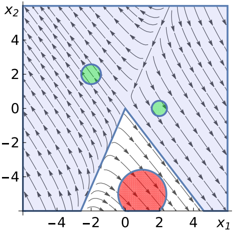


Qualitative analysis

In essence: discrete abstraction using information in the problem.

Some sources of predicates:

- right-hand sides of ODEs, their factors, etc.
- functions defining the pre/postcondition
- physically meaningful quantities (e.g. divergence of the vector field)



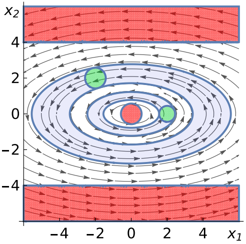
First integrals

and Darboux polynomials

Conserved quantities in the continuous system.

Functions p such that $p' = 0$ (i.e. the rate of change of p w.r.t. f is 0).

Searching for **polynomial** first integrals (of bounded degree) can be done using linear algebra.



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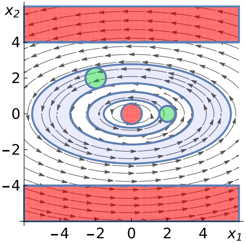
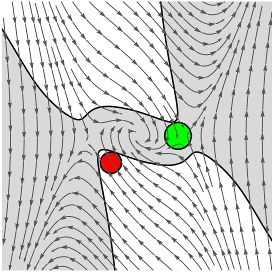


Darboux polynomials: $p' = \alpha p$, where α is a polynomial.

Barrier certificates

Main idea: find a continuous invariant $p \leq 0$ using

- **differential inequalities**, e.g. $p' \leq 0$, $p' \leq \lambda p$ ($\lambda \in \mathbb{R}$), and
- **sum-of-squares decomposition** (via semidefinite programming).

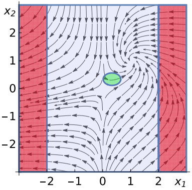


First described by Prajna and Jadbabaie (HSCC 2004).
Generalizes to **vector barrier certificates** (our work, FM 2018).

Differential saturation

A strategy for combining invariant generation methods.

Iteratively refine the invariant using available methods.



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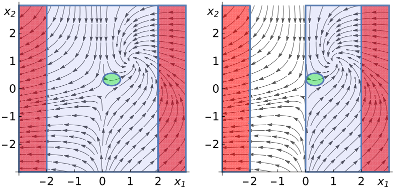

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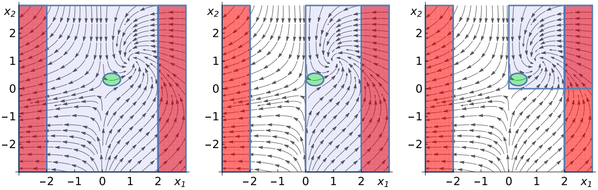

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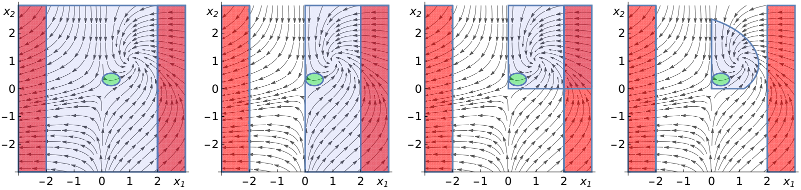
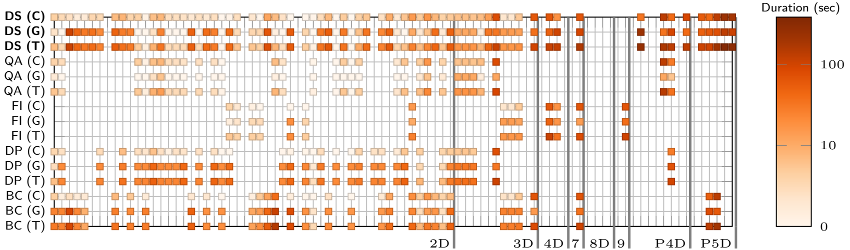


- Refinement 1 (using a Darboux polynomial)
- Refinement 2 (using Qualitative analysis)
- Refinement 3 (using a barrier certificate)

Some results

Non-linear systems

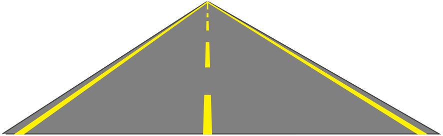
- 90 benchmark safety verification problems from the literature.
- 71 problem could be solved by the combined strategy.



Non-linear problems (dimension: 2D-9D, followed by 4D and 5D product systems)

- A few problems were **only** solved by the combined strategy (no individual method succeeded by itself).

Conclusion & future outlook



The results we observe are thus far very encouraging.

- Many more invariant generation methods to implement.
- Generation strategies that work solely in tractable theories.
- Larger corpus of continuous verification problems needed.

Goal: to make hybrid systems theorem proving more or less **automatic**.

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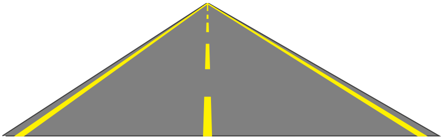

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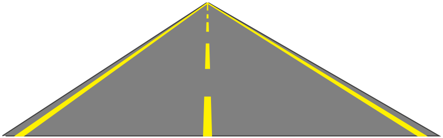

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