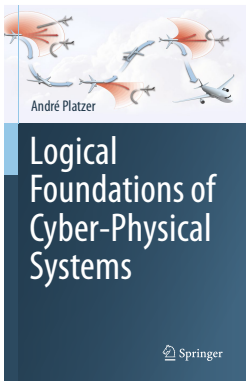


# 16: Winning & Proving Hybrid Games

## Logical Foundations of Cyber-Physical Systems



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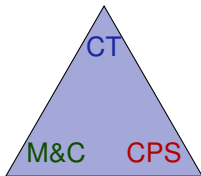
Computer Science Department  
Carnegie Mellon University

- 1 Learning Objectives
- 2 Semantical Considerations
- 3 Dynamic Axioms for Hybrid Games
  - Assignments
  - Differential Equations
  - Challenge Games
  - Choice Games
  - Sequential Games
  - Dual Games
  - Example Proof: Demon's Choice
- 4 Repetitions
  - Proofs for Loops
  - Example Proof: Dual Filibuster
  - Example Proof: Push-around Cart
- 5 Axiomatization
- 6 Summary



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rigorous reasoning for adversarial dynamics  
compositional reasoning from compositional semantics  
modular addition of adversarial dynamics  
axiomatization of dGL



analytical&semantical interaction  
discrete+continuous+adversarial  
fixpoints

CPS semantics  
align semantics&reasoning  
operational CPS effects



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Definition (Hybrid game  $\alpha$ )

$$\alpha, \beta ::= x := e \mid ?Q \mid x' = f(x) \& Q \mid \alpha \cup \beta \mid \alpha; \beta \mid \alpha^* \mid \alpha^d$$

Definition (dGL Formula  $P$ )

$$P, Q ::= e \geq \tilde{e} \mid \neg P \mid P \wedge Q \mid \forall x P \mid \exists x P \mid \langle \alpha \rangle P \mid [\alpha] P$$

Discrete  
Assign

Test  
Game

Differential  
Equation

Choice  
Game

Seq.  
Game

Repeat  
Game

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All  
Reals

Some  
Reals

Discrete  
Assign

Test  
Game

Differential  
Equation

Choice  
Game

Seq.  
Game

Repeat  
Game

Dual  
Game

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All  
Reals

Some  
Reals



Discrete  
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All  
Reals

Some  
Reals

Angel  
Wins

Discrete  
Assign

Test  
Game

Differential  
Equation

Choice  
Game

Seq.  
Game

Repeat  
Game

Dual  
Game

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$$P, Q ::= e \geq \tilde{e} \mid \neg P \mid P \wedge Q \mid \forall x P \mid \exists x P \mid \langle \alpha \rangle P \mid [ \alpha ] P$$

All  
Reals

Some  
Reals

Angel  
Wins

Demon  
Wins

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Test  
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Definition (Hybrid game  $\alpha$ )

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Definition (dGL Formula  $P$ )

$$P, Q ::= e \geq \tilde{e} \mid \neg P \mid P \wedge Q \mid \forall x P \mid \exists x P \mid \langle \alpha \rangle P \mid [ \alpha ] P$$

“Angel has Wings  $\langle \alpha \rangle$ ”

All  
Reals

Some  
Reals

Angel  
Wins

Demon  
Wins

Definition (Hybrid game  $\alpha$ )

$\llbracket \cdot \rrbracket : \text{HG} \rightarrow (\wp(\mathcal{S}) \rightarrow \wp(\mathcal{S}))$

$$\begin{aligned} \zeta_{x:=e}(X) &= \{\omega \in \mathcal{S} : \omega_x^{\omega[e]} \in X\} \\ \zeta_{x'=f(x)}(X) &= \{\varphi(0) \in \mathcal{S} : \varphi(r) \in X \text{ for some } \varphi:[0,r] \rightarrow \mathcal{S}, \varphi \models x' = f(x)\} \\ \zeta_{?Q}(X) &= \llbracket Q \rrbracket \cap X \\ \zeta_{\alpha \cup \beta}(X) &= \zeta_{\alpha}(X) \cup \zeta_{\beta}(X) \\ \zeta_{\alpha;\beta}(X) &= \zeta_{\alpha}(\zeta_{\beta}(X)) \\ \zeta_{\alpha^*}(X) &= \bigcap \{Z \subseteq \mathcal{S} : X \cup \zeta_{\alpha}(Z) \subseteq Z\} \\ \zeta_{\alpha^d}(X) &= (\zeta_{\alpha}(X^c))^c \end{aligned}$$

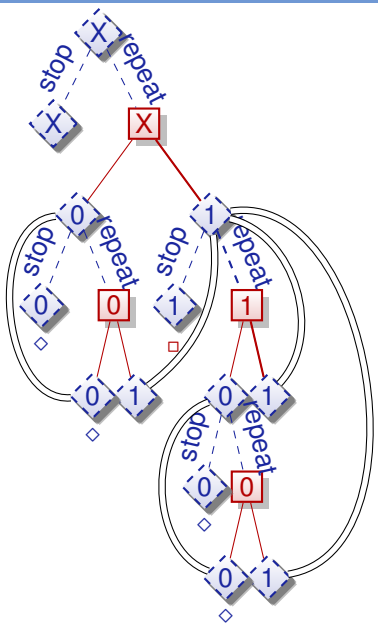
Definition (dGL Formula  $P$ )

$\llbracket \cdot \rrbracket : \text{Fml} \rightarrow \wp(\mathcal{S})$

$$\begin{aligned} \llbracket e_1 \geq e_2 \rrbracket &= \{\omega \in \mathcal{S} : \omega[e_1] \geq \omega[e_2]\} \\ \llbracket \neg P \rrbracket &= (\llbracket P \rrbracket)^c \\ \llbracket P \wedge Q \rrbracket &= \llbracket P \rrbracket \cap \llbracket Q \rrbracket \\ \llbracket \langle \alpha \rangle P \rrbracket &= \zeta_{\alpha}(\llbracket P \rrbracket) \\ \llbracket [\alpha] P \rrbracket &= \delta_{\alpha}(\llbracket P \rrbracket) \end{aligned}$$



$\langle (x := 0 \cap x := 1)^* \rangle x = 0$   
 $\overset{\text{wfd}}{\rightsquigarrow} \text{false unless } x = 0$

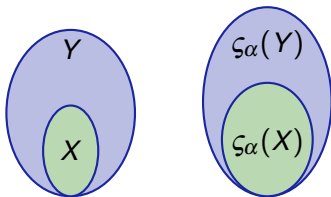


Theorem (Consistency & determinacy)

*Hybrid games are consistent and determined, i.e.,  $\models \neg\langle\alpha\rangle\neg P \leftrightarrow [\alpha]P$ .*

Lemma (Monotonicity)

$\zeta_\alpha(X) \subseteq \zeta_\alpha(Y)$  and  $\delta_\alpha(X) \subseteq \delta_\alpha(Y)$  for all  $X \subseteq Y$



## Theorem (Consistency & determinacy)

Hybrid games are consistent and determined, i.e.,  $\models \neg\langle\alpha\rangle\neg P \leftrightarrow [\alpha]P$ .

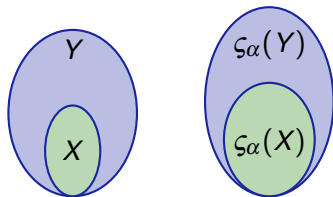
## Corollary

*Determined: At least one player wins:*  $\neg\langle\alpha\rangle\neg P \rightarrow [\alpha]P$  so  $\langle\alpha\rangle\neg P \vee [\alpha]P$

*Consistent: At most one player wins:*  $[\alpha]P \rightarrow \neg\langle\alpha\rangle\neg P$  so  $\neg([\alpha]P \wedge \langle\alpha\rangle\neg P)$

## Lemma (Monotonicity)

$\zeta_\alpha(X) \subseteq \zeta_\alpha(Y)$  and  $\delta_\alpha(X) \subseteq \delta_\alpha(Y)$  for all  $X \subseteq Y$





## Theorem (Consistency &amp; determinacy)

Hybrid games are consistent and determined, i.e.,  $\models \neg\langle\alpha\rangle\neg P \leftrightarrow [\alpha]P$ .

## Proof Sketch.

$$\zeta_{\alpha\cup\beta}(X^{\complement})^{\complement} = (\zeta_{\alpha}(X^{\complement}) \cup \zeta_{\beta}(X^{\complement}))^{\complement} = \zeta_{\alpha}(X^{\complement})^{\complement} \cap \zeta_{\beta}(X^{\complement})^{\complement} = \delta_{\alpha}(X) \cap \delta_{\beta}(X) = \delta_{\alpha\cup\beta}(X) \quad \square$$

## Lemma (Monotonicity)

$\zeta_{\alpha}(X) \subseteq \zeta_{\alpha}(Y)$  and  $\delta_{\alpha}(X) \subseteq \delta_{\alpha}(Y)$  for all  $X \subseteq Y$

## Proof Sketch.

- $X \subseteq Y$  so  $X^{\complement} \supseteq Y^{\complement}$  so  $\zeta_{\alpha}(X^{\complement}) \supseteq \zeta_{\alpha}(Y^{\complement})$  so  $\zeta_{\alpha}^{\text{d}}(X) = (\zeta_{\alpha}(X^{\complement}))^{\complement} \subseteq (\zeta_{\alpha}(Y^{\complement}))^{\complement} = \zeta_{\alpha}^{\text{d}}(Y)$ .
- $\zeta_{\alpha}^*(X) = \bigcap \{Z \subseteq \mathcal{S} : X \cup \zeta_{\alpha}(Z) \subseteq Z\} \subseteq \bigcap \{Z \subseteq \mathcal{S} : Y \cup \zeta_{\alpha}(Z) \subseteq Z\} = \zeta_{\alpha}^*(Y)$  because  $X \subseteq Y$  □

## Theorem (Consistency & determinacy)

*Hybrid games are consistent and determined, i.e.,  $\models \neg\langle\alpha\rangle\neg P \leftrightarrow [\alpha]P$ .*

## Lemma (Monotonicity)

$\zeta_\alpha(X) \subseteq \zeta_\alpha(Y)$  and  $\delta_\alpha(X) \subseteq \delta_\alpha(Y)$  for all  $X \subseteq Y$

Theorem (Consistency & determinacy)

*Hybrid games are consistent and determined, i.e.,  $\models \neg\langle\alpha\rangle\neg P \leftrightarrow [\alpha]P$ .*

Corollary (Axiom: Determinacy)

$[\cdot] [\alpha]P \leftrightarrow \neg\langle\alpha\rangle\neg P$

Lemma (Monotonicity)

$\zeta_\alpha(X) \subseteq \zeta_\alpha(Y)$  and  $\delta_\alpha(X) \subseteq \delta_\alpha(Y)$  for all  $X \subseteq Y$

Corollary (Rule: Monotonicity)

$M \frac{P \rightarrow Q}{\langle\alpha\rangle P \rightarrow \langle\alpha\rangle Q}$ 
     
  $M[\cdot] \frac{P \rightarrow Q}{[\alpha]P \rightarrow [\alpha]Q}$

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$$\langle := \rangle \langle x := e \rangle p(x) \leftrightarrow$$

$\zeta_{x:=e}(X)$



$$\langle := \rangle \langle x := e \rangle p(x) \leftrightarrow p(e)$$

$\zeta_{x:=e}(X)$

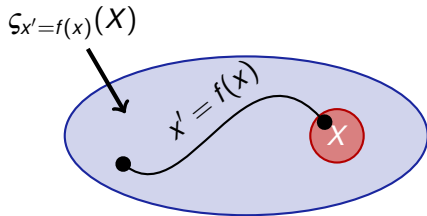


Axiom (Differential Equation)

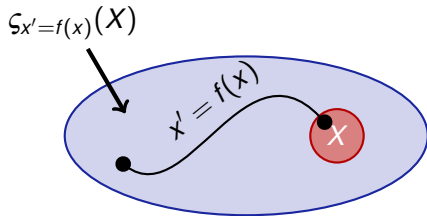
TOCL'15

$\langle \langle ' \rangle \langle x' = f(x) \rangle p(x) \leftrightarrow$

$(y'(t) = f(y))$

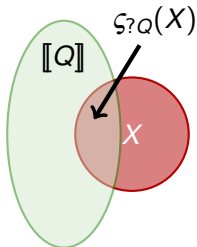


$$\langle \langle \rangle \langle x' = f(x) \rangle p(x) \leftrightarrow \exists t \geq 0 \langle x := y(t) \rangle p(x) \quad (y'(t) = f(y))$$

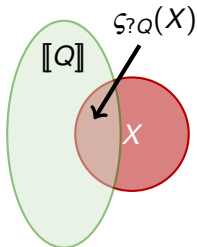




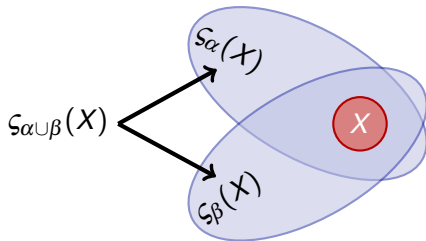
$$\langle ? \rangle \langle ?Q \rangle P \leftrightarrow$$



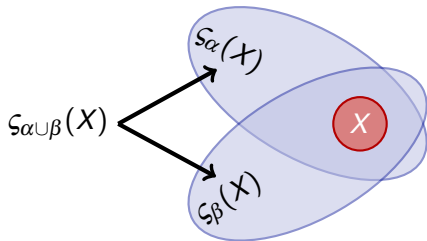
$$\langle ? \rangle \langle ?Q \rangle P \leftrightarrow Q \wedge P$$



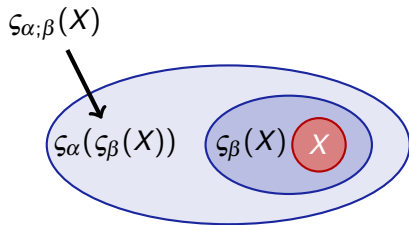
$$\langle U \rangle \langle \alpha \cup \beta \rangle P \leftrightarrow$$



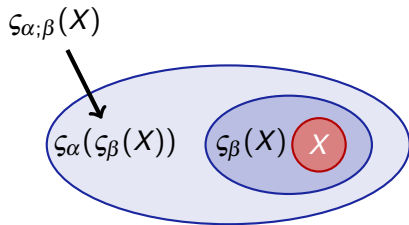
$$\langle U \rangle \langle \alpha \cup \beta \rangle P \leftrightarrow \langle \alpha \rangle P \vee \langle \beta \rangle P$$



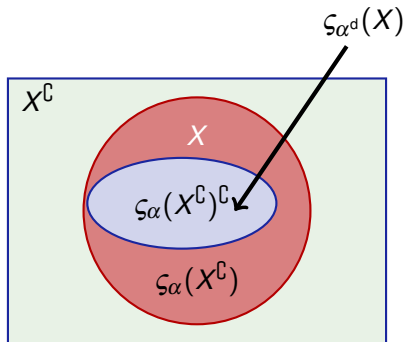
$$\langle ; \rangle \langle \alpha; \beta \rangle P \leftrightarrow$$



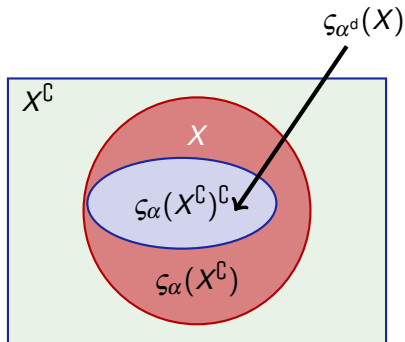
$$\langle ; \rangle \langle \alpha; \beta \rangle P \leftrightarrow \langle \alpha \rangle \langle \beta \rangle P$$



$$\langle d \rangle \langle \alpha^d \rangle P \leftrightarrow$$



$$\langle d \rangle \langle \alpha^d \rangle P \leftrightarrow \neg \langle \alpha \rangle \neg P$$







---

$$\vdash \langle \alpha \cap \beta \rangle P \leftrightarrow$$



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$$\vdash \langle \alpha \cap \beta \rangle P \leftrightarrow \langle \alpha \rangle P \wedge \langle \beta \rangle P$$



$$\alpha \cap \beta \equiv (\alpha^d \cup \beta^d)^d$$

$$\frac{\langle^d \rangle \vdash \langle (\alpha^d \cup \beta^d)^d \rangle P \leftrightarrow \langle \alpha \rangle P \wedge \langle \beta \rangle P}{\vdash \langle \alpha \cap \beta \rangle P \leftrightarrow \langle \alpha \rangle P \wedge \langle \beta \rangle P}$$

$$\langle^d \rangle \langle \alpha^d \rangle P \leftrightarrow \neg \langle \alpha \rangle \neg P$$

$$\frac{\langle^U \rangle \vdash \neg \langle \alpha^d \cup \beta^d \rangle \neg P \leftrightarrow \langle \alpha \rangle P \wedge \langle \beta \rangle P}{\langle^d \rangle \vdash \langle (\alpha^d \cup \beta^d)^d \rangle P \leftrightarrow \langle \alpha \rangle P \wedge \langle \beta \rangle P}$$


---


$$\vdash \langle \alpha \cap \beta \rangle P \leftrightarrow \langle \alpha \rangle P \wedge \langle \beta \rangle P$$

$$\langle U \rangle \langle \alpha \cup \beta \rangle P \leftrightarrow \langle \alpha \rangle P \vee \langle \beta \rangle P$$

$$\begin{array}{c} \langle^d \rangle \frac{}{\vdash \neg(\langle \alpha^d \rangle \neg P \vee \langle \beta^d \rangle \neg P) \leftrightarrow \langle \alpha \rangle P \wedge \langle \beta \rangle P} \\ \langle U \rangle \frac{}{\vdash \neg \langle \alpha^d \cup \beta^d \rangle \neg P \leftrightarrow \langle \alpha \rangle P \wedge \langle \beta \rangle P} \\ \langle^d \rangle \frac{}{\vdash \langle (\alpha^d \cup \beta^d)^d \rangle P \leftrightarrow \langle \alpha \rangle P \wedge \langle \beta \rangle P} \\ \vdash \langle \alpha \cap \beta \rangle P \leftrightarrow \langle \alpha \rangle P \wedge \langle \beta \rangle P \end{array}$$

$$\langle^d \rangle \langle \alpha^d \rangle P \leftrightarrow \neg \langle \alpha \rangle \neg P$$

$$\frac{}{\vdash \neg(\neg \langle \alpha \rangle \neg \neg P \vee \neg \langle \beta \rangle \neg \neg P) \leftrightarrow \langle \alpha \rangle P \wedge \langle \beta \rangle P}$$

$$\frac{\langle^d \rangle \vdash \neg(\langle \alpha^d \rangle \neg P \vee \langle \beta^d \rangle \neg P) \leftrightarrow \langle \alpha \rangle P \wedge \langle \beta \rangle P}{\langle^U \rangle \vdash \neg \langle \alpha^d \cup \beta^d \rangle \neg P \leftrightarrow \langle \alpha \rangle P \wedge \langle \beta \rangle P}$$

$$\frac{\langle^d \rangle \vdash \langle (\alpha^d \cup \beta^d)^d \rangle P \leftrightarrow \langle \alpha \rangle P \wedge \langle \beta \rangle P}{\vdash \langle \alpha \cap \beta \rangle P \leftrightarrow \langle \alpha \rangle P \wedge \langle \beta \rangle P}$$

$$\begin{array}{c}
 \hline
 \vdash \langle \alpha \rangle P \wedge \langle \beta \rangle P \leftrightarrow \langle \alpha \rangle P \wedge \langle \beta \rangle P \\
 \hline
 \vdash \neg(\neg \langle \alpha \rangle \neg \neg P \vee \neg \langle \beta \rangle \neg \neg P) \leftrightarrow \langle \alpha \rangle P \wedge \langle \beta \rangle P \\
 \hline
 \langle^d \rangle \vdash \neg(\langle \alpha^d \rangle \neg P \vee \langle \beta^d \rangle \neg P) \leftrightarrow \langle \alpha \rangle P \wedge \langle \beta \rangle P \\
 \hline
 \langle^U \rangle \vdash \neg \langle \alpha^d \cup \beta^d \rangle \neg P \leftrightarrow \langle \alpha \rangle P \wedge \langle \beta \rangle P \\
 \hline
 \langle^d \rangle \vdash \langle (\alpha^d \cup \beta^d)^d \rangle P \leftrightarrow \langle \alpha \rangle P \wedge \langle \beta \rangle P \\
 \hline
 \vdash \langle \alpha \cap \beta \rangle P \leftrightarrow \langle \alpha \rangle P \wedge \langle \beta \rangle P
 \end{array}$$

$$\begin{array}{c}
 * \\
 \hline
 \vdash \langle \alpha \rangle P \wedge \langle \beta \rangle P \leftrightarrow \langle \alpha \rangle P \wedge \langle \beta \rangle P \\
 \hline
 \vdash \neg(\neg \langle \alpha \rangle \neg \neg P \vee \neg \langle \beta \rangle \neg \neg P) \leftrightarrow \langle \alpha \rangle P \wedge \langle \beta \rangle P \\
 \hline
 \langle^d \rangle \vdash \neg(\langle \alpha^d \rangle \neg P \vee \langle \beta^d \rangle \neg P) \leftrightarrow \langle \alpha \rangle P \wedge \langle \beta \rangle P \\
 \hline
 \langle^U \rangle \vdash \neg \langle \alpha^d \cup \beta^d \rangle \neg P \leftrightarrow \langle \alpha \rangle P \wedge \langle \beta \rangle P \\
 \hline
 \langle^d \rangle \vdash \langle (\alpha^d \cup \beta^d)^d \rangle P \leftrightarrow \langle \alpha \rangle P \wedge \langle \beta \rangle P \\
 \hline
 \vdash \langle \alpha \cap \beta \rangle P \leftrightarrow \langle \alpha \rangle P \wedge \langle \beta \rangle P
 \end{array}$$



$$\begin{array}{c}
 * \\
 \hline
 \vdash \langle \alpha \rangle P \wedge \langle \beta \rangle P \leftrightarrow \langle \alpha \rangle P \wedge \langle \beta \rangle P \\
 \hline
 \vdash \neg(\neg\langle \alpha \rangle \neg\neg P \vee \neg\langle \beta \rangle \neg\neg P) \leftrightarrow \langle \alpha \rangle P \wedge \langle \beta \rangle P \\
 \hline
 \langle^d \rangle \vdash \neg(\langle \alpha^d \rangle \neg P \vee \langle \beta^d \rangle \neg P) \leftrightarrow \langle \alpha \rangle P \wedge \langle \beta \rangle P \\
 \hline
 \langle^U \rangle \vdash \neg\langle \alpha^d \cup \beta^d \rangle \neg P \leftrightarrow \langle \alpha \rangle P \wedge \langle \beta \rangle P \\
 \hline
 \langle^d \rangle \vdash \langle (\alpha^d \cup \beta^d)^d \rangle P \leftrightarrow \langle \alpha \rangle P \wedge \langle \beta \rangle P \\
 \hline
 \vdash \langle \alpha \cap \beta \rangle P \leftrightarrow \langle \alpha \rangle P \wedge \langle \beta \rangle P
 \end{array}$$

Derived axiom:

$$\langle \cap \rangle \langle \alpha \cap \beta \rangle P \leftrightarrow \langle \alpha \rangle P \wedge \langle \beta \rangle P$$



$$[\cdot] \frac{}{\vdash [\alpha \cap \beta] P \leftrightarrow}$$



$$[\cdot] \frac{}{\vdash [\alpha \wedge \beta]P \leftrightarrow [\alpha]P \vee [\beta]P}$$



$$[\cdot] [\alpha]P \leftrightarrow \neg\langle\alpha\rangle\neg P$$

$$\frac{\langle\cap\rangle \frac{}{\vdash \neg\langle\alpha\cap\beta\rangle\neg P \leftrightarrow [\alpha]P \vee [\beta]P}}{[\cdot] \frac{}{\vdash [\alpha\cap\beta]P \leftrightarrow [\alpha]P \vee [\beta]P}}$$

$$\langle \cap \rangle \langle \alpha \cap \beta \rangle P \leftrightarrow \langle \alpha \rangle P \wedge \langle \beta \rangle P$$

$$\frac{\frac{\frac{\vdash \neg(\langle \alpha \rangle \neg P \wedge \langle \beta \rangle \neg P) \leftrightarrow [\alpha] P \vee [\beta] P}{\langle \cap \rangle \vdash \neg \langle \alpha \cap \beta \rangle \neg P \leftrightarrow [\alpha] P \vee [\beta] P}}{[\cdot] \vdash [\alpha \cap \beta] P \leftrightarrow [\alpha] P \vee [\beta] P}}$$



# Example: Demon's Choice Derives by Duality

$$\begin{array}{l} \frac{[\cdot] \overline{\vdash \neg\langle\alpha\rangle\neg P \vee \neg\langle\beta\rangle\neg P \leftrightarrow [\alpha]P \vee [\beta]P}}{\vdash \neg(\langle\alpha\rangle\neg P \wedge \langle\beta\rangle\neg P) \leftrightarrow [\alpha]P \vee [\beta]P} \\ \frac{\langle\cap\rangle \overline{\vdash \neg\langle\alpha \cap \beta\rangle\neg P \leftrightarrow [\alpha]P \vee [\beta]P}}{[\cdot] \overline{\vdash [\alpha \cap \beta]P \leftrightarrow [\alpha]P \vee [\beta]P}} \end{array}$$

$$[\cdot] \quad [\alpha]P \leftrightarrow \neg\langle\alpha\rangle\neg P$$

$$\frac{}{\vdash [\alpha]P \vee [\beta]P \leftrightarrow [\alpha]P \vee [\beta]P}$$

$$\frac{[\cdot] \quad \vdash \neg\langle\alpha\rangle\neg P \vee \neg\langle\beta\rangle\neg P \leftrightarrow [\alpha]P \vee [\beta]P}{\vdash \neg(\langle\alpha\rangle\neg P \wedge \langle\beta\rangle\neg P) \leftrightarrow [\alpha]P \vee [\beta]P}$$

$$\frac{\langle\cap\rangle \quad \vdash \neg\langle\alpha \cap \beta\rangle\neg P \leftrightarrow [\alpha]P \vee [\beta]P}{[\cdot] \quad \vdash [\alpha \cap \beta]P \leftrightarrow [\alpha]P \vee [\beta]P}$$

$$\begin{array}{c}
 * \\
 \hline
 \vdash [\alpha]P \vee [\beta]P \leftrightarrow [\alpha]P \vee [\beta]P \\
 \hline
 [\cdot] \vdash \neg \langle \alpha \rangle \neg P \vee \neg \langle \beta \rangle \neg P \leftrightarrow [\alpha]P \vee [\beta]P \\
 \hline
 \vdash \neg (\langle \alpha \rangle \neg P \wedge \langle \beta \rangle \neg P) \leftrightarrow [\alpha]P \vee [\beta]P \\
 \hline
 \langle \cap \rangle \vdash \neg \langle \alpha \cap \beta \rangle \neg P \leftrightarrow [\alpha]P \vee [\beta]P \\
 \hline
 [\cdot] \vdash [\alpha \cap \beta]P \leftrightarrow [\alpha]P \vee [\beta]P
 \end{array}$$



$$\begin{array}{c}
 * \\
 \hline
 \vdash [\alpha]P \vee [\beta]P \leftrightarrow [\alpha]P \vee [\beta]P \\
 \hline
 [\cdot] \vdash \neg \langle \alpha \rangle \neg P \vee \neg \langle \beta \rangle \neg P \leftrightarrow [\alpha]P \vee [\beta]P \\
 \hline
 \vdash \neg (\langle \alpha \rangle \neg P \wedge \langle \beta \rangle \neg P) \leftrightarrow [\alpha]P \vee [\beta]P \\
 \hline
 \langle \cap \rangle \vdash \neg \langle \alpha \cap \beta \rangle \neg P \leftrightarrow [\alpha]P \vee [\beta]P \\
 \hline
 [\cdot] \vdash [\alpha \cap \beta]P \leftrightarrow [\alpha]P \vee [\beta]P
 \end{array}$$

Derived axioms:

$$[\cap] [\alpha \cap \beta]P \leftrightarrow [\alpha]P \vee [\beta]P$$

$$\langle \cap \rangle \langle \alpha \cap \beta \rangle P \leftrightarrow \langle \alpha \rangle P \wedge \langle \beta \rangle P$$

$$[\cdot] [\alpha]P \leftrightarrow \neg \langle \alpha \rangle \neg P$$

$$\langle := \rangle \langle x := e \rangle p(x) \leftrightarrow p(e)$$

$$\langle ' \rangle \langle x' = f(x) \rangle p(x) \leftrightarrow \exists t \geq 0 \langle x := y(t) \rangle p(x)$$

$$\langle ? \rangle \langle ?Q \rangle P \leftrightarrow Q \wedge P$$

$$\langle \cup \rangle \langle \alpha \cup \beta \rangle P \leftrightarrow \langle \alpha \rangle P \vee \langle \beta \rangle P$$

$$\langle ; \rangle \langle \alpha ; \beta \rangle P \leftrightarrow \langle \alpha \rangle \langle \beta \rangle P$$

$$\langle ^d \rangle \langle \alpha^d \rangle P \leftrightarrow \neg \langle \alpha \rangle \neg P$$

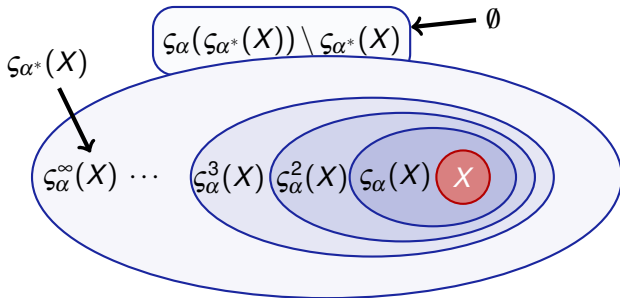


# Outline

- 1 Learning Objectives
- 2 Semantical Considerations
- 3 Dynamic Axioms for Hybrid Games
  - Assignments
  - Differential Equations
  - Challenge Games
  - Choice Games
  - Sequential Games
  - Dual Games
  - Example Proof: Demon's Choice
- 4 Repetitions**
  - Proofs for Loops
  - Example Proof: Dual Filibuster
  - Example Proof: Push-around Cart
- 5 Axiomatization
- 6 Summary

## Definition (Hybrid game $\alpha$ )

$$\mathcal{S}_{\alpha^*}(X) = \bigcap \{Z \subseteq \mathcal{S} : X \cup \mathcal{S}_{\alpha}(Z) = Z\}$$





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$$\zeta_{\alpha^*}(X) = \bigcap \{Z \subseteq \mathcal{S} : X \cup \zeta_{\alpha}(Z) \subseteq Z\}$$

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## Lemma (Axiom: Iteration)

$$\langle * \rangle \langle \alpha^* \rangle P \leftrightarrow P \vee \langle \alpha \rangle \langle \alpha^* \rangle P$$



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$$FP \frac{}{\langle \alpha^* \rangle P \rightarrow Q}$$

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Lemma (Axiom: Iteration)

$$\langle * \rangle \langle \alpha^* \rangle P \leftrightarrow P \vee \langle \alpha \rangle \langle \alpha^* \rangle P$$

Lemma (Rule: Least Fixpoint)

$$FP \frac{P \vee \langle \alpha \rangle Q \rightarrow Q}{\langle \alpha^* \rangle P \rightarrow Q}$$

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## Corollary (Derived Rule: Loop Invariant)

$$loop \frac{P \rightarrow [\alpha]P}{P \rightarrow [\alpha^*]P}$$



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Corollary (Derived Rule: Loop Invariant)

$$\text{loop} \frac{P \rightarrow [\alpha]P}{P \rightarrow [\alpha^*]P}$$

Proof

$$\frac{\vdash P \rightarrow [\alpha]P}{\vdash P \rightarrow P \wedge [\alpha]P}$$

$$\frac{[\cdot] \vdash P \rightarrow P \wedge \neg \langle \alpha \rangle \neg P}{\vdash \neg P \vee \langle \alpha \rangle \neg P \rightarrow \neg P}$$

$$FP \frac{\vdash \langle \alpha^* \rangle \neg P \rightarrow \neg P}{\vdash P \rightarrow \neg \langle \alpha^* \rangle \neg P}$$

$$[\cdot] \frac{\vdash P \rightarrow \neg \langle \alpha^* \rangle \neg P}{\vdash P \rightarrow [\alpha^*]P}$$

$$[\cdot] \quad [\alpha]P \leftrightarrow \neg \langle \alpha \rangle \neg P$$

$$\langle := \rangle \quad \langle x := e \rangle p(x) \leftrightarrow p(e)$$

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$$\langle \cup \rangle \quad \langle \alpha \cup \beta \rangle P \leftrightarrow \langle \alpha \rangle P \vee \langle \beta \rangle P$$

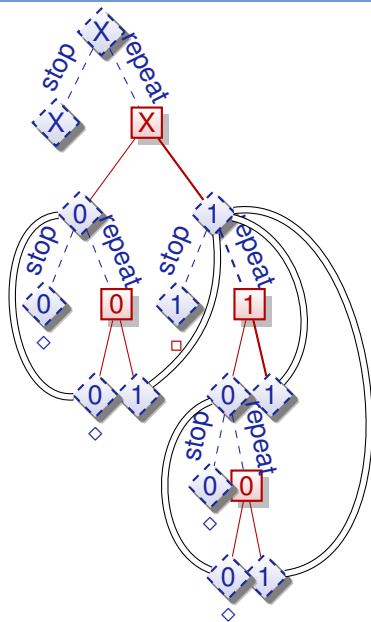
$$\langle ; \rangle \quad \langle \alpha ; \beta \rangle P \leftrightarrow \langle \alpha \rangle \langle \beta \rangle P$$

$$\langle * \rangle \quad \langle \alpha^* \rangle P \leftrightarrow P \vee \langle \alpha \rangle \langle \alpha^* \rangle P$$

$$\langle ^d \rangle \quad \langle \alpha^d \rangle P \leftrightarrow \neg \langle \alpha \rangle \neg P$$

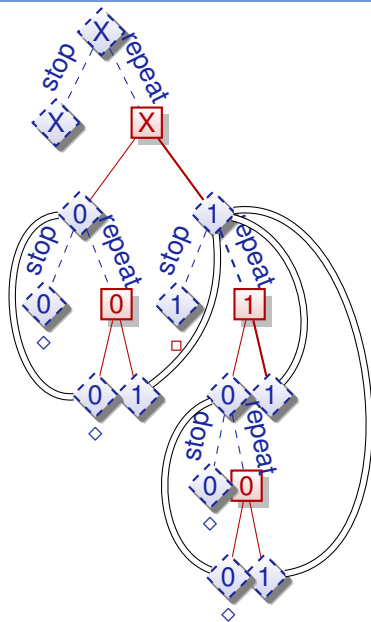
$$\text{M} \quad \frac{P \rightarrow Q}{\langle \alpha \rangle P \rightarrow \langle \alpha \rangle Q}$$

$$\text{FP} \quad \frac{P \vee \langle \alpha \rangle Q \rightarrow Q}{\langle \alpha^* \rangle P \rightarrow Q}$$



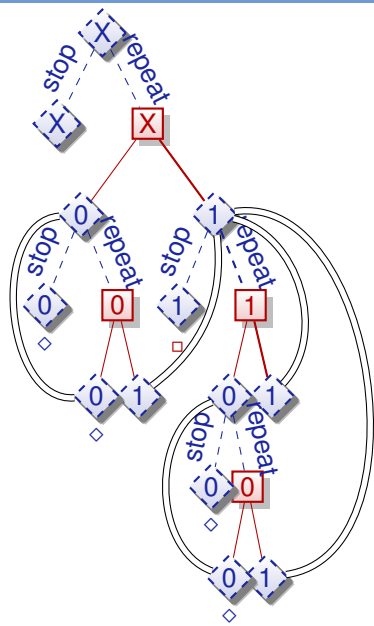
$$\langle^d \rangle \frac{}{x = 0 \vdash \langle (x := 0 \cup x := 1)^x \rangle x = 0}$$

$$\frac{\text{ind} \frac{}{x = 0 \vdash [(x := 0 \wedge x := 1)^*] x = 0}}{\langle^d \rangle \frac{}{x = 0 \vdash \langle (x := 0 \cup x := 1)^x \rangle x = 0}}$$

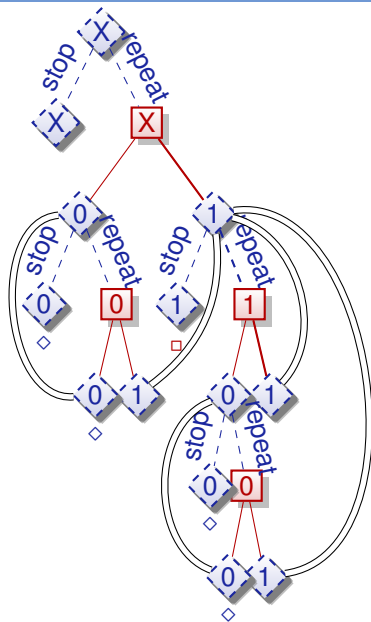




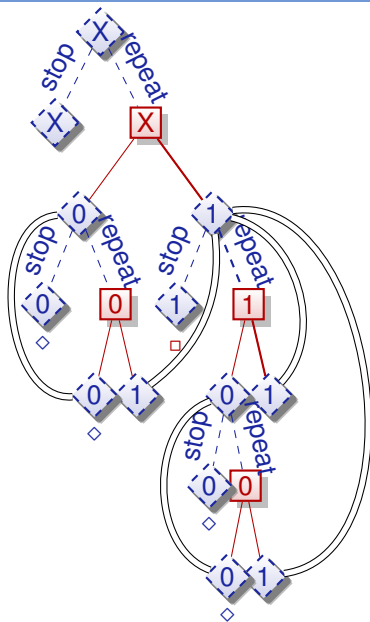
$$\begin{array}{c}
 \frac{[\cdot]}{x = 0 \vdash [x := 0 \wedge x := 1]x = 0} \\
 \text{ind} \frac{x = 0 \vdash [(x := 0 \wedge x := 1)^*]x = 0}{\langle^d \rangle x = 0 \vdash \langle (x := 0 \cup x := 1)^x \rangle x = 0}
 \end{array}$$



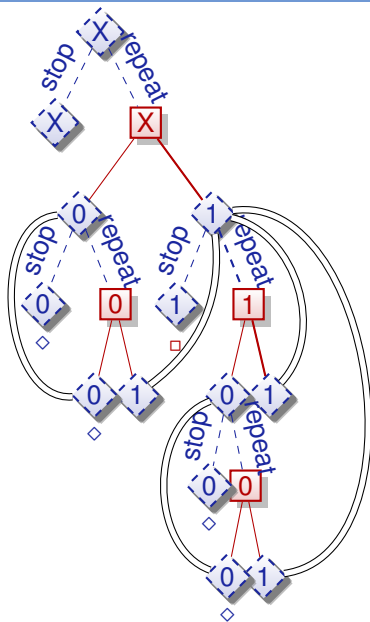
$$\begin{array}{c}
 \langle^d \rangle \frac{}{x = 0 \vdash \neg \langle x := 0 \wedge x := 1 \rangle \neg x = 0} \\
 [ \cdot ] \frac{}{x = 0 \vdash [x := 0 \wedge x := 1] x = 0} \\
 \text{ind} \frac{}{x = 0 \vdash [(x := 0 \wedge x := 1)^*] x = 0} \\
 \langle^d \rangle \frac{}{x = 0 \vdash \langle (x := 0 \cup x := 1)^x \rangle x = 0}
 \end{array}$$



$$\begin{array}{c}
 \langle \cup \rangle \frac{}{x = 0 \vdash \langle x := 0 \cup x := 1 \rangle x = 0} \\
 \langle d \rangle \frac{}{x = 0 \vdash \neg \langle x := 0 \cap x := 1 \rangle \neg x = 0} \\
 [\cdot] \frac{}{x = 0 \vdash [x := 0 \cap x := 1] x = 0} \\
 \text{ind} \frac{}{x = 0 \vdash [(x := 0 \cap x := 1)^*] x = 0} \\
 \langle d \rangle \frac{}{x = 0 \vdash \langle (x := 0 \cup x := 1)^x \rangle x = 0}
 \end{array}$$



$$\begin{array}{l}
 \langle := \rangle \frac{}{x = 0 \vdash \langle x := 0 \rangle x = 0 \vee \langle x := 1 \rangle x = 0} \\
 \langle \cup \rangle \frac{}{x = 0 \vdash \langle x := 0 \cup x := 1 \rangle x = 0} \\
 \langle \text{d} \rangle \frac{}{x = 0 \vdash \neg \langle x := 0 \cap x := 1 \rangle \neg x = 0} \\
 [\cdot] \frac{}{x = 0 \vdash [x := 0 \cap x := 1] x = 0} \\
 \text{ind} \frac{}{x = 0 \vdash [(x := 0 \cap x := 1)^*] x = 0} \\
 \langle \text{d} \rangle \frac{}{x = 0 \vdash \langle (x := 0 \cup x := 1)^x \rangle x = 0}
 \end{array}$$



$$\frac{}{\mathbb{R} \quad x = 0 \vdash 0 = 0 \vee 1 = 0}$$

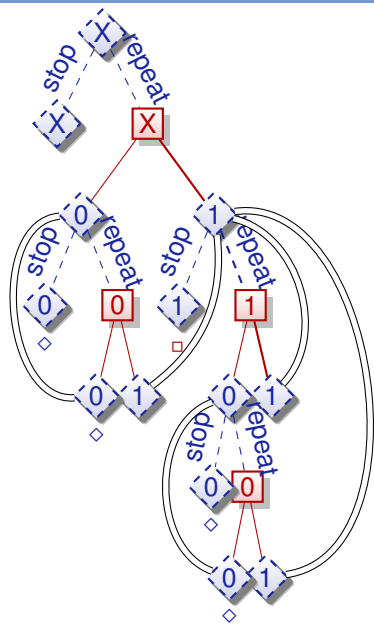
$$\frac{}{\langle := \rangle \quad x = 0 \vdash \langle x := 0 \rangle x = 0 \vee \langle x := 1 \rangle x = 0}$$

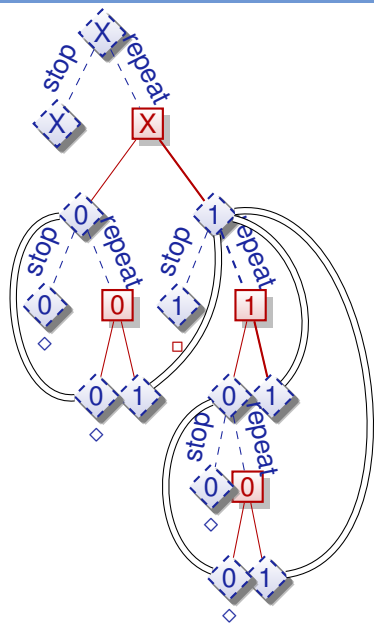
$$\frac{}{\langle \cup \rangle \quad x = 0 \vdash \langle x := 0 \cup x := 1 \rangle x = 0}$$

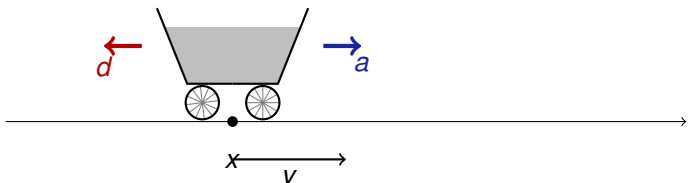
$$\frac{}{\langle d \rangle \quad x = 0 \vdash \neg \langle x := 0 \cap x := 1 \rangle \neg x = 0}$$

$$\frac{}{[\cdot] \quad x = 0 \vdash [x := 0 \cap x := 1] x = 0}$$

$$\frac{}{\text{ind} \quad x = 0 \vdash [(x := 0 \cap x := 1)^*] x = 0}$$

$$\frac{}{\langle d \rangle \quad x = 0 \vdash \langle (x := 0 \cup x := 1)^x \rangle x = 0}$$


$$\begin{array}{c}
 \mathbb{R} \quad \frac{}{x = 0 \vdash 0 = 0 \vee 1 = 0} \\
 \langle := \rangle \quad \frac{}{x = 0 \vdash \langle x := 0 \rangle x = 0 \vee \langle x := 1 \rangle x = 0} \\
 \langle \cup \rangle \quad \frac{}{x = 0 \vdash \langle x := 0 \cup x := 1 \rangle x = 0} \\
 \langle \text{d} \rangle \quad \frac{}{x = 0 \vdash \neg \langle x := 0 \cap x := 1 \rangle \neg x = 0} \\
 [\cdot] \quad \frac{}{x = 0 \vdash [x := 0 \cap x := 1] x = 0} \\
 \text{ind} \quad \frac{}{x = 0 \vdash [(x := 0 \cap x := 1)^*] x = 0} \\
 \langle \text{d} \rangle \quad \frac{}{x = 0 \vdash \langle (x := 0 \cup x := 1)^{\times} \rangle x = 0}
 \end{array}$$




$$\text{ind } \overline{J \vdash [((d := 1 \wedge d := -1); (a := 1 \vee a := -1); \{x' = v, v' = a + d\})^*] x \geq 0}$$

$$\text{ind} \frac{[;] \frac{}{J \vdash [(d:=1 \wedge d:= -1); (a:=1 \vee a:= -1); \{x' = v, v' = a + d\}] J}}{J \vdash [((d:=1 \wedge d:= -1); (a:=1 \vee a:= -1); \{x'=v, v'=a+d\})^*] x \geq 0}}$$



$$\begin{array}{l}
 [\cap] \quad \frac{}{J \vdash [d := 1 \cap d := -1][a := 1 \cup a := -1]; \{x' = v, v' = a + d\}} J \\
 [;] \quad \frac{}{J \vdash [(d := 1 \cap d := -1); (a := 1 \cup a := -1)]; \{x' = v, v' = a + d\}} J \\
 \text{ind} \quad \frac{}{J \vdash [((d := 1 \cap d := -1); (a := 1 \cup a := -1)); \{x' = v, v' = a + d\}]^*] x \geq 0}
 \end{array}$$

$$\begin{array}{l}
 \text{VR,WR} \\
 \hline
 J \vdash [d := 1] [(a := 1 \cup a := -1); \{x' = v, v' = a + d\}] J \vee [d := -1] \dots \\
 \hline
 [\cap] \\
 J \vdash [d := 1 \cap d := -1] [(a := 1 \cup a := -1); \{x' = v, v' = a + d\}] J \\
 \hline
 [;] \\
 J \vdash [(d := 1 \cap d := -1); (a := 1 \cup a := -1); \{x' = v, v' = a + d\}] J \\
 \hline
 \text{ind} \\
 J \vdash [((d := 1 \cap d := -1); (a := 1 \cup a := -1); \{x' = v, v' = a + d\})^*] x \geq 0
 \end{array}$$

$$\begin{array}{l}
 \text{[:=]} \frac{}{J \vdash [d := 1][(a := 1 \cup a := -1); \{x' = v, v' = a + d\}]J} \\
 \text{VR,WR} \frac{}{J \vdash [d := 1][(a := 1 \cup a := -1); \{x' = v, v' = a + d\}]J \vee [d := -1] \dots} \\
 \text{[\cap]} \frac{}{J \vdash [d := 1 \cap d := -1][(a := 1 \cup a := -1); \{x' = v, v' = a + d\}]J} \\
 \text{[;]} \frac{}{J \vdash [(d := 1 \cap d := -1); (a := 1 \cup a := -1); \{x' = v, v' = a + d\}]J} \\
 \text{ind} \frac{}{J \vdash [((d := 1 \cap d := -1); (a := 1 \cup a := -1); \{x' = v, v' = a + d\})^*]x \geq 0}
 \end{array}$$

$$\begin{array}{c}
 \frac{[;]}{J \vdash [(a:=1 \cup a:=-1); \{x' = v, v' = a+1\}]J} \\
 \frac{[:=]}{J \vdash [d:=1][(a:=1 \cup a:=-1); \{x' = v, v' = a+d\}]J} \\
 \frac{\vee R, \vee W}{J \vdash [d:=1][(a:=1 \cup a:=-1); \{x' = v, v' = a+d\}]J \vee [d:= -1] \dots} \\
 \frac{[\cap]}{J \vdash [d:=1 \cap d:= -1][(a:=1 \cup a:=-1); \{x' = v, v' = a+d\}]J} \\
 \frac{[;]}{J \vdash [(d:=1 \cap d:= -1); (a:=1 \cup a:=-1); \{x' = v, v' = a+d\}]J} \\
 \frac{\text{ind}}{J \vdash [((d:=1 \cap d:= -1); (a:=1 \cup a:=-1); \{x'=v, v'=a+d\})^*]x \geq 0}
 \end{array}$$

$$\begin{array}{l}
\frac{[U] \quad J \vdash [a := 1 \cup a := -1][\{x' = v, v' = a + 1\}]J}{[U]} \\
\frac{[:] \quad J \vdash [(a := 1 \cup a := -1); \{x' = v, v' = a + 1\}]J}{[:]} \\
\frac{[:=] \quad J \vdash [d := 1][(a := 1 \cup a := -1); \{x' = v, v' = a + d\}]J}{[:=]} \\
\frac{\text{VR,WR} \quad J \vdash [d := 1][(a := 1 \cup a := -1); \{x' = v, v' = a + d\}]J \vee [d := -1] \dots}{\text{VR,WR}} \\
\frac{[\cap] \quad J \vdash [d := 1 \cap d := -1][(a := 1 \cup a := -1); \{x' = v, v' = a + d\}]J}{[\cap]} \\
\frac{[;] \quad J \vdash [(d := 1 \cap d := -1); (a := 1 \cup a := -1); \{x' = v, v' = a + d\}]J}{[;]} \\
\frac{\text{ind} \quad J \vdash [((d := 1 \cap d := -1); (a := 1 \cup a := -1); \{x' = v, v' = a + d\})^*]x \geq 0}{\text{ind}}
\end{array}$$

$$\begin{array}{l}
\text{[:=]} \quad \frac{J \vdash [a := 1][\{x' = v, v' = a + 1\}]J \wedge [a := -1][\{x' = v, v' = a + 1\}]J}{J \vdash [a := 1 \cup a := -1][\{x' = v, v' = a + 1\}]J} \\
\text{[U]} \quad \frac{J \vdash [(a := 1 \cup a := -1); \{x' = v, v' = a + 1\}]J}{J \vdash [d := 1][(a := 1 \cup a := -1); \{x' = v, v' = a + d\}]J} \\
\text{[:=]} \quad \frac{J \vdash [d := 1][(a := 1 \cup a := -1); \{x' = v, v' = a + d\}]J}{J \vdash [d := 1][(a := 1 \cup a := -1); \{x' = v, v' = a + d\}]J \vee [d := -1] \dots} \\
\text{VR,WR} \quad \frac{J \vdash [d := 1 \cap d := -1][(a := 1 \cup a := -1); \{x' = v, v' = a + d\}]J}{J \vdash [(d := 1 \cap d := -1); (a := 1 \cup a := -1); \{x' = v, v' = a + d\}]J} \\
\text{[∩]} \quad \frac{J \vdash [(d := 1 \cap d := -1); (a := 1 \cup a := -1); \{x' = v, v' = a + d\}]J}{J \vdash [((d := 1 \cap d := -1); (a := 1 \cup a := -1); \{x' = v, v' = a + d\})^*]x \geq 0} \\
\text{[∗]} \quad \text{ind}
\end{array}$$

$$\begin{array}{l}
\frac{J \vdash [\{x' = v, v' = 1 + 1\}]J \wedge [\{x' = v, v' = -1 + 1\}]J}{[:=]} J \vdash [a := 1][\{x' = v, v' = a + 1\}]J \wedge [a := -1][\{x' = v, v' = a + 1\}]J \\
\frac{J \vdash [a := 1 \cup a := -1][\{x' = v, v' = a + 1\}]J}{[U]} J \vdash [(a := 1 \cup a := -1); \{x' = v, v' = a + 1\}]J \\
\frac{J \vdash [(a := 1 \cup a := -1); \{x' = v, v' = a + 1\}]J}{[:=]} J \vdash [d := 1][(a := 1 \cup a := -1); \{x' = v, v' = a + d\}]J \\
\frac{J \vdash [d := 1][(a := 1 \cup a := -1); \{x' = v, v' = a + d\}]J \vee [d := -1] \dots}{\vee R, \vee R} J \vdash [d := 1 \cap d := -1][(a := 1 \cup a := -1); \{x' = v, v' = a + d\}]J \\
\frac{J \vdash [d := 1 \cap d := -1][(a := 1 \cup a := -1); \{x' = v, v' = a + d\}]J}{[I]} J \vdash [(d := 1 \cap d := -1); (a := 1 \cup a := -1); \{x' = v, v' = a + d\}]J \\
\frac{J \vdash [(d := 1 \cap d := -1); (a := 1 \cup a := -1); \{x' = v, v' = a + d\}]^* J}{\text{ind}} J \vdash [((d := 1 \cap d := -1); (a := 1 \cup a := -1); \{x' = v, v' = a + d\})^*]x \geq 0
\end{array}$$

$$\begin{array}{l}
 \frac{J \vdash [\{x' = v, v' = 1 + 1\}]J \wedge [\{x' = v, v' = -1 + 1\}]J}{[:=]} J \vdash [a := 1][\{x' = v, v' = a + 1\}]J \wedge [a := -1][\{x' = v, v' = a + 1\}]J \\
 \frac{[U]} J \vdash [a := 1 \cup a := -1][\{x' = v, v' = a + 1\}]J \\
 \frac{[;]} J \vdash [(a := 1 \cup a := -1); \{x' = v, v' = a + 1\}]J \\
 \frac{[:=]} J \vdash [d := 1][(a := 1 \cup a := -1); \{x' = v, v' = a + d\}]J \\
 \frac{VR, WR}{J \vdash [d := 1][(a := 1 \cup a := -1); \{x' = v, v' = a + d\}]J \vee [d := -1] \dots} \\
 \frac{[\cap]} J \vdash [d := 1 \cap d := -1][(a := 1 \cup a := -1); \{x' = v, v' = a + d\}]J \\
 \frac{[;]} J \vdash [(d := 1 \cap d := -1); (a := 1 \cup a := -1); \{x' = v, v' = a + d\}]J \\
 \frac{ind}{J \vdash [((d := 1 \cap d := -1); (a := 1 \cup a := -1); \{x' = v, v' = a + d\})^*]x \geq 0}
 \end{array}$$

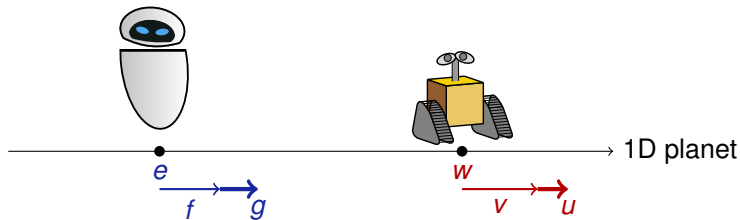
$$J \stackrel{\text{def}}{\equiv} x \geq 0 \wedge v \geq 0$$



$$\begin{array}{l}
 \frac{J \vdash [\{x' = v, v' = 1 + 1\}]J \wedge [\{x' = v, v' = -1 + 1\}]J}{[:=]} J \vdash [a := 1][\{x' = v, v' = a + 1\}]J \wedge [a := -1][\{x' = v, v' = a + 1\}]J \\
 \frac{[U]} J \vdash [a := 1 \cup a := -1][\{x' = v, v' = a + 1\}]J \\
 \frac{[;]} J \vdash [(a := 1 \cup a := -1); \{x' = v, v' = a + 1\}]J \\
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 \frac{[\cap]} J \vdash [d := 1 \cap d := -1][(a := 1 \cup a := -1); \{x' = v, v' = a + d\}]J \\
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 \frac{ind}{J \vdash [((d := 1 \cap d := -1); (a := 1 \cup a := -1); \{x' = v, v' = a + d\})^*]x \geq 0}
 \end{array}$$

$$J \stackrel{\text{def}}{=} x \geq 0 \wedge v \geq 0 \quad [;].[:=] \frac{x \geq 0 \wedge v \geq 0 \vdash \forall t \geq 0 (x + vt + t^2 \geq 0 \wedge v + 2t \geq 0)}{J \vdash [\{x' = v, v' = 1 + 1\}]J}$$

$$[;].[:=] \frac{x \geq 0 \wedge v \geq 0 \vdash \forall t \geq 0 (x + vt \geq 0 \wedge v \geq 0)}{J \vdash [\{x' = v, v' = 0\}]J}$$



$$(w - e)^2 \leq 1 \wedge v = f \rightarrow$$

$$\langle ((u := 1 \cap u := -1);$$

$$(g := 1 \cup g := -1);$$

$$t := 0;$$

$$\{w' = v, v' = u, e' = f, f' = g, t' = 1 \& t \leq 1\}^d$$

$$\rangle^x \rangle (w - e)^2 \leq 1$$

EVE at  $e$  plays Angel's part controlling  $g$

WALL·E at  $w$  plays Demon's part controlling  $u$



# Outline

- 1 Learning Objectives
- 2 Semantical Considerations
- 3 Dynamic Axioms for Hybrid Games
  - Assignments
  - Differential Equations
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  - Choice Games
  - Sequential Games
  - Dual Games
  - Example Proof: Demon's Choice
- 4 Repetitions
  - Proofs for Loops
  - Example Proof: Dual Filibuster
  - Example Proof: Push-around Cart
- 5 **Axiomatization**
- 6 Summary

$$[\cdot] \quad [\alpha]P \leftrightarrow \neg \langle \alpha \rangle \neg P$$

$$\langle := \rangle \quad \langle x := e \rangle p(x) \leftrightarrow p(e)$$

$$\langle ' \rangle \quad \langle x' = f(x) \rangle p(x) \leftrightarrow \exists t \geq 0 \langle x := y(t) \rangle p(x)$$

$$\langle ? \rangle \quad \langle ?Q \rangle P \leftrightarrow Q \wedge P$$

$$\langle \cup \rangle \quad \langle \alpha \cup \beta \rangle P \leftrightarrow \langle \alpha \rangle P \vee \langle \beta \rangle P$$

$$\langle ; \rangle \quad \langle \alpha ; \beta \rangle P \leftrightarrow \langle \alpha \rangle \langle \beta \rangle P$$

$$\langle * \rangle \quad \langle \alpha^* \rangle P \leftrightarrow P \vee \langle \alpha \rangle \langle \alpha^* \rangle P$$

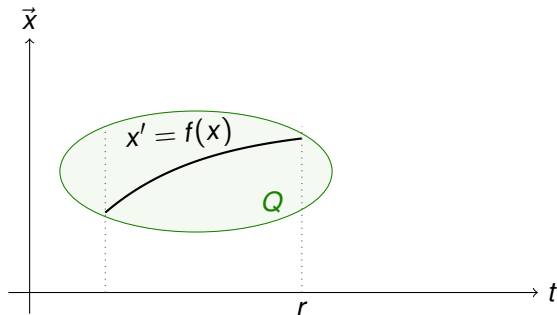
$$\langle ^d \rangle \quad \langle \alpha^d \rangle P \leftrightarrow \neg \langle \alpha \rangle \neg P$$

$$\text{M} \quad \frac{P \rightarrow Q}{\langle \alpha \rangle P \rightarrow \langle \alpha \rangle Q}$$

$$\text{FP} \quad \frac{P \vee \langle \alpha \rangle Q \rightarrow Q}{\langle \alpha^* \rangle P \rightarrow Q}$$

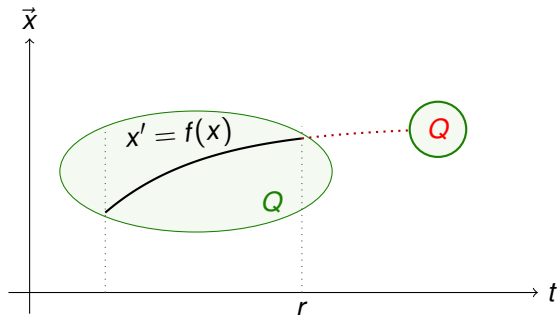
$$x' = f(x) \& Q$$

$$x' = f(x); ?(Q)$$



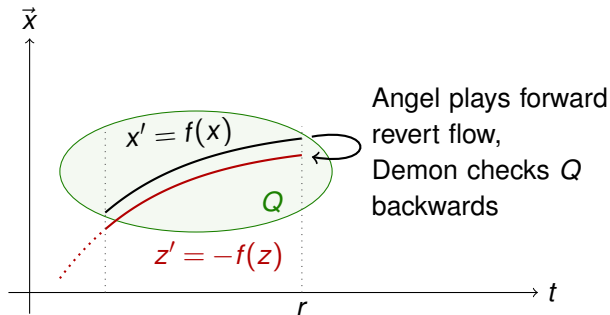
$$x' = f(x) \& Q$$

$$x' = f(x); ?(Q)$$



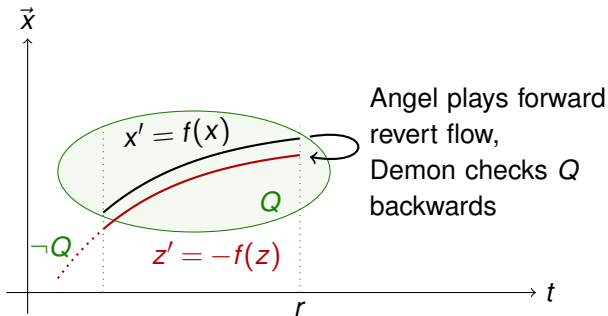
$$x' = f(x) \& Q$$

$$x' = f(x); (z := x; z' = -f(z))^d; ?(Q(z))$$



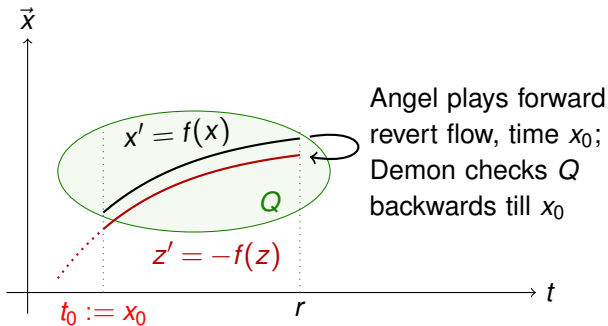
$$x' = f(x) \& Q$$

$$x' = f(x); (z := x; z' = -f(z))^d; ?(Q(z))$$



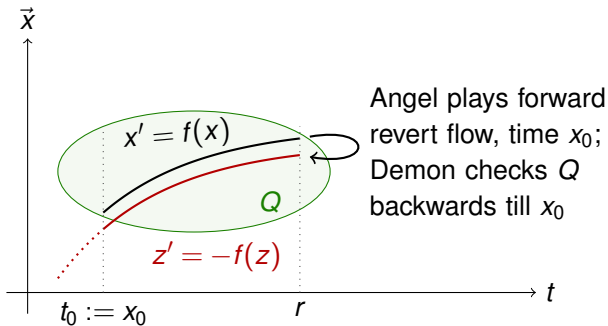


$$x' = f(x) \ \& \ Q \equiv t_0 := x_0; x' = f(x); (z := x; z' = -f(z))^d; ?(z_0 \geq t_0 \rightarrow Q(z))$$



# “There and Back Again” Game

$$x' = f(x) \ \& \ Q \equiv t_0 := x_0; x' = f(x); (z := x; z' = -f(z))^d; ?(z_0 \geq t_0 \rightarrow Q(z))$$



## Lemma

*Evolution domains definable by games*



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Definition (Hybrid game  $\alpha$ )

$\llbracket \cdot \rrbracket : \text{HG} \rightarrow (\wp(\mathcal{S}) \rightarrow \wp(\mathcal{S}))$

$$\begin{aligned} \zeta_{x:=e}(X) &= \{\omega \in \mathcal{S} : \omega_x^{\omega[e]} \in X\} \\ \zeta_{x'=f(x)}(X) &= \{\varphi(0) \in \mathcal{S} : \varphi(r) \in X \text{ for some } \varphi: [0, r] \rightarrow \mathcal{S}, \varphi \models x' = f(x)\} \\ \zeta_{?Q}(X) &= \llbracket Q \rrbracket \cap X \\ \zeta_{\alpha \cup \beta}(X) &= \zeta_{\alpha}(X) \cup \zeta_{\beta}(X) \\ \zeta_{\alpha; \beta}(X) &= \zeta_{\alpha}(\zeta_{\beta}(X)) \\ \zeta_{\alpha^*}(X) &= \bigcap \{Z \subseteq \mathcal{S} : X \cup \zeta_{\alpha}(Z) \subseteq Z\} \\ \zeta_{\alpha^d}(X) &= (\zeta_{\alpha}(X^c))^c \end{aligned}$$

Definition (dGL Formula  $P$ )

$\llbracket \cdot \rrbracket : \text{Fml} \rightarrow \wp(\mathcal{S})$

$$\begin{aligned} \llbracket e_1 \geq e_2 \rrbracket &= \{\omega \in \mathcal{S} : \omega[e_1] \geq \omega[e_2]\} \\ \llbracket \neg P \rrbracket &= (\llbracket P \rrbracket)^c \\ \llbracket P \wedge Q \rrbracket &= \llbracket P \rrbracket \cap \llbracket Q \rrbracket \\ \llbracket \langle \alpha \rangle P \rrbracket &= \zeta_{\alpha}(\llbracket P \rrbracket) \\ \llbracket [\alpha] P \rrbracket &= \delta_{\alpha}(\llbracket P \rrbracket) \end{aligned}$$

$$[\cdot] \quad [\alpha]P \leftrightarrow \neg \langle \alpha \rangle \neg P$$

$$\langle := \rangle \quad \langle x := e \rangle p(x) \leftrightarrow p(e)$$

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$$\langle * \rangle \quad \langle \alpha^* \rangle P \leftrightarrow P \vee \langle \alpha \rangle \langle \alpha^* \rangle P$$

$$\langle ^d \rangle \quad \langle \alpha^d \rangle P \leftrightarrow \neg \langle \alpha \rangle \neg P$$

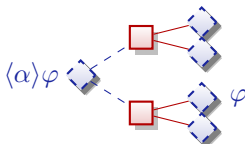
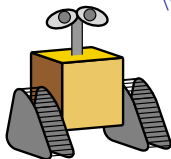
$$\text{M} \quad \frac{P \rightarrow Q}{\langle \alpha \rangle P \rightarrow \langle \alpha \rangle Q}$$

$$\text{FP} \quad \frac{P \vee \langle \alpha \rangle Q \rightarrow Q}{\langle \alpha^* \rangle P \rightarrow Q}$$



differential game logic

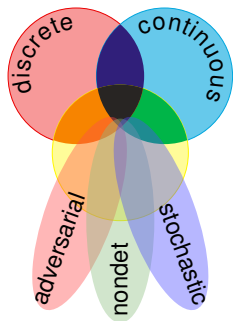
$$\text{dGL} = \text{GL} + \text{HG} = \text{dL} + \text{d}$$



- Axiomatics for hybrid games
- Proving winning strategies

Next chapter

- 1 Soundness
- 2 Proofs
- 3 Separations





André Platzer.

*Logical Foundations of Cyber-Physical Systems.*

Springer, Cham, 2018.

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André Platzer.

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*ACM Trans. Comput. Log.*, 17(1):1:1–1:51, 2015.

doi:10.1145/2817824.