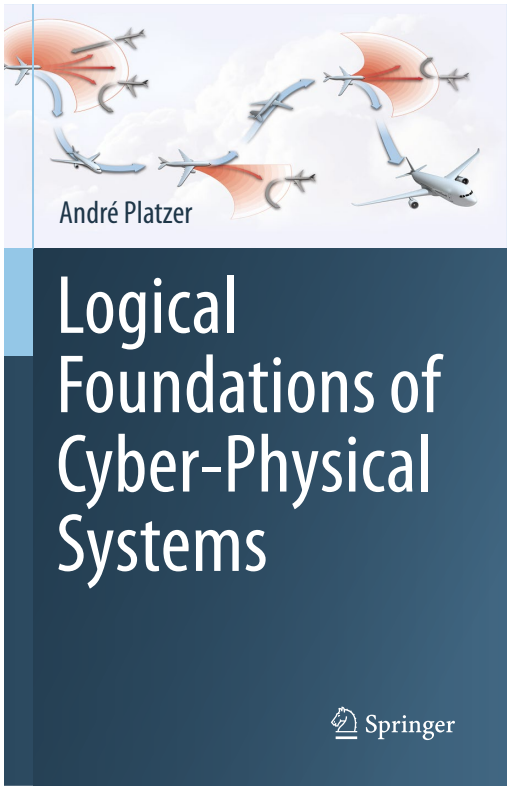
17: Game Proofs & Separations

Logical Foundations of Cyber-Physical Systems



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Department of Informatics

Computer Science Department
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Outline

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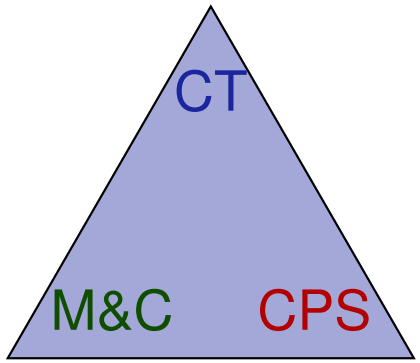
rigorous reasoning for adversarial dynamics
miracle of soundness
separations
axiomatization of dGL
multi-dynamical systems
differential game invariants



differential games
systems vs. games

CPS semantics
multi-scale feedback

Differential Game Logic: Syntax

Definition (Hybrid game α)

$$\alpha, \beta \ ::= \ x := e \mid ?Q \mid x' = f(x) \,\&\, Q \mid \alpha \cup \beta \mid \alpha; \beta \mid \alpha^* \mid \alpha^{\mathrm{d}}$$

Definition (dGL Formula P)

$$P, Q \ ::= \ e \geq \tilde{e} \mid \neg P \mid P \wedge Q \mid \forall x\, P \mid \exists x\, P \mid \langle \alpha \rangle P \mid [\alpha] P$$

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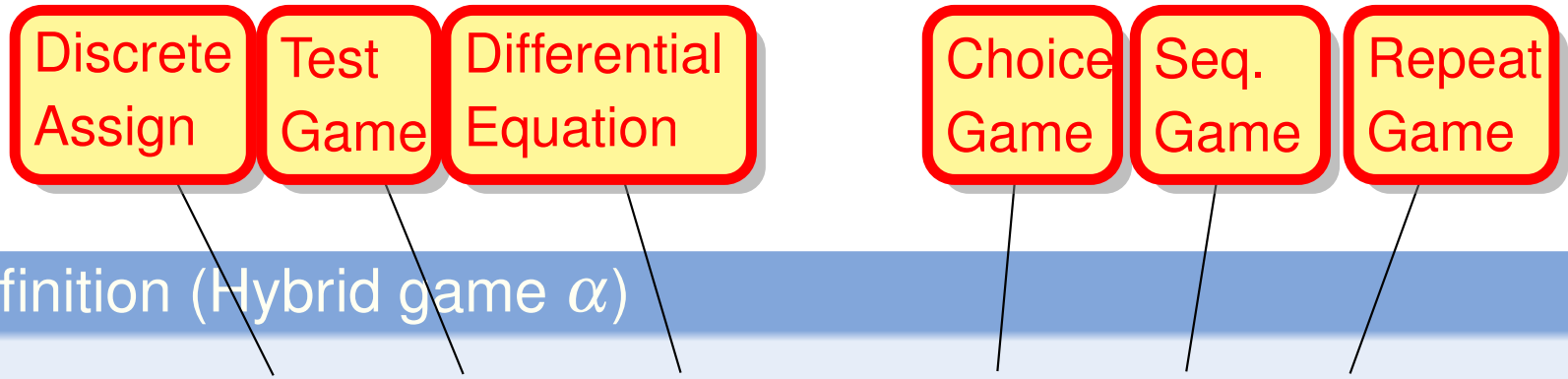

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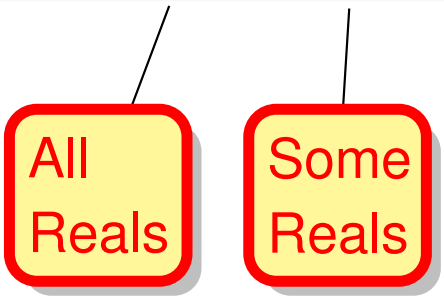

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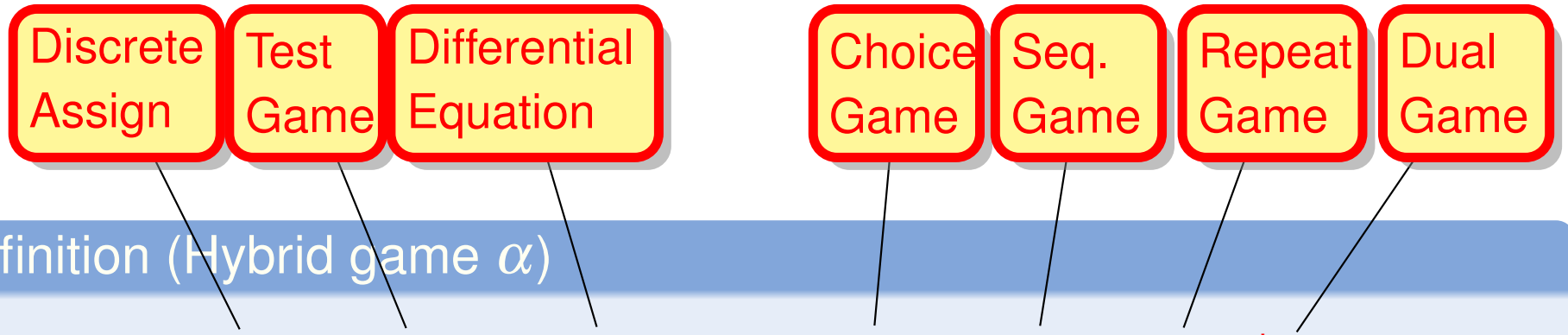

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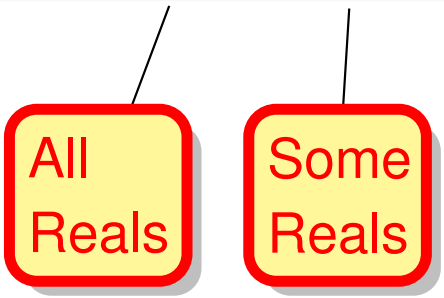

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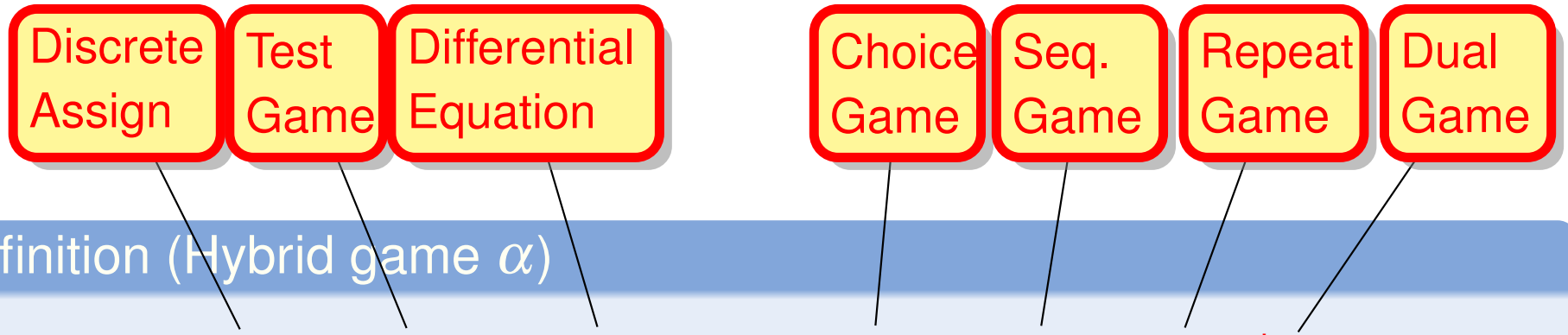

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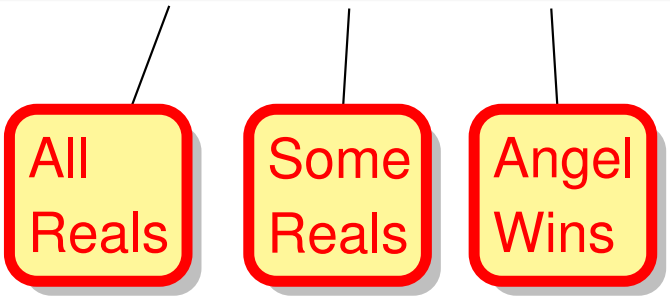

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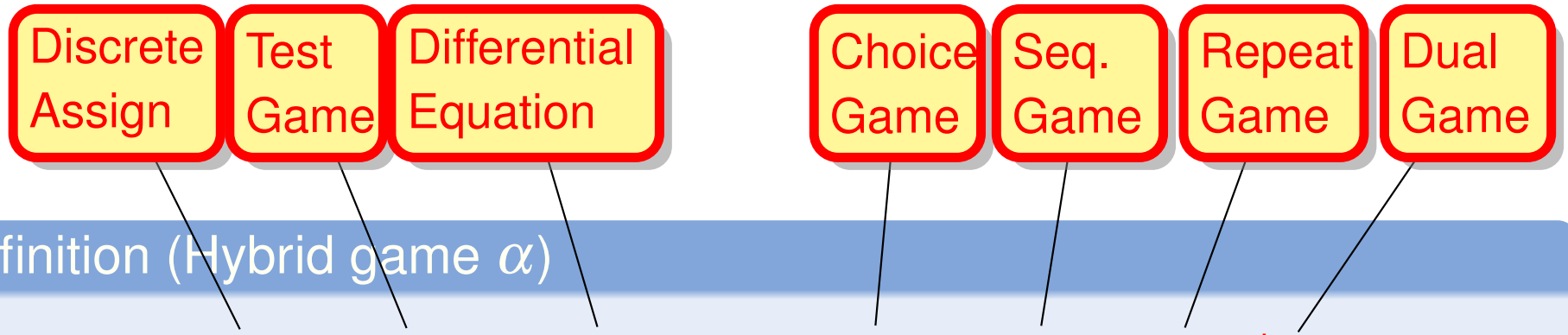

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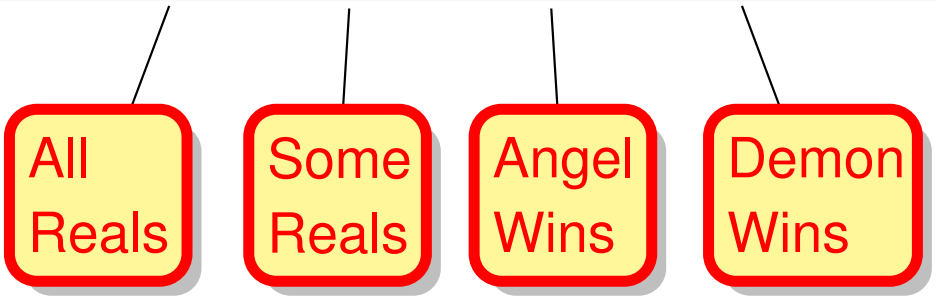

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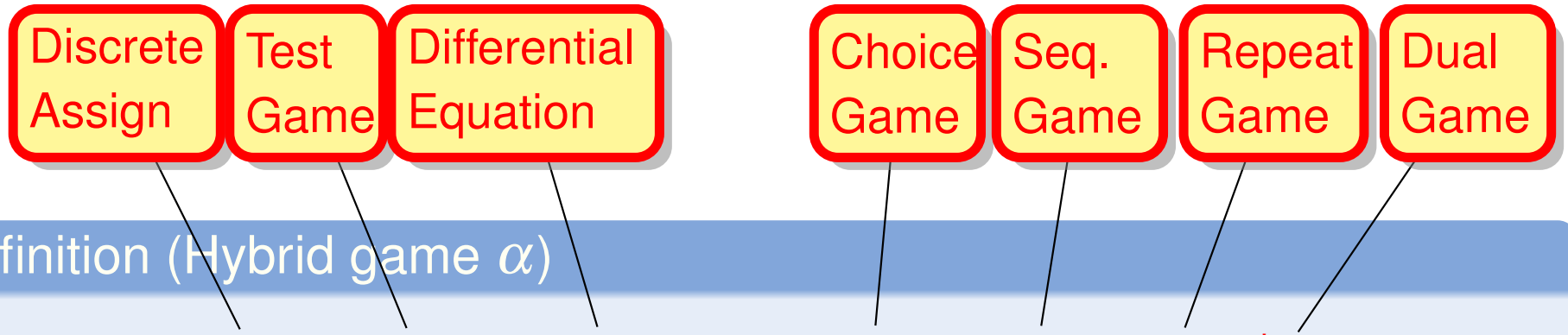

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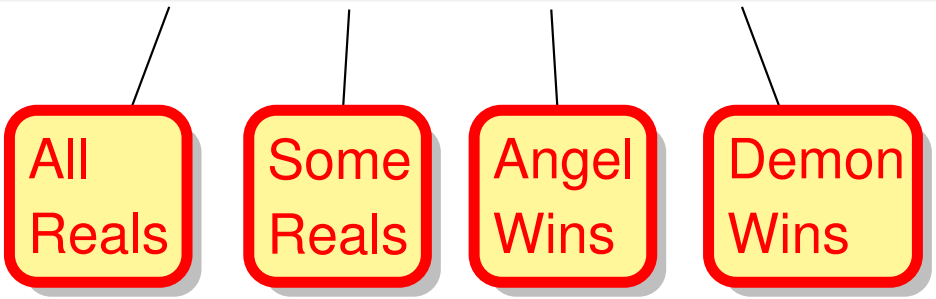
"Angel has Wings $\langle \alpha \rangle$"



Differential Game Logic: Denotational Semantics

Definition (Hybrid game α) $[\![\cdot]\!] : \mathrm{HG} \to (\wp(\mathscr{S}) \to \wp(\mathscr{S}))$

$$
\begin{aligned}
\varsigma_{x:=e}(X) &= \{\omega \in \mathscr{S} : \omega_x^{\omega[\![e]\!]} \in X\} \\
\varsigma_{x'=f(x)}(X) &= \{\varphi(0) \in \mathscr{S} : \varphi(r) \in X \text{ for some } \varphi{:}[0,r] \to \mathscr{S},\ \varphi \models x' = f(x)\} \\
\varsigma_{?Q}(X) &= [\![Q]\!] \cap X \\
\varsigma_{\alpha \cup \beta}(X) &= \varsigma_\alpha(X) \cup \varsigma_\beta(X) \\
\varsigma_{\alpha;\beta}(X) &= \varsigma_\alpha(\varsigma_\beta(X)) \\
\varsigma_{\alpha^*}(X) &= \bigcap\{Z \subseteq \mathscr{S} : X \cup \varsigma_\alpha(Z) \subseteq Z\} \\
\varsigma_{\alpha^d}(X) &= (\varsigma_\alpha(X^\complement))^\complement
\end{aligned}
$$

Definition (dGL Formula P) $[\![\cdot]\!] : \mathrm{Fml} \to \wp(\mathscr{S})$

$$
\begin{aligned}
[\![e_1 \geq e_2]\!] &= \{\omega \in \mathscr{S} : \omega[\![e_1]\!] \geq \omega[\![e_2]\!]\} \\
[\![\neg P]\!] &= ([\![P]\!])^\complement \\
[\![P \wedge Q]\!] &= [\![P]\!] \cap [\![Q]\!] \\
[\![\langle\alpha\rangle P]\!] &= \varsigma_\alpha([\![P]\!]) \\
[\![[\alpha]P]\!] &= \delta_\alpha([\![P]\!])
\end{aligned}
$$

Differential Game Logic: Axiomatization

$$[\cdot]\quad [\alpha]P \leftrightarrow \neg\langle\alpha\rangle\neg P$$

$$\langle:=\rangle\quad \langle x:=e\rangle p(x) \leftrightarrow p(e)$$

$$\langle'\rangle\quad \langle x'=f(x)\rangle P \leftrightarrow \exists t{\geq}0\, \langle x:=y(t)\rangle P$$

$$\langle ?\rangle\quad \langle ?Q\rangle P \leftrightarrow (Q\wedge P)$$

$$\langle\cup\rangle\quad \langle\alpha\cup\beta\rangle P \leftrightarrow \langle\alpha\rangle P\vee\langle\beta\rangle P$$

$$\langle;\rangle\quad \langle\alpha;\beta\rangle P \leftrightarrow \langle\alpha\rangle\langle\beta\rangle P$$

$$\langle^*\rangle\quad \langle\alpha^*\rangle P \leftrightarrow P\vee\langle\alpha\rangle\langle\alpha^*\rangle P$$

$$\langle^d\rangle\quad \langle\alpha^{\mathrm{d}}\rangle P \leftrightarrow \neg\langle\alpha\rangle\neg P$$

$$\mathrm{M}\ \frac{P\to Q}{\langle\alpha\rangle P\to\langle\alpha\rangle Q}$$

$$\mathrm{FP}\ \frac{P\vee\langle\alpha\rangle Q\to Q}{\langle\alpha^*\rangle P\to Q}$$

$$\mathrm{MP}\ \frac{P\quad P\to Q}{Q}$$

$$\forall\ \frac{p\to Q}{p\to\forall x\, Q}\qquad (x\notin \mathrm{FV}(p))$$

$$\mathrm{US}\ \frac{\varphi}{\varphi^{\psi(\cdot)}_{p(\cdot)}}$$

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Soundness

Theorem (Soundness)

dGL *proof calculus is sound*

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Do we have to prove anything at all?

More Axioms

K $[\alpha](P \to Q) \to ([\alpha]P \to [\alpha]Q)$

M$_{[\cdot]}$ $\dfrac{P \to Q}{[\alpha]P \to [\alpha]Q}$

$\overleftarrow{\text{M}}$ $\langle\alpha\rangle(P \vee Q) \to \langle\alpha\rangle P \vee \langle\alpha\rangle Q$

M $\langle\alpha\rangle P \vee \langle\alpha\rangle Q \to \langle\alpha\rangle(P \vee Q)$

I $[\alpha^*]P \leftrightarrow P \wedge [\alpha^*](P \to [\alpha]P)$

ind $\dfrac{P \to [\alpha]P}{P \to [\alpha^*]P}$

B $\langle\alpha\rangle\exists x\, P \to \exists x \langle\alpha\rangle P$ $(x \notin \alpha)$

$\overleftarrow{\text{B}}$ $\exists x \langle\alpha\rangle P \to \langle\alpha\rangle\exists x\, P$

G $\dfrac{P}{[\alpha]P}$

M$_{[\cdot]}$ $\dfrac{P \to Q}{[\alpha]P \to [\alpha]Q}$

R $\dfrac{P_1 \wedge P_2 \to Q}{[\alpha]P_1 \wedge [\alpha]P_2 \to [\alpha]Q}$

M$_{[\cdot]}$ $\dfrac{P_1 \wedge P_2 \to Q}{[\alpha](P_1 \wedge P_2) \to [\alpha]Q}$

FA $\langle\alpha^*\rangle P \to P \vee \langle\alpha^*\rangle(\neg P \wedge \langle\alpha\rangle P)$

$\overleftarrow{[*]}$ $[\alpha^*]P \leftrightarrow P \wedge [\alpha^*][\alpha]P$

More Axioms ???

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Separating Axioms

Theorem (Axiomatic separation: hybrid systems vs. hybrid games)

Axiomatic separation is K, I, C, B, V, G. So, dGL *is a subregular, sub-Barcan, monotonic modal logic without loop induction axioms.*

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One game's boxes are another game's diamonds.
Don't use axioms that do not belong to you!

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Theorem (Soundness)

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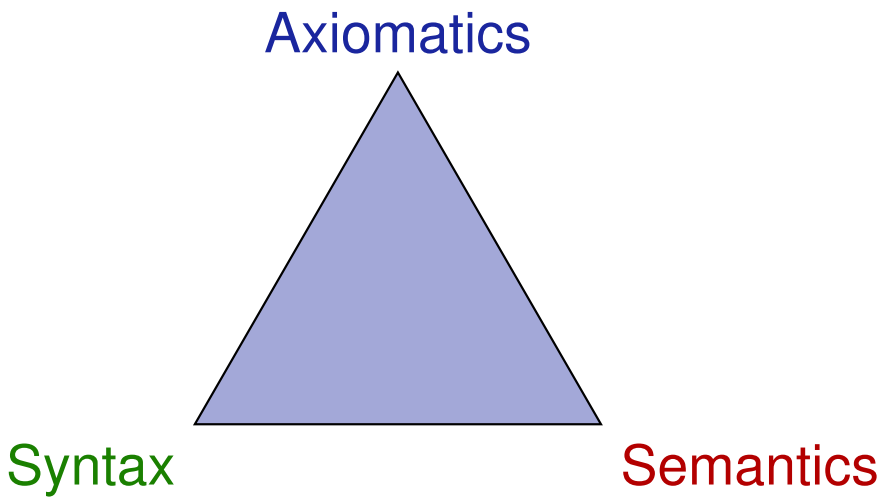

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Proof.

$$\langle\cup\rangle\ \langle\alpha\cup\beta\rangle P \leftrightarrow \langle\alpha\rangle P \vee \langle\beta\rangle P$$

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□

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$\langle;\rangle$ $\langle\alpha;\beta\rangle P \leftrightarrow \langle\alpha\rangle\langle\beta\rangle P$

$[\cdot]$ is sound by determinacy

$[\cdot]$ $[\alpha]P \leftrightarrow \neg\langle\alpha\rangle\neg P$

M Assume the premise $P\to Q$ is valid, i.e., $[\![P]\!]\subseteq[\![Q]\!]$.
Then the conclusion $\langle\alpha\rangle P\to\langle\alpha\rangle Q$ is valid, i.e.,
$[\![\langle\alpha\rangle P]\!] = \varsigma_\alpha([\![P]\!])\subseteq\varsigma_\alpha([\![Q]\!]) = [\![\langle\alpha\rangle Q]\!]$ by monotonicity.

$$\text{M}\ \frac{P\to Q}{\langle\alpha\rangle P\to\langle\alpha\rangle Q}$$

□

The Miracle of Soundness

Soundness links semantics and axiomatics in perfect unison!

Compositional Soundness

- Soundness: If P provable then P valid — $\vdash P$ implies $\models P$
- *Conditio sine qua non* for logic
- *Every* formula that it proves with *any* proof has to be valid.
- Fortunately, proofs are composed from axioms by proof rules.

Sufficient:

1. All axioms are sound: valid formulas.
2. All proof rules are sound: take valid premises to valid conclusions.

Then

- Proof is a long combination of many simple arguments.
- Each individual step is a sound axiom or sound proof rule, so sound.

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Soundness+Completeness links semantics and axiomatics in perfect unison!

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Theorem (Completeness)

dGL *calculus is a sound & complete axiomatization of hybrid games relative to any (differentially) expressive*[1] *logic L.*

$$\models \varphi \quad \textit{iff} \quad L \vdash \varphi$$

[1] $\forall \varphi \in \text{dGL}\ \exists \varphi^\flat \in L \quad \models \varphi \leftrightarrow \varphi^\flat$
$\langle x' = f(x) \rangle G \leftrightarrow (\langle x' = f(x) \rangle G)^\flat$ provable for $G \in L$

Soundness & Completeness: Consequences

Corollary (Constructive)

Constructive and Moschovakis-coding-free. (Minimal: $x' = f(x)$, $\exists$, $[\alpha^*]$)

Corollary (Characterization of hybrid game challenges)

- $[\alpha^*]G$: *Succinct invariants* — *discrete* Π^0_2
- $[x' = f(x)]G$ *and* $\langle x' = f(x)\rangle G$: *Succinct differential (in)variants* — Δ^1_1
- $\exists x\, G$: *Complexity depends on Herbrand disjunctions:* — *discrete* Π^1_1

  ✓ *uninterpreted* ✓ *reals* × $\exists x\,[\alpha^*]G$ Π^1_1*-complete for discrete* α

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  ✓ *uninterpreted* ✓ *reals* × $\exists x\,[\alpha^*]G$ Π^1_1*-complete for discrete* α

set is Π^0_n iff it's $\{x : \forall y_1\, \exists y_2\, \forall y_3 \ldots y_n\, \varphi(x, y_1, \ldots, y_n)\}$ for a decidable φ
set is Σ^0_n iff it's $\{x : \exists y_1\, \forall y_2\, \exists y_3 \ldots y_n\, \varphi(x, y_1, \ldots, y_n)\}$ for a decidable φ
set is Π^1_1 iff it's $\{x : \forall f\, \exists y\, \varphi(x, y, f)\}$ for a decidable φ and functions f
set is Σ^1_1 iff it's $\{x : \exists f\, \forall y\, \varphi(x, y, f)\}$ for a decidable φ and functions f
$\Delta^i_n = \Sigma^i_n \cap \Pi^i_n$

Expressiveness

Theorem (Expressive Power: hybrid systems $<$ hybrid games)

dGL *for hybrid games strictly more expressive than* dL *for hybrid systems:*

$$\text{dL} < \text{dGL}$$

"$\leq$" For every dL formula φ there is a dGL formula $\tilde{\varphi}$ that is equivalent.

"$\not\geq$" Not the other way around.

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$$\text{dL} < \text{dGL}$$

"$\leq$" For every dL formula φ there is a dGL formula $\tilde{\varphi}$ that is equivalent.
Easy: same formula where Angel plays for nondeterminism.

"$\not\geq$" Not the other way around.
Hard: see proof. TOCL'15

Expressiveness

Theorem (Expressive Power: hybrid systems $<$ hybrid games)

dGL *for hybrid games strictly more expressive than* dL *for hybrid systems:*

$$\text{dL} < \text{dGL}$$

"$\leq$" For every dL formula φ there is a dGL formula $\tilde{\varphi}$ that is equivalent.
Easy: same formula where Angel plays for nondeterminism.

"$\not\geq$" Not the other way around.
Hard: see proof. TOCL'15

Corollary

Hybrid games are strictly more than hybrid systems.

$$\text{con}\ \frac{}{\Gamma \vdash \langle \alpha^* \rangle Q, \Delta}$$

$$\frac{}{\vdash x \geq 0 \rightarrow \langle (x := x - 1)^* \rangle x < 1}$$

$$\text{con}\ \frac{\Gamma \vdash \exists v\, p(v), \Delta \quad \vdash \forall v{>}0\,(p(v) \to \langle\alpha\rangle p(v-1)) \quad \exists v{\leq}0\, p(v) \vdash Q}{\Gamma \vdash \langle\alpha^*\rangle Q, \Delta}$$

$$\vdash x \geq 0 \to \langle (x := x-1)^* \rangle x < 1$$

$$\text{con}\ \frac{\Gamma \vdash \exists v\, p(v), \Delta \quad \vdash \forall v{>}0\,(p(v) \to \langle\alpha\rangle p(v-1)) \quad \exists v{\leq}0\, p(v) \vdash Q}{\Gamma \vdash \langle\alpha^*\rangle Q, \Delta}(v \notin \alpha)$$

$$\vdash x \geq 0 \to \langle (x := x-1)^* \rangle x < 1$$

$$\text{con}\ \frac{\Gamma \vdash \exists v\, p(v), \Delta \quad \vdash \forall v{>}0\,(p(v) \to \langle\alpha\rangle p(v-1)) \quad \exists v{\leq}0\, p(v) \vdash Q}{\Gamma \vdash \langle\alpha^*\rangle Q, \Delta}(v \notin \alpha)$$

$$\to\text{R}\ \frac{}{\dfrac{x \geq 0 \vdash \langle (x := x-1)^* \rangle x < 1}{\vdash x \geq 0 \to \langle (x := x-1)^* \rangle x < 1}}$$

$$\text{con}\ \frac{\Gamma \vdash \exists v\, p(v), \Delta \quad \vdash \forall v{>}0\,(p(v) \to \langle\alpha\rangle p(v-1)) \quad \exists v{\leq}0\, p(v) \vdash Q}{\Gamma \vdash \langle\alpha^*\rangle Q, \Delta}(v \notin \alpha)$$

$$\text{con}\ \frac{\dfrac{\overline{x{\geq}0 \vdash \exists n\, x{<}n{+}1} \quad \overline{x{<}n{+}1 \wedge n{>}0 \vdash \langle x := x{-}1\rangle x{<}n{-}1{+}1} \quad \overline{\exists n{\leq}0\, x{<}n{+}1 \vdash x{<}1}}{x \geq 0 \vdash \langle (x := x-1)^*\rangle x < 1}}{\to\text{R}\ \vdash x \geq 0 \to \langle (x := x-1)^*\rangle x < 1}$$

$p(n) \equiv x < n+1$

$$\text{con}\ \frac{\Gamma \vdash \exists v\, p(v), \Delta \quad \vdash \forall v{>}0\,(p(v) \to \langle\alpha\rangle p(v-1)) \quad \exists v{\leq}0\, p(v) \vdash Q}{\Gamma \vdash \langle\alpha^*\rangle Q, \Delta}(v \notin \alpha)$$

$$\to\text{R}\,\frac{\text{con}\,\dfrac{\mathbb{R}\,\dfrac{*}{x{\geq}0 \vdash \exists n\, x{<}n{+}1} \quad \dfrac{}{x{<}n{+}1 \wedge n{>}0 \vdash \langle x := x{-}1\rangle x{<}n{-}1{+}1} \quad \dfrac{}{\exists n{\leq}0\, x{<}n{+}1 \vdash x{<}1}}{x \geq 0 \vdash \langle (x := x-1)^*\rangle x < 1}}{\vdash x \geq 0 \to \langle (x := x-1)^*\rangle x < 1}$$

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$$\text{con}\ \frac{\Gamma \vdash \exists v\, p(v), \Delta \quad \vdash \forall v{>}0\,(p(v) \to \langle\alpha\rangle p(v-1)) \quad \exists v{\leq}0\, p(v) \vdash Q}{\Gamma \vdash \langle\alpha^*\rangle Q, \Delta}(v \notin \alpha)$$

$$\text{con}\ \frac{\mathbb{R}\ \dfrac{*}{x{\geq}0 \vdash \exists n\, x{<}n{+}1} \langle{:=}\rangle \dfrac{\dfrac{}{x{<}n{+}1 \wedge n{>}0 \vdash x{-}1{<}n{-}1{+}1}}{x{<}n{+}1 \wedge n{>}0 \vdash \langle x := x{-}1\rangle x{<}n{-}1{+}1} \quad \dfrac{}{\exists n{\leq}0\, x{<}n{+}1 \vdash x{<}1}}{x \geq 0 \vdash \langle (x := x-1)^*\rangle x < 1}$$
$$\to\text{R}\ \frac{}{\vdash x \geq 0 \to \langle (x := x-1)^*\rangle x < 1}$$

$p(n) \equiv x < n+1$

$$\text{con}\ \frac{\Gamma \vdash \exists v\, p(v), \Delta \quad \vdash \forall v{>}0\,(p(v) \to \langle\alpha\rangle p(v-1)) \quad \exists v{\leq}0\, p(v) \vdash Q}{\Gamma \vdash \langle\alpha^*\rangle Q, \Delta}(v \notin \alpha)$$

$$\to\text{R}\,\frac{\text{con}\,\dfrac{\mathbb{R}\,\dfrac{*}{x{\geq}0 \vdash \exists n\, x{<}n{+}1}\langle{:=}\rangle\dfrac{\mathbb{R}\,\dfrac{*}{x{<}n{+}1 \wedge n{>}0 \vdash x{-}1{<}n{-}1{+}1}}{x{<}n{+}1 \wedge n{>}0 \vdash \langle x := x{-}1\rangle x{<}n{-}1{+}1} \quad \dfrac{}{\exists n{\leq}0\, x{<}n{+}1 \vdash x{<}1}}{x \geq 0 \vdash \langle (x := x-1)^*\rangle x < 1}}{\vdash x \geq 0 \to \langle (x := x-1)^*\rangle x < 1}$$

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$$\text{con}\frac{\mathbb{R}\dfrac{*}{x{\geq}0 \vdash \exists n\, x{<}n{+}1}\langle{:=}\rangle\dfrac{\mathbb{R}\dfrac{*}{x{<}n{+}1 \wedge n{>}0 \vdash x{-}1{<}n{-}1{+}1}}{x{<}n{+}1 \wedge n{>}0 \vdash \langle x := x{-}1\rangle x{<}n{-}1{+}1}\ \mathbb{R}\dfrac{*}{\exists n{\leq}0\, x{<}n{+}1 \vdash x{<}1}}{x \geq 0 \vdash \langle (x := x-1)^*\rangle x < 1}$$
$$\to\text{R}\frac{}{\vdash x \geq 0 \to \langle (x := x-1)^*\rangle x < 1}$$

$p(n) \equiv x < n+1$

Example Proof: 2-Nim-type Game

$$x \geq 0 \rightarrow \langle \underbrace{(\underbrace{x := x-1}_{\beta} \cap \underbrace{x := x-2}_{\gamma})^*}_{\alpha} \rangle 0 \leq x < 2$$

- Fixpoint style proof technique

$$\frac{}{x \geq 0 \rightarrow \langle \alpha^* \rangle 0 \leq x < 2} \langle^*\rangle, \forall, \text{MP}$$

Example Proof: 2-Nim-type Game

$$x \geq 0 \rightarrow \langle (\underbrace{\underbrace{x := x-1}_{\beta} \cap \underbrace{x := x-2}_{\gamma}}_{\alpha})^* \rangle 0 \leq x < 2$$

- Fixpoint style proof technique

$$\text{US}\ \frac{}{\forall x\,(0{\leq}x{<}2 \vee \langle\alpha\rangle\langle\alpha^*\rangle 0{\leq}x{<}2 \rightarrow \langle\alpha^*\rangle 0{\leq}x{<}2) \rightarrow (x \geq 0 \rightarrow \langle\alpha^*\rangle 0{\leq}x{<}2)}$$

$$\langle^*\rangle,\forall,\text{MP}\ \frac{}{x \geq 0 \rightarrow \langle\alpha^*\rangle 0{\leq}x{<}2}$$

Example Proof: 2-Nim-type Game

$$x \geq 0 \rightarrow \langle \underbrace{(\underbrace{x := x-1}_{\beta} \cap \underbrace{x := x-2}_{\gamma})^*}_{\alpha} \rangle 0 \leq x < 2$$

- Fixpoint style proof technique

$$\frac{\forall x\,(0{\leq}x{<}2 \vee \langle\alpha\rangle p(x) \rightarrow p(x)) \rightarrow (x \geq 0 \rightarrow p(x))}{\langle\cup\rangle,\langle^d\rangle}$$

$$\frac{\forall x\,(0{\leq}x{<}2 \vee \langle\alpha\rangle\langle\alpha^*\rangle 0{\leq}x{<}2 \rightarrow \langle\alpha^*\rangle 0{\leq}x{<}2) \rightarrow (x \geq 0 \rightarrow \langle\alpha^*\rangle 0{\leq}x{<}2)}{\text{US}}$$

$$\frac{x \geq 0 \rightarrow \langle\alpha^*\rangle 0{\leq}x{<}2}{\langle^*\rangle,\forall,\text{MP}}$$

Example Proof: 2-Nim-type Game

$$x \geq 0 \rightarrow \langle (\underbrace{\underbrace{x := x-1}_{\beta} \cap \underbrace{x := x-2}_{\gamma}}_{\alpha})^* \rangle 0 \leq x < 2$$

- Fixpoint style proof technique

$$\langle := \rangle \quad \frac{}{\forall x\,(0 \leq x < 2 \vee \langle \beta \rangle p(x) \wedge \langle \gamma \rangle p(x) \rightarrow p(x)) \rightarrow (x \geq 0 \rightarrow p(x))}$$

$$\langle \cup \rangle, \langle ^d \rangle \quad \frac{}{\forall x\,(0 \leq x < 2 \vee \langle \alpha \rangle p(x) \rightarrow p(x)) \rightarrow (x \geq 0 \rightarrow p(x))}$$

$$\text{US} \quad \frac{}{\forall x\,(0 \leq x < 2 \vee \langle \alpha \rangle \langle \alpha^* \rangle 0 \leq x < 2 \rightarrow \langle \alpha^* \rangle 0 \leq x < 2) \rightarrow (x \geq 0 \rightarrow \langle \alpha^* \rangle 0 \leq x < 2)}$$

$$\langle ^* \rangle, \forall, \text{MP} \quad \frac{}{x \geq 0 \rightarrow \langle \alpha^* \rangle 0 \leq x < 2}$$

Example Proof: 2-Nim-type Game

$$x \geq 0 \rightarrow \langle (\underbrace{\underbrace{x := x-1}_{\beta} \cap \underbrace{x := x-2}_{\gamma}}_{\alpha})^* \rangle 0 \leq x < 2$$

- Fixpoint style proof technique

$$\mathbb{R}\ \frac{\forall x\,(0\leq x<2 \vee p(x-1) \wedge p(x-2) \rightarrow p(x)) \rightarrow (x \geq 0 \rightarrow p(x))}{\forall x\,(0\leq x<2 \vee \langle \beta \rangle p(x) \wedge \langle \gamma \rangle p(x) \rightarrow p(x)) \rightarrow (x \geq 0 \rightarrow p(x))}\ \langle := \rangle$$

$$\frac{\forall x\,(0\leq x<2 \vee \langle \beta \rangle p(x) \wedge \langle \gamma \rangle p(x) \rightarrow p(x)) \rightarrow (x \geq 0 \rightarrow p(x))}{\forall x\,(0\leq x<2 \vee \langle \alpha \rangle p(x) \rightarrow p(x)) \rightarrow (x \geq 0 \rightarrow p(x))}\ \langle \cup \rangle, \langle ^d \rangle$$

$$\text{US}\ \frac{\forall x\,(0\leq x<2 \vee \langle \alpha \rangle \langle \alpha^* \rangle 0\leq x<2 \rightarrow \langle \alpha^* \rangle 0\leq x<2) \rightarrow (x \geq 0 \rightarrow \langle \alpha^* \rangle 0\leq x<2)}{x \geq 0 \rightarrow \langle \alpha^* \rangle 0\leq x<2}\ \langle ^* \rangle, \forall, \text{MP}$$

Example Proof: 2-Nim-type Game

$$x \geq 0 \rightarrow \langle (\underbrace{\underbrace{x := x-1}_{\beta} \cap \underbrace{x := x-2}_{\gamma}}_{\alpha})^* \rangle 0 \leq x < 2$$

- Fixpoint style proof technique

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\forall x\,(0\leq x<2 \vee p(x-1)\wedge p(x-2) \rightarrow p(x)) \rightarrow (x\geq 0 \rightarrow p(x))}\,\mathbb{R}}{\forall x\,(0\leq x<2 \vee \langle\beta\rangle p(x)\wedge\langle\gamma\rangle p(x) \rightarrow p(x)) \rightarrow (x\geq 0 \rightarrow p(x))}\,\langle:=\rangle}{\forall x\,(0\leq x<2 \vee \langle\alpha\rangle p(x) \rightarrow p(x)) \rightarrow (x\geq 0 \rightarrow p(x))}\,\langle\cup\rangle,\langle^d\rangle}{\forall x\,(0\leq x<2 \vee \langle\alpha\rangle\langle\alpha^*\rangle 0\leq x<2 \rightarrow \langle\alpha^*\rangle 0\leq x<2) \rightarrow (x\geq 0 \rightarrow \langle\alpha^*\rangle 0\leq x<2)}\,\mathrm{US}}{x\geq 0 \rightarrow \langle\alpha^*\rangle 0\leq x<2}\,\langle^*\rangle,\forall,\mathrm{MP}$$

*

Example Proof: Hybrid Game

$$\underbrace{\langle(\underbrace{x:=1;x'=1^{\mathsf{d}}}_{\beta}\cup\underbrace{x:=x-1}_{\gamma})^*\rangle 0\le x<1}_{\alpha}$$

- Fixpoint style proof technique

$$\langle^*\rangle\ \frac{}{true\rightarrow\langle\alpha^*\rangle 0\le x<1}$$

Example Proof: Hybrid Game

$$\underbrace{\langle(\underbrace{x:=1;x'=1^{\mathrm{d}}}_{\beta}\cup\underbrace{x:=x-1}_{\gamma})^*\rangle 0\le x<1}_{\alpha}$$

- Fixpoint style proof technique

$$\text{US}\ \frac{\forall x\,(0\le x<1\vee\langle\alpha\rangle\langle\alpha^*\rangle 0\le x<1\to\langle\alpha^*\rangle 0\le x<1)\to(\mathit{true}\to\langle\alpha^*\rangle 0\le x<1)}{\langle^*\rangle\ \mathit{true}\to\langle\alpha^*\rangle 0\le x<1}$$

Example Proof: Hybrid Game

$$\langle \underbrace{(\underbrace{x := 1; x' = 1^{\mathrm{d}}}_{\beta} \cup \underbrace{x := x - 1}_{\gamma})^*}_{\alpha} \rangle 0 \leq x < 1$$

- Fixpoint style proof technique

$$\langle\cup\rangle \quad \frac{}{\forall x\,(0{\leq}x{<}1 \lor \langle \beta \cup \gamma \rangle p(x) \to p(x)) \to (\mathit{true} \to p(x))}$$

$$\mathrm{US} \quad \frac{}{\forall x\,(0{\leq}x{<}1 \lor \langle \alpha \rangle \langle \alpha^* \rangle 0{\leq}x{<}1 \to \langle \alpha^* \rangle 0{\leq}x{<}1) \to (\mathit{true} \to \langle \alpha^* \rangle 0{\leq}x{<}1)}$$

$$\langle^*\rangle \quad \frac{}{\mathit{true} \to \langle \alpha^* \rangle 0{\leq}x{<}1}$$

Example Proof: Hybrid Game

$$\underbrace{\langle(\underbrace{x:=1;x'=1^{\mathrm{d}}}_{\beta}\cup\underbrace{x:=x-1}_{\gamma})^*\rangle 0\le x<1}_{\alpha}$$

- Fixpoint style proof technique

$$\langle;\rangle,\langle^d\rangle\ \frac{}{\forall x\,(0\le x<1\vee\langle\beta\rangle p(x)\vee\langle\gamma\rangle p(x)\to p(x))\to(true\to p(x))}$$

$$\langle\cup\rangle\ \frac{}{\forall x\,(0\le x<1\vee\langle\beta\cup\gamma\rangle p(x)\to p(x))\to(true\to p(x))}$$

$$\mathrm{US}\ \frac{}{\forall x\,(0\le x<1\vee\langle\alpha\rangle\langle\alpha^*\rangle 0\le x<1\to\langle\alpha^*\rangle 0\le x<1)\to(true\to\langle\alpha^*\rangle 0\le x<1)}$$

$$\langle^*\rangle\ \frac{}{true\to\langle\alpha^*\rangle 0\le x<1}$$

Example Proof: Hybrid Game

$$\underbrace{\langle(\underbrace{x:=1;x'=1^{\mathrm{d}}}_{\beta}\cup\underbrace{x:=x-1}_{\gamma})^*\rangle 0\le x<1}_{\alpha}$$

- Fixpoint style proof technique

$$\frac{}{\forall x\,(0\le x<1\vee\langle x:=1\rangle\neg\langle x'=1\rangle\neg p(x)\vee p(x-1)\to p(x))\to(\mathit{true}\to p(x))}\ \langle'\rangle$$

$$\frac{\forall x\,(0\le x<1\vee\langle\beta\rangle p(x)\vee\langle\gamma\rangle p(x)\to p(x))\to(\mathit{true}\to p(x))}{\forall x\,(0\le x<1\vee\langle\beta\cup\gamma\rangle p(x)\to p(x))\to(\mathit{true}\to p(x))}\ \langle;\rangle,\langle^d\rangle$$

$$\frac{\forall x\,(0\le x<1\vee\langle\beta\cup\gamma\rangle p(x)\to p(x))\to(\mathit{true}\to p(x))}{\forall x\,(0\le x<1\vee\langle\alpha\rangle\langle\alpha^*\rangle 0\le x<1\to\langle\alpha^*\rangle 0\le x<1)\to(\mathit{true}\to\langle\alpha^*\rangle 0\le x<1)}\ \langle\cup\rangle$$

$$\frac{\forall x\,(0\le x<1\vee\langle\alpha\rangle\langle\alpha^*\rangle 0\le x<1\to\langle\alpha^*\rangle 0\le x<1)\to(\mathit{true}\to\langle\alpha^*\rangle 0\le x<1)}{\mathit{true}\to\langle\alpha^*\rangle 0\le x<1}\ \langle^*\rangle\ \text{US}$$

Example Proof: Hybrid Game

$$\underbrace{\langle(\underbrace{x:=1;x'=1^{\mathrm{d}}}_{\beta}\cup\underbrace{x:=x-1}_{\gamma})^*\rangle 0\le x<1}_{\alpha}$$

- Fixpoint style proof technique

$$\langle:=\rangle\ \frac{\forall x\,(0\le x<1\vee\langle x:=1\rangle\neg\exists t{\ge}0\,\langle x:=x+t\rangle\neg p(x)\vee p(x-1)\to p(x))\to(true\to p(x))}{}$$

$$\langle'\rangle\ \frac{\forall x\,(0\le x<1\vee\langle x:=1\rangle\neg\langle x'=1\rangle\neg p(x)\vee p(x-1)\to p(x))\to(true\to p(x))}{}$$

$$\langle;\rangle,\langle^d\rangle\ \frac{\forall x\,(0\le x<1\vee\langle\beta\rangle p(x)\vee\langle\gamma\rangle p(x)\to p(x))\to(true\to p(x))}{}$$

$$\langle\cup\rangle\ \frac{\forall x\,(0\le x<1\vee\langle\beta\cup\gamma\rangle p(x)\to p(x))\to(true\to p(x))}{}$$

$$\mathrm{US}\ \frac{\forall x\,(0\le x<1\vee\langle\alpha\rangle\langle\alpha^*\rangle 0\le x<1\to\langle\alpha^*\rangle 0\le x<1)\to(true\to\langle\alpha^*\rangle 0\le x<1)}{}$$

$$\langle^*\rangle\ \frac{}{true\to\langle\alpha^*\rangle 0\le x<1}$$

Example Proof: Hybrid Game

$$\langle \underbrace{(\underbrace{x:=1;x'=1^{\mathrm{d}}}_{\beta} \cup \underbrace{x:=x-1}_{\gamma})^*}_{\alpha} \rangle 0 \leq x < 1$$

- Fixpoint style proof technique

$$\mathbb{R}\ \frac{}{\forall x\,(0\leq x<1 \vee \forall t\geq 0\, p(1+t) \vee p(x-1) \to p(x)) \to (\mathit{true} \to p(x))}$$

$$\langle := \rangle\ \frac{}{\forall x\,(0\leq x<1 \vee \langle x:=1\rangle \neg \exists t\geq 0\, \langle x:=x+t\rangle \neg p(x) \vee p(x-1) \to p(x)) \to (\mathit{true} \to p(x))}$$

$$\langle' \rangle\ \frac{}{\forall x\,(0\leq x<1 \vee \langle x:=1\rangle \neg \langle x'=1\rangle \neg p(x) \vee p(x-1) \to p(x)) \to (\mathit{true} \to p(x))}$$

$$\langle ; \rangle, \langle ^d \rangle\ \frac{}{\forall x\,(0\leq x<1 \vee \langle \beta\rangle p(x) \vee \langle \gamma\rangle p(x) \to p(x)) \to (\mathit{true} \to p(x))}$$

$$\langle \cup \rangle\ \frac{}{\forall x\,(0\leq x<1 \vee \langle \beta \cup \gamma\rangle p(x) \to p(x)) \to (\mathit{true} \to p(x))}$$

$$\mathrm{US}\ \frac{}{\forall x\,(0\leq x<1 \vee \langle \alpha\rangle\langle \alpha^*\rangle 0\leq x<1 \to \langle \alpha^*\rangle 0\leq x<1) \to (\mathit{true} \to \langle \alpha^*\rangle 0\leq x<1)}$$

$$\langle ^* \rangle\ \frac{}{\mathit{true} \to \langle \alpha^*\rangle 0\leq x<1}$$

Example Proof: Hybrid Game

$$\langle \underbrace{(\underbrace{x:=1;x'=1^{\mathrm{d}}}_{\beta} \cup \underbrace{x:=x-1}_{\gamma})^*}_{\alpha} \rangle 0 \leq x < 1$$

- Fixpoint style proof technique

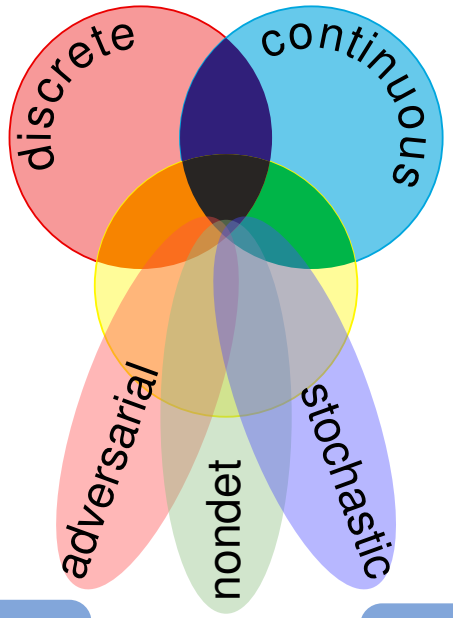
$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\forall x\,(0\leq x<1 \vee \forall t\geq 0\, p(1+t) \vee p(x-1) \to p(x)) \to (true \to p(x))}\,*}{\forall x\,(0\leq x<1 \vee \langle x:=1\rangle \neg \exists t\geq 0\, \langle x:=x+t\rangle \neg p(x) \vee p(x-1) \to p(x)) \to (true \to p(x))}\,\mathbb{R}}{\forall x\,(0\leq x<1 \vee \langle x:=1\rangle \neg \langle x'=1\rangle \neg p(x) \vee p(x-1) \to p(x)) \to (true \to p(x))}\,\langle := \rangle}{\forall x\,(0\leq x<1 \vee \langle \beta\rangle p(x) \vee \langle \gamma\rangle p(x) \to p(x)) \to (true \to p(x))}\,\langle ' \rangle}{\forall x\,(0\leq x<1 \vee \langle \beta \cup \gamma\rangle p(x) \to p(x)) \to (true \to p(x))}\,\langle ;\rangle, \langle ^d \rangle}{\forall x\,(0\leq x<1 \vee \langle \alpha\rangle\langle \alpha^*\rangle 0\leq x<1 \to \langle \alpha^*\rangle 0\leq x<1) \to (true \to \langle \alpha^*\rangle 0\leq x<1)}\,\langle \cup \rangle}{true \to \langle \alpha^*\rangle 0\leq x<1}\,\mathrm{US}\,\langle ^* \rangle$$

Outline

CPSs are Multi-Dynamical Systems

CPS Dynamics

CPS are characterized by multiple facets of dynamical systems.



CPS Compositions

CPS combines multiple simple dynamical effects.

Descriptive simplification

Tame Parts

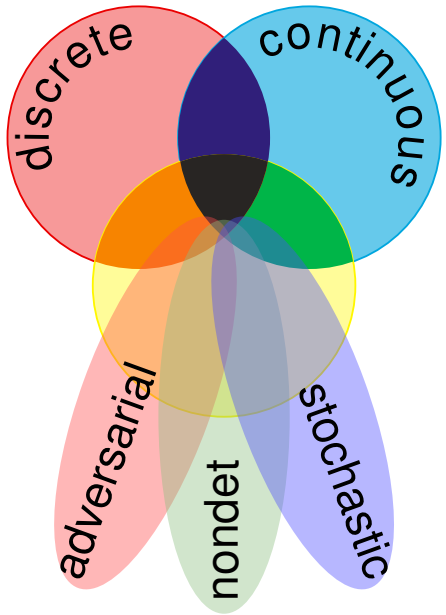
Exploiting compositionality tames CPS complexity.

Analytic simplification

CPSs are Multi-Dynamical Systems

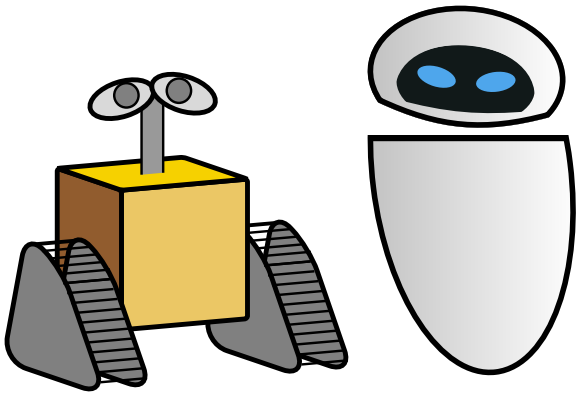
hybrid systems

HS = discrete + ODE



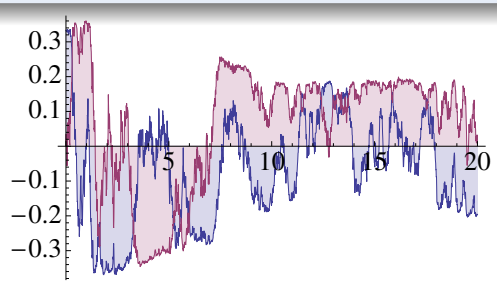
hybrid games

HG = HS + adversary



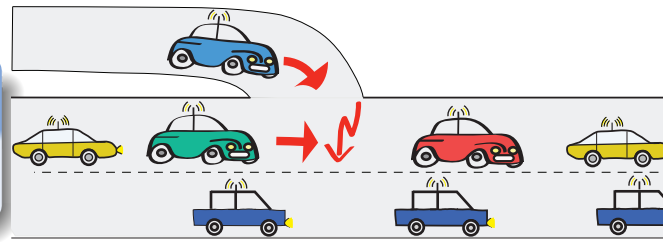
stochastic hybrid sys.

SHS = HS + stochastics

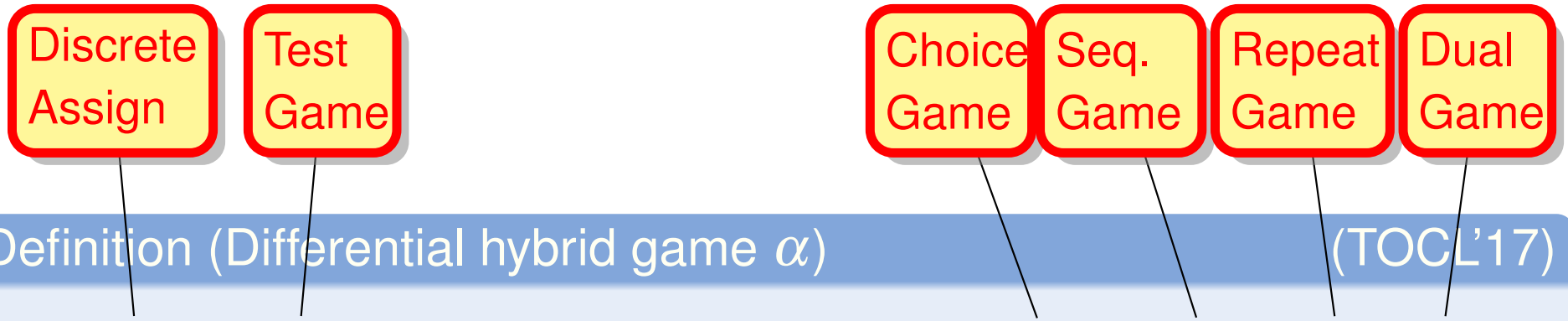


distributed hybrid sys.

DHS = HS + distributed



Differential Game Logic: Syntax

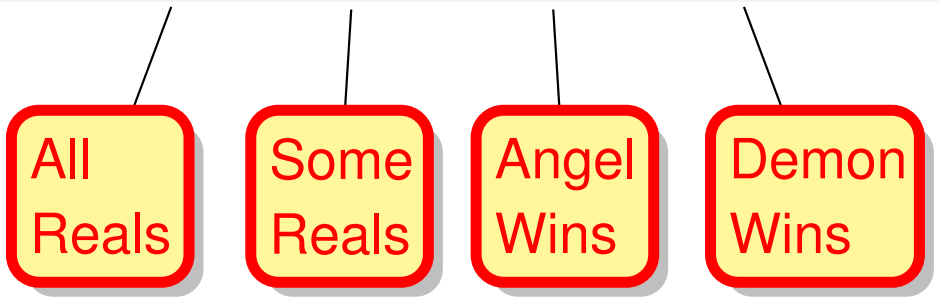


Definition (Differential hybrid game α) (TOCL'17)

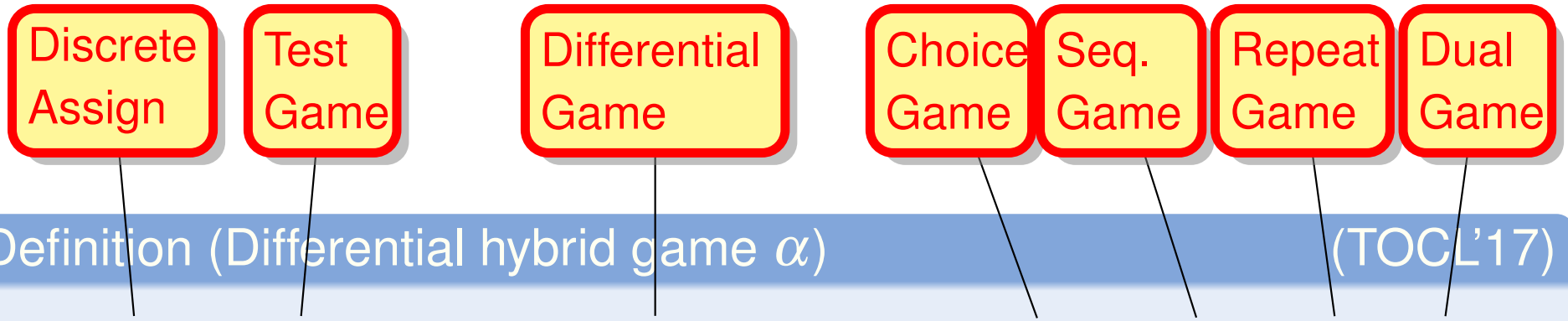
$$x := e \mid ?Q \mid x' = f(x,y,z) \,\&^{g}\, y \in Y \,\&\, z \in Z \mid \alpha \cup \beta \mid \alpha;\beta \mid \alpha^* \mid \alpha^{\mathrm{d}}$$

Definition (dGL Formula P)

$$e \geq \tilde{e} \mid \neg P \mid P \wedge Q \mid \forall x\, P \mid \exists x\, P \mid \langle \alpha \rangle P \mid [\alpha] P$$



Differential Game Logic: Syntax

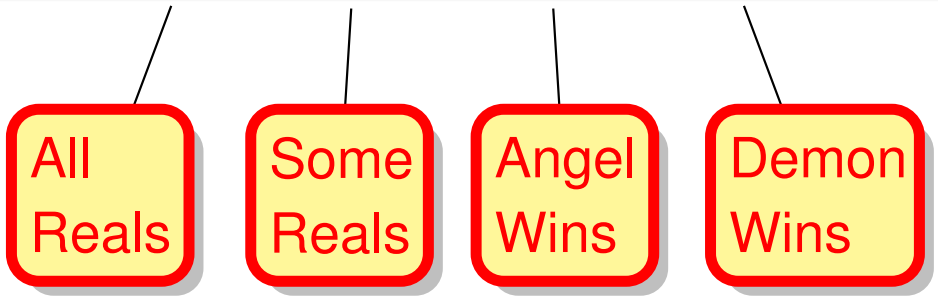


Definition (Differential hybrid game α) (TOCL'17)

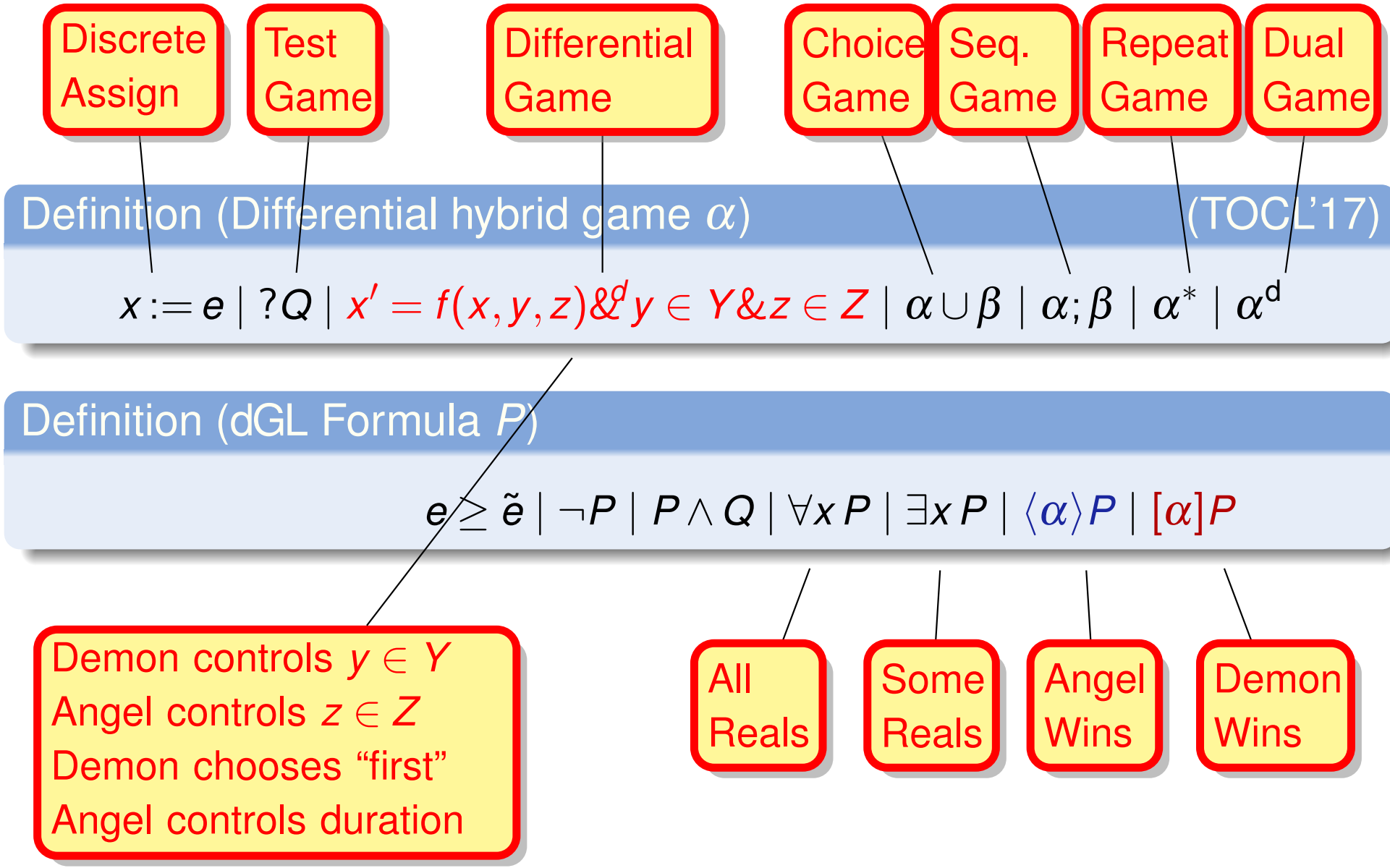
$$x := e \mid ?Q \mid x' = f(x,y,z) \,\&^{q}\, y \in Y \,\&\, z \in Z \mid \alpha \cup \beta \mid \alpha;\beta \mid \alpha^* \mid \alpha^{\mathrm{d}}$$

Definition (dGL Formula P)

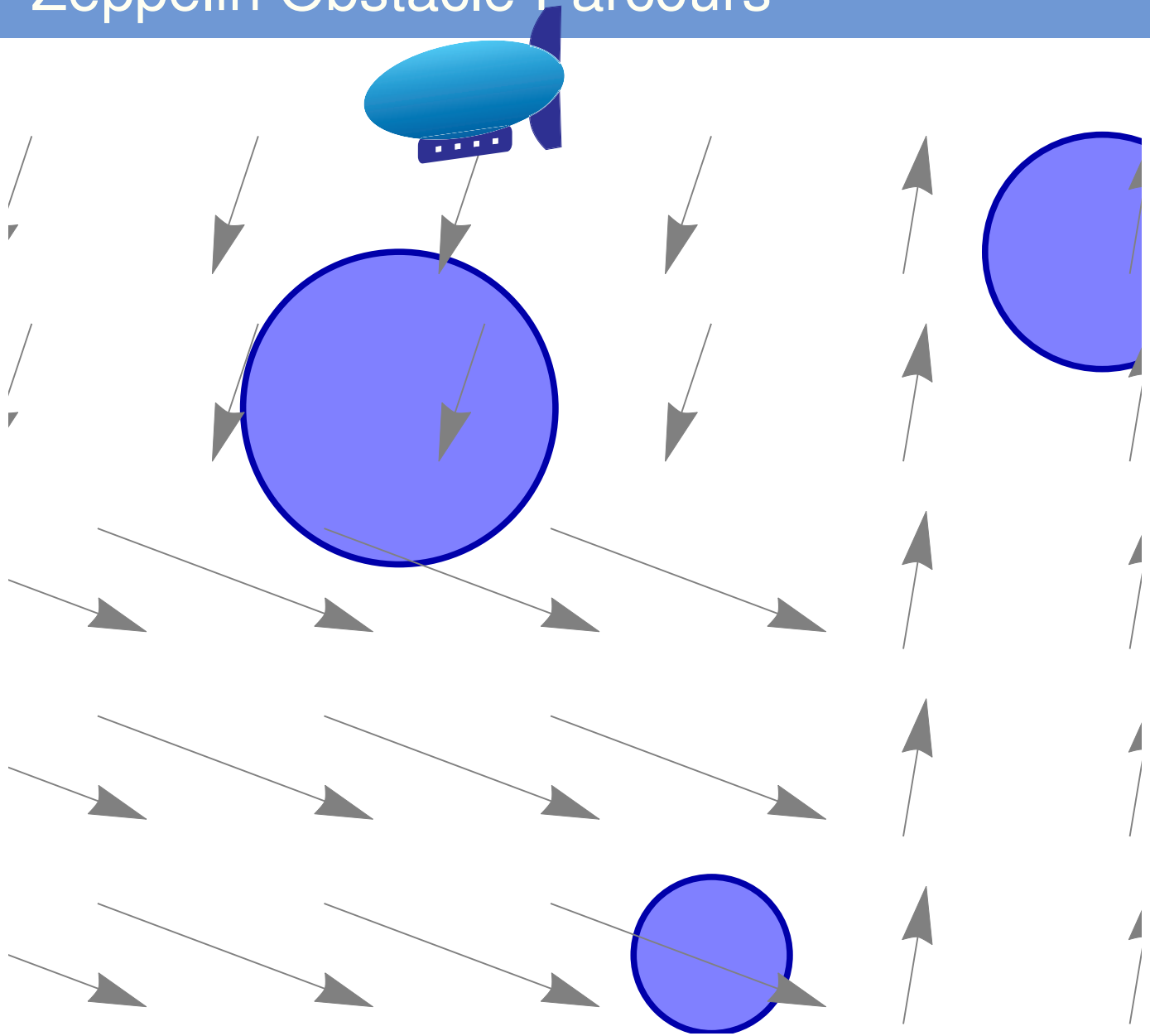
$$e \geq \tilde{e} \mid \neg P \mid P \wedge Q \mid \forall x\, P \mid \exists x\, P \mid \langle \alpha \rangle P \mid [\alpha] P$$



Differential Game Logic: Syntax

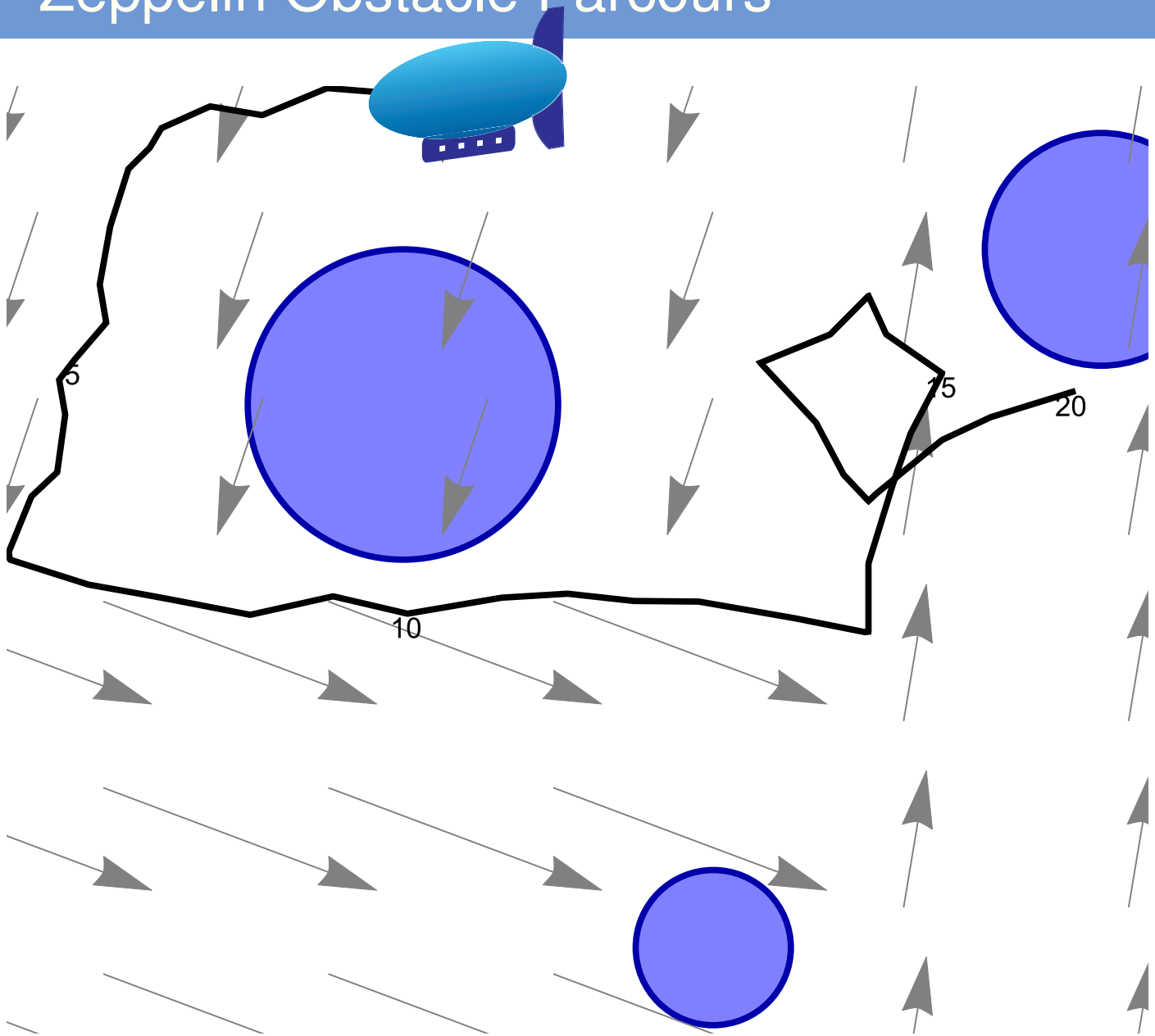


Zeppelin Obstacle Parcours



avoid obstacles
changing wind
local turbulence

Zeppelin Obstacle Parcours



avoid obstacles
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Zeppelin Obstacle Parcours

$$c > 0 \land \|x - o\|^2 \geq c^2 \rightarrow$$
$$\big[\big(v := *; o := *; c := *; ?C;$$
$$\{x' = v + py + rz \,\&^d\, y \in B \,\&\, z \in B\}$$
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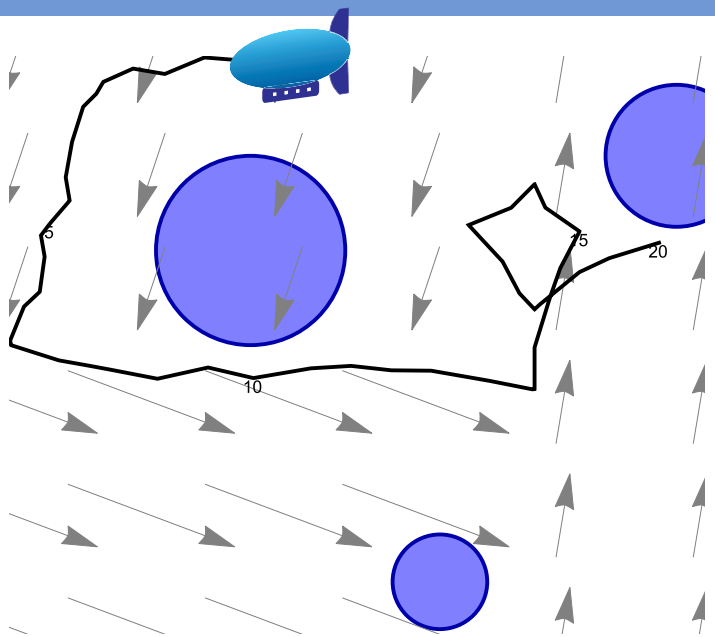

- ✓ airship at $x \in \mathbb{R}^2$
- ✓ propeller p controlled in any direction $y \in B$, i.e., $y_1^2 + y_2^2 \leq 1$
- × sporadically changing homogeneous wind field $v \in \mathbb{R}^2$
- × sporadically changing obstacle $o \in \mathbb{R}^2$ of size c subject to C
- × continuously local turbulence of magnitude r in any direction $z \in B$

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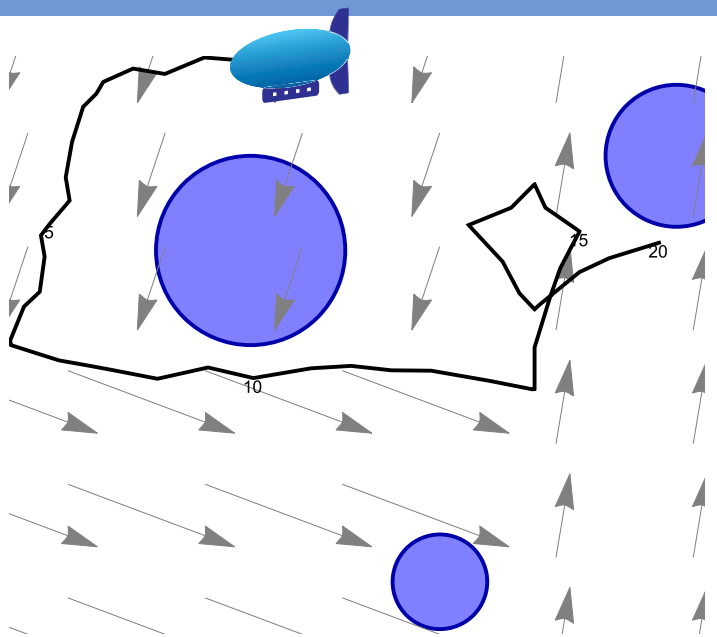
- If $r > p$
- If $p > \|v\| + r$
- If $\|v\| + r > p > r$



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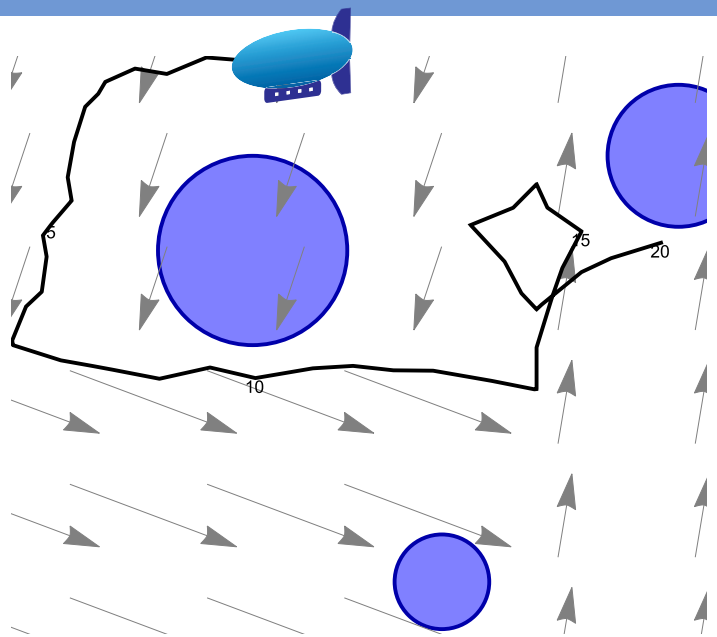

- × If $r > p$ hopeless turbulence
- If $p > \|v\| + r$
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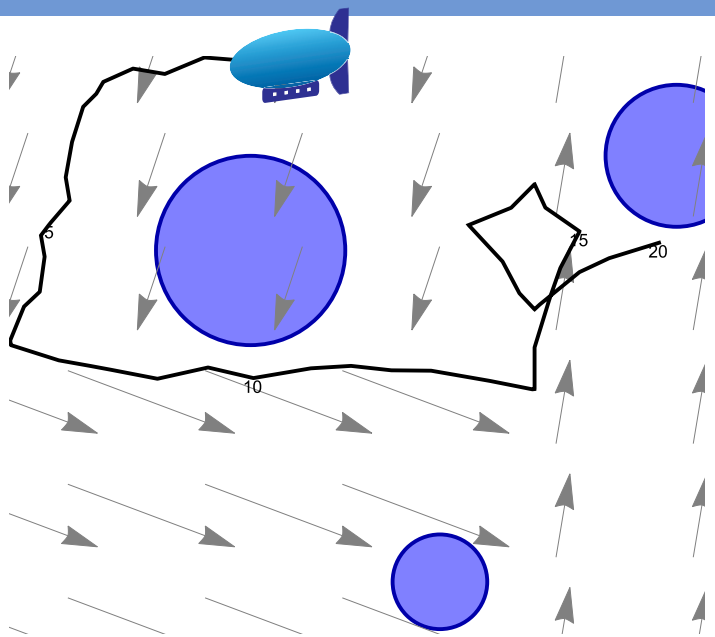

- × If $r > p$ hopeless turbulence
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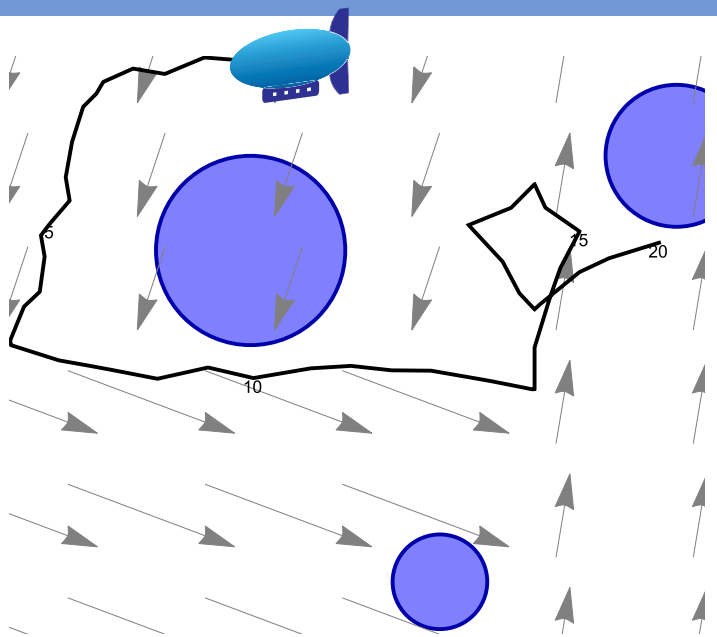

× If $r > p$ hopeless turbulence

✓ If $p > \|v\| + r$ super-powered prop

? If $\|v\| + r > p > r$ our challenge

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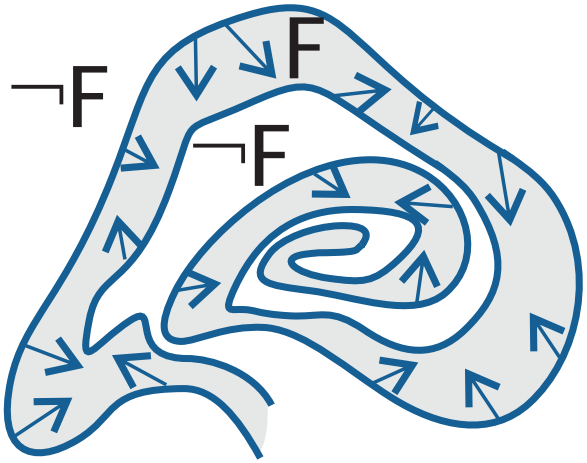
Differential Game Invariants

Theorem (Differential Game Invariants)

DGI $\dfrac{}{F \to [x' = f(x,y,z) \&^d y \in Y \& z \in Z]F}$

Theorem (Differential Game Refinement)



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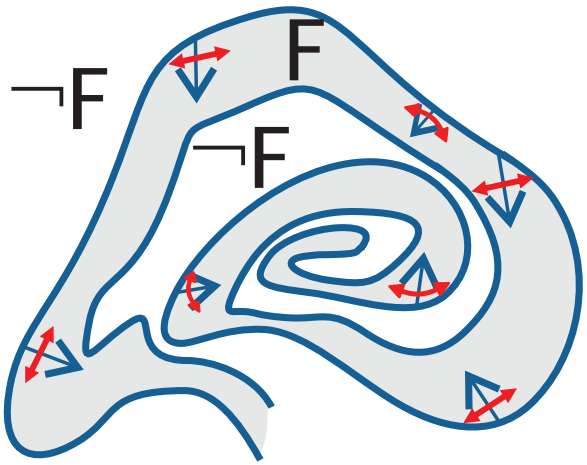

Differential Game Invariants

Theorem (Differential Game Invariants)

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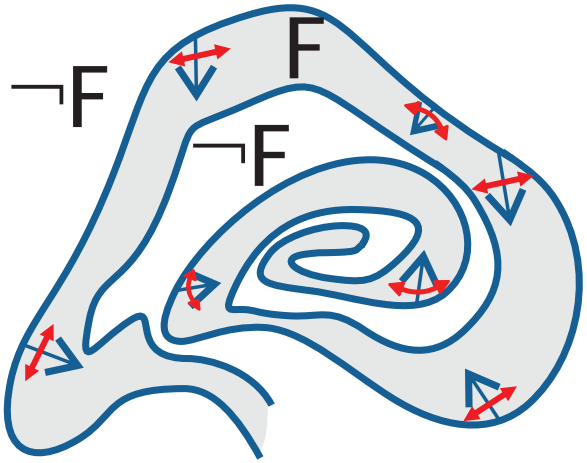
Theorem (Differential Game Refinement)



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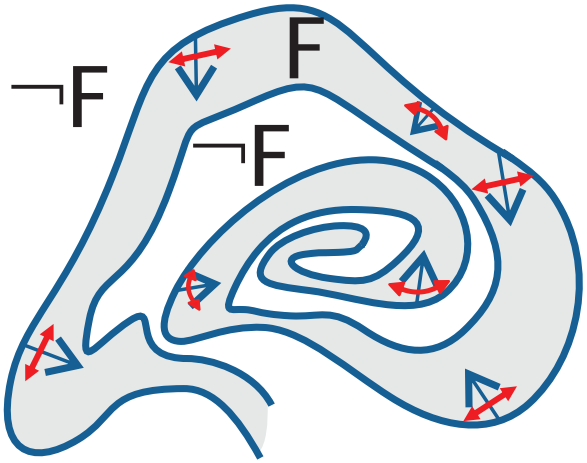

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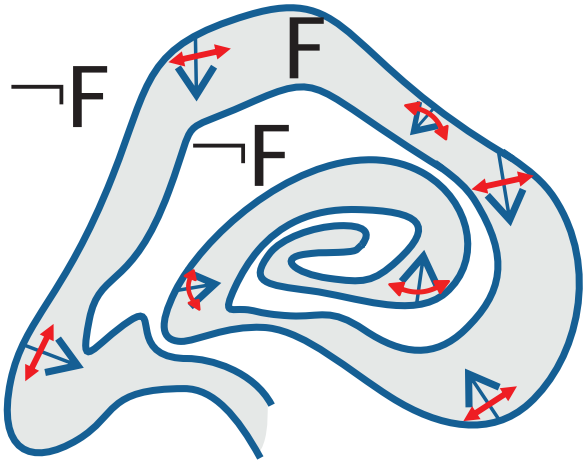

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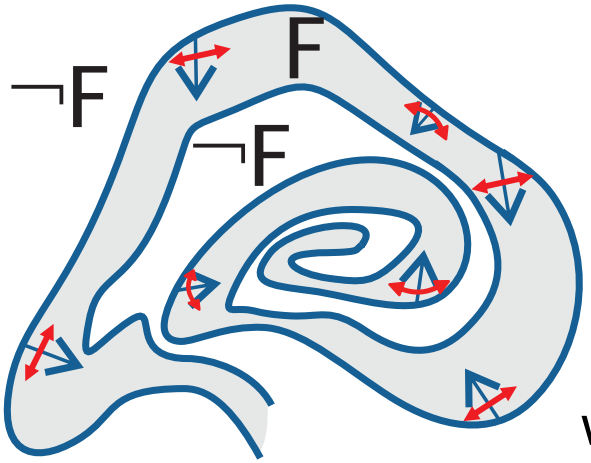

$$\text{DGI}\ \frac{}{1 \leq x^3 \vdash [x' = -1+2y+z \&^d y \in I \& z \in I]1 \leq x^3}$$

where $y \in I \equiv -1 \leq y \leq 1$

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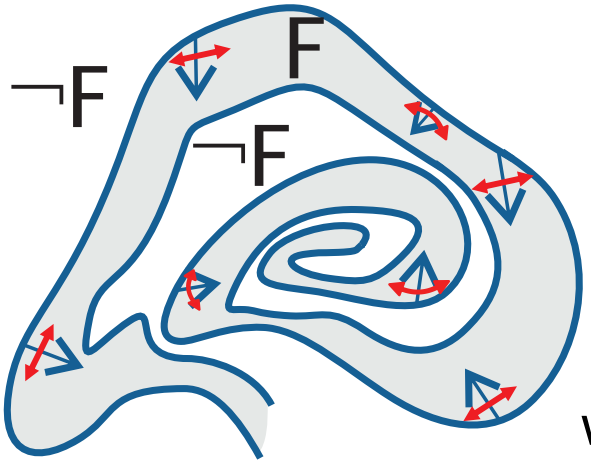

$$\text{[:=]}\ \frac{}{\vdash \exists y{\in}I \forall z{\in}I\, [x' := -1+2y+z]0 \leq 3x^2 x'}$$
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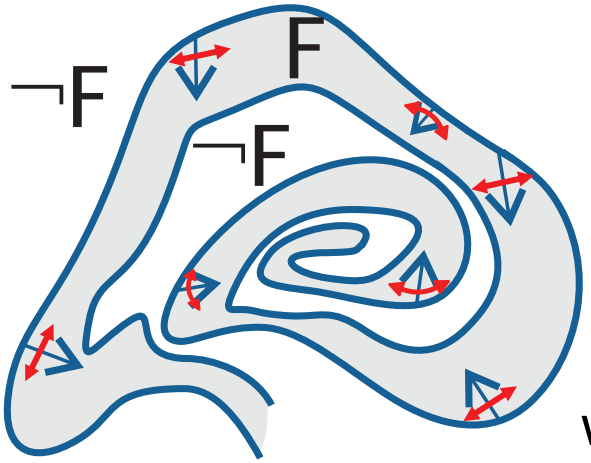

$$\mathbb{R}\ \frac{}{\vdash \exists y \in I \forall z \in I\, 0 \le 3x^2(-1+2y+z)}$$

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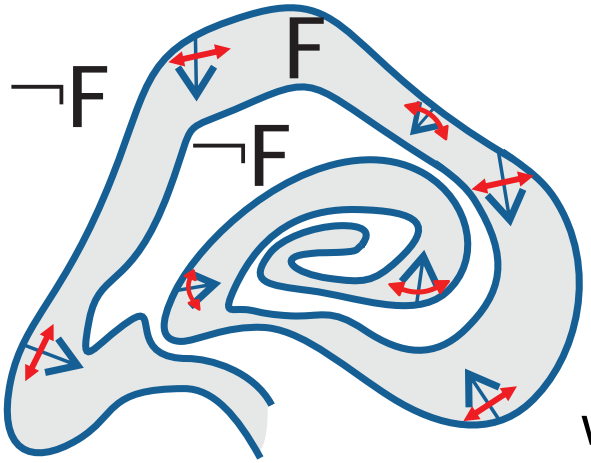

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$$\text{DGI}\frac{}{\|l-m\|^2>0 \vdash [m' = My, l' = Lz \&^d y \in B \& z \in B]\, \|l-m\|^2>0}$$

if $L \leq M$

Differential Game Invariants

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\mathbb{R}\ \dfrac{}{\vdash \exists y \in B \forall z \in B\, (2(l-m)\cdot(Lz - My) \geq 0)} \\
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Differential Game Variants

Theorem (Differential Game Variants)

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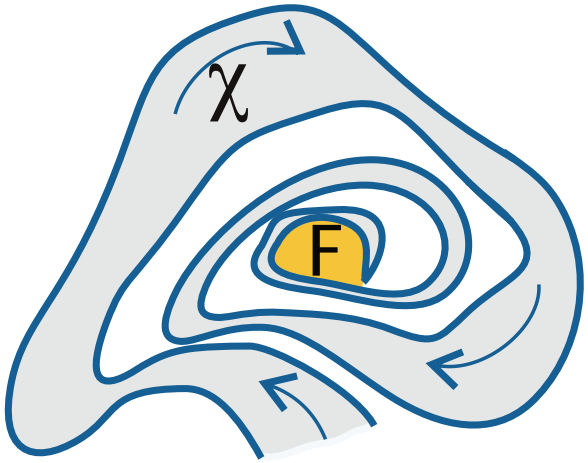

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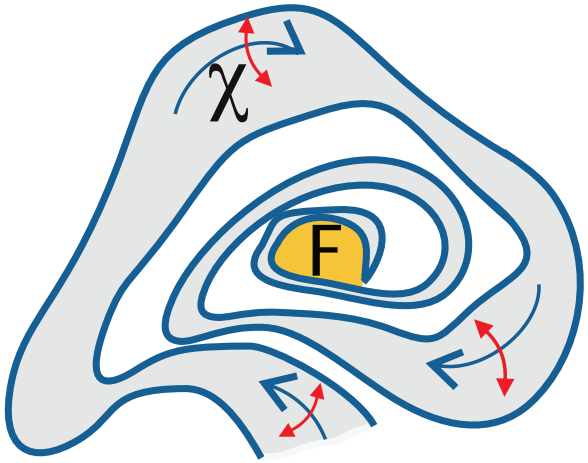

Differential Game Variants

Theorem (Differential Game Variants)

$$\text{DGV}\ \frac{\exists\varepsilon>0\,\forall x\,\exists z\in Z\,\forall y\in Y\,(g\leq 0\rightarrow[x':=f(x,y,z)](g)'\geq\varepsilon)}{\langle x'=f(x,y,z)\&^d y\in Y\&z\in Z\rangle g\geq 0}$$

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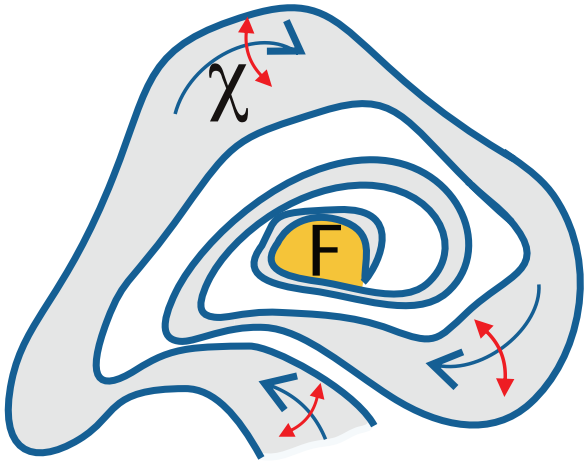

Differential Game Variants

Theorem (Differential Game Variants)

$$\text{DGV}\ \frac{\exists\varepsilon>0\,\forall x\,\exists z\in Z\,\forall y\in Y\,(g\leq 0\rightarrow[x':=f(x,y,z)](g)'\geq\varepsilon)}{\langle x'=f(x,y,z)\&^d y\in Y\&z\in Z\rangle g\geq 0}$$

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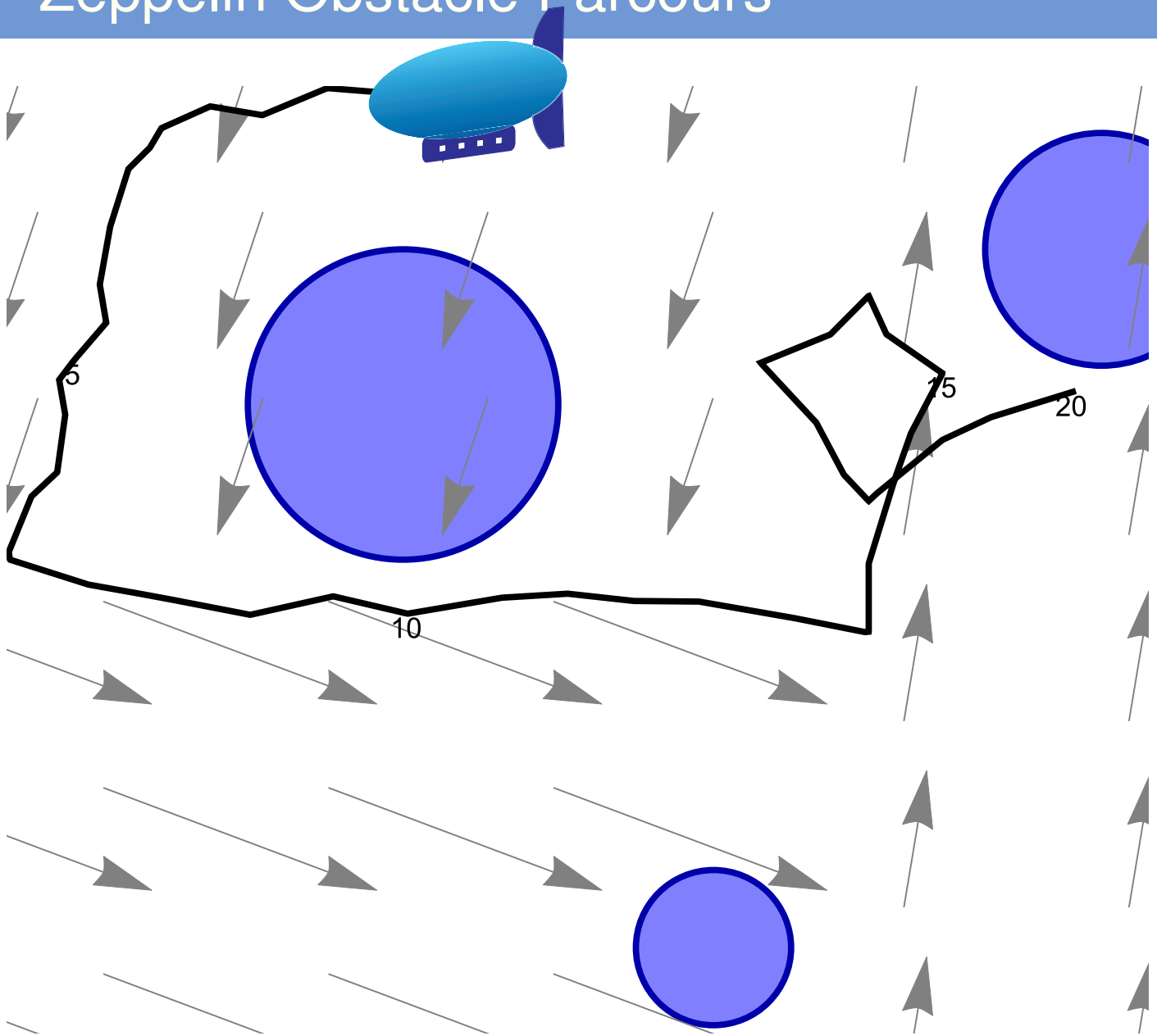
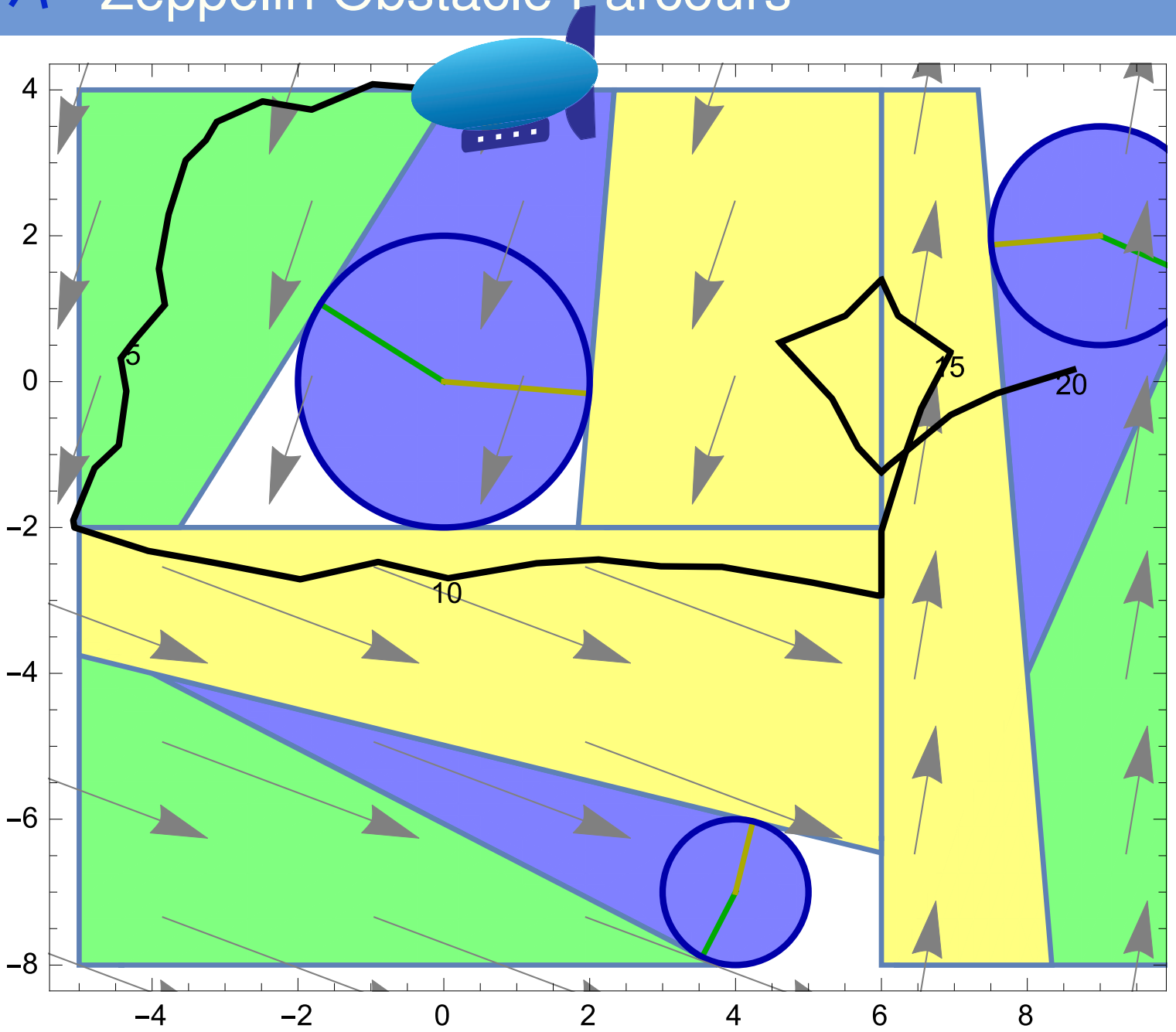
Zeppelin Obstacle Parcours



avoid obstacles
changing wind
local turbulence

Zeppelin Obstacle Parcours



avoid obstacles
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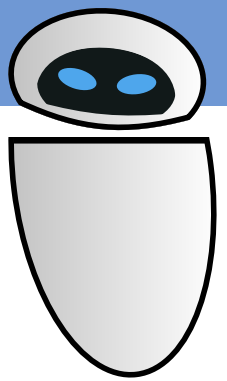
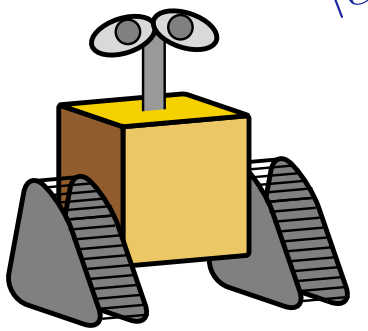
Outline

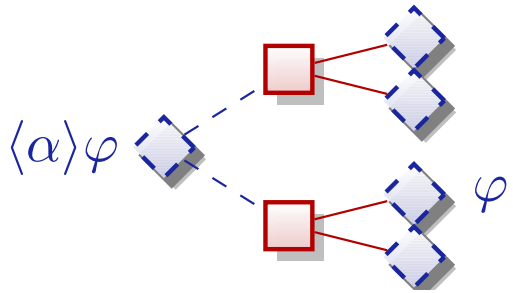
Summary



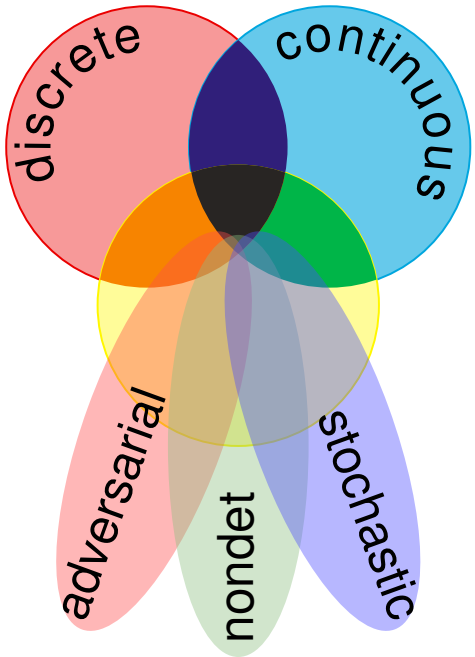
differential game logic

$$\mathsf{dGL} = \mathsf{GL} + \mathsf{HG} = \mathsf{dL} + {}^{\mathsf{d}}$$





- Logic for hybrid games
- Compositional PL + logic
- Discrete + continuous + adversarial
- Winning regions iterate $\geq\omega^\omega$
- Sound & rel. complete axiomatization
- Hybrid games $>$ hybrid systems
- ${}^{\mathsf{d}}$ radical challenge yet smooth extension
- Don't use systems thinking for games

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