15-819N/18-879L Logical Analysis of Hybrid Systems

Assignment 2 ($\sum 60$) due 02/24/11 in class

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Disclaimer: No solution will be accepted that comes without an **explanation**!

Exercise 1 Hybrid Systems in the Real World (5p)

1. Give three new examples of hybrid systems (not from class or Bestiarium collection). Explain where their discrete and continuous dynamics and their hybrid interactions come from.
2. Discuss safety-critical properties of these systems and explain to what extent hybridness is crucial in establishing or analyzing these properties faithfully.

Exercise 2 Semantics of First-Order Logic (15p)

In class, we have seen several important semantic concepts that have to do with the semantics of first-order logic (or logic in general):

- an interpretation I that assigns relations and functions to symbols
- variable assignment β that assigns objects to variables
- the valuation function $[\![\cdot]\!]_{I,\beta}$ that assigns a meaning to formulas.

In this exercise, we try to understand the relation between the two.

1. Given an interpretation I and variable assignment β, is there a valuation function $[\![\cdot]\!] : FOL \to \{true, false\}$ that corresponds to it? And in what sense does it correspond to it? Prove or disprove.
2. Given a valuation function $[\![\cdot]\!] : FOL \to \{true, false\}$ are there an interpretation I and a variable assignment β that corresponds to it? And in what sense does it correspond to it? Prove or disprove.
3. Is there a bijection between valuation functions and interpretations/assignments? Prove or disprove. What properties does it preserve?

Exercise 3 Logic and the Reals (16p)

1. Give a quantifier-free formula in $\mathrm{FOL}_{\mathbb{R}}$ of the same vocabulary that is equivalent to each of the following formulas and explain why they are equivalent.
   a) $\exists x\,(ax^2 + bx + c = 0)$
   b) $\forall x\,(y < x^2) \to \exists z\,(a = yz^2)$
   c) $\forall x\,(\exists y\,(ax < y^2) \to bx < z)$
   d) $\forall a\,\big((\exists x\ ax^3 + 2x^2 - 5x + 10 = 0) \ \Rightarrow\ \forall b \exists c \neg\exists x\ ax^2 + bx - c = 0\big)$
   e) $\forall a\,\big((\exists x\ ax^3 + 2x^2 - 5x + 10 = 0) \ \Rightarrow\ \forall c \exists b \neg\exists x\ ax^2 + bx + c = 0\big)$

2. Are the following two formulas (ϕ and ψ) equivalent in first-order real arithmetic?

$$\phi \equiv \forall x_0 \forall y_0 \forall u_0 \exists v_0 \forall b \exists p \forall b \forall T\ (b \geq 0 \land p > 0 \land T \geq 0 \land (x_0 - y_0)^2 + (u_0 - v_0)^2 \geq (p + 2bT)^2 \rightarrow$$
$$\forall d_2 \forall d_1 \forall x \forall u \forall e_2 \forall e_1 \forall y \forall v \forall t (t \leq T \land d_1^2 + d_2^2 \leq b^2 \land e_1^2 + e_2^2 \leq b^2$$
$$\land -t * b \leq x - x_0 \land x - x_0 \leq t * b \land -t * b \leq u - u_0 \land u - u_0 \leq t * b \land -t * b \leq y - y_0$$
$$\land\, y - y_0 \leq t * b \land -t * b \leq v - v_0 \land v - v_0 \leq t * b \rightarrow (x - y)^2 + (u - v)^2 \geq p^2)$$

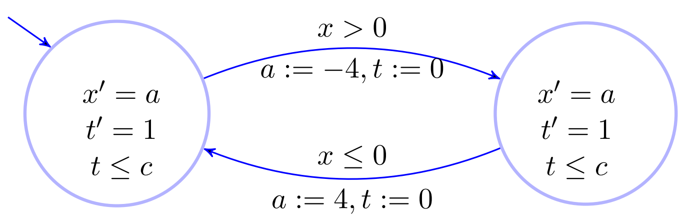
$$\psi \equiv \forall b \forall c \forall d \exists x\ ax^4 + 2bx^3 - 5cx^2 + dx - 10 = 0$$

Exercise 4 Definability and First-Order Real Arithmetic (18p)

1. Prove that the sets definable in first-order real arithmetic using $\exists, \forall, \land, \lor, \rightarrow, \leftrightarrow, \neg, >, =, \geq, \leq, <, \neq$ with polynomial terms are exactly the semialgebraic sets.
2. Are the logics $\mathrm{FOL}_{\mathbb{R}}[+, \cdot, =]$ and $\mathrm{FOL}_{\mathbb{R}}[+, \cdot, =, <, \leq]$ equally expressive, i.e., any formula in one logic can be stated equivalently using a formula in the other? Prove or disprove.
3. Can you give a quantifier free formula in $\mathrm{FOL}_{\mathbb{R}}[+, \cdot, =]$ that is equivalent to $\exists x\,(ax > b)$? If so, prove equivalence. Otherwise, explain why.
4. Show that division ($/$) is definable in first-order real arithmetic, i.e., give a translation from the logic $\mathrm{FOL}_{\mathbb{R}}[+, \cdot, /, =]$ to the logic $\mathrm{FOL}_{\mathbb{R}}[+, \cdot, =]$. Explain why your translation is correct. How does quantifier elimination work for this logic? Give a construction and explain.

Exercise 5 Hybrid Systems and $\mathbb{C}$ (6p)

Your advisor has implemented a quantifier elimination procedure for the first-order logic of complex arithmetic. He asks you to use it for verifying the following hybrid automaton and show that $|x|$ stays bounded by $4c$.



What do you do now? How do you accomplish this task? When you meet with him in the next meeting, what do you tell him?