15-819N/18-879L Logical Analysis of Hybrid Systems

Assignment 1 ($\sum 60$) due Tue 2/8/2011 in class

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Disclaimer: No solution will be accepted that comes without an **explanation**!

Exercise 1 Differential Equations (12p)

1. Find one solution of the following initial value problem (for $y_0, z_0 \in \mathbb{R}$), explain and prove why it is a solution:

$$\begin{bmatrix} x' = y & y' = z + u & z' = a & u' = b \\ x(0) = 0 & y(0) = y_0 & z(0) = z_0 & u(0) = 0 \end{bmatrix} \tag{1}$$

2. How many solutions does equation (1) have?
3. Show that every continuous function f on a domain $D \subseteq \mathbb{R} \times \mathbb{R}^n$ for which the derivative $\frac{\mathsf{d}f(x,y)}{\mathsf{d}y}$ exists and is bounded on D is Lipschitz-continuous. What is the corresponding Lipschitz-constant?
4. Show that continuously differentiable functions are locally Lipschitz-continuous.
5. Prove or disprove the following conjecture:
   Let f be a smooth function (i.e., continuously differentiable arbitrarily often) defined on $\mathbb{R} \times \mathbb{R}^n$. Then there is a solution of $x'(t) = f(t, x(t))$ that is defined on $\mathbb{R}$.

Exercise 2 First-Order Logic (8p)

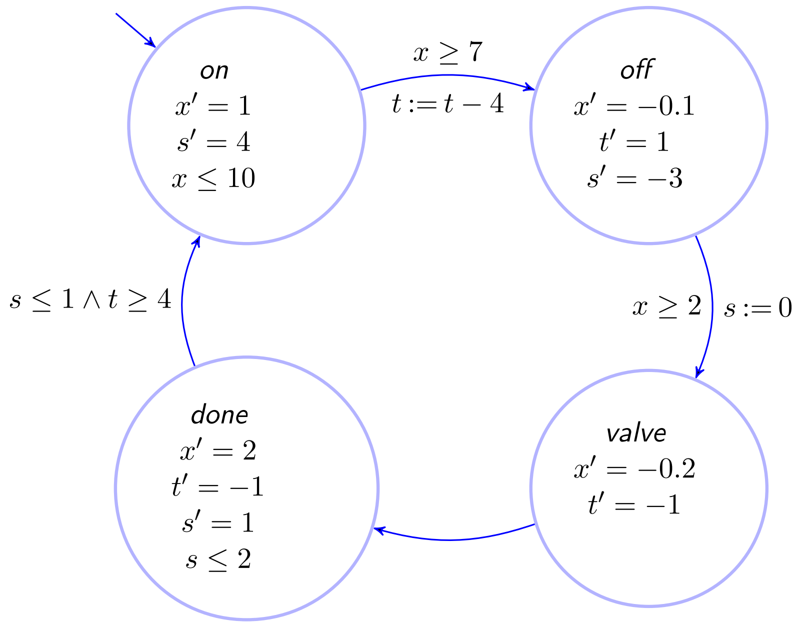
1. Are the following formulas valid/invalid/satisfiable/unsatisfiable?
   a) $(a \leftrightarrow b) \leftrightarrow (a \rightarrow \neg b) \wedge (\neg b \rightarrow a)$
   b) $(c \leftrightarrow d) \rightarrow ((a \leftrightarrow c) \leftrightarrow (a \leftrightarrow d))$
   c) $\neg a \rightarrow (a \rightarrow b)$
   d) $\forall a \forall b \forall c (r(a,b) \wedge r(b,c) \rightarrow r(a,c)) \wedge \forall a \neg r(a,a) \ \rightarrow\ \forall a \forall b (r(a,b) \rightarrow \neg r(b,a))$

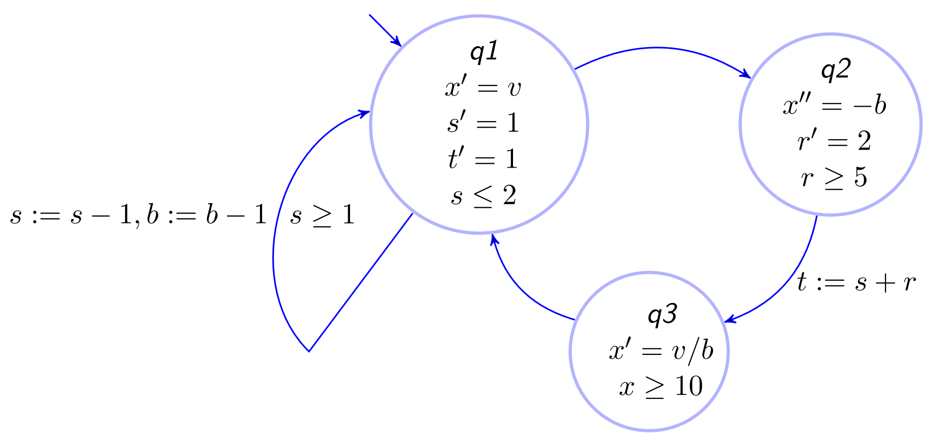
Exercise 3 Hybrid Automata (20p)

Consider the two automata below. Note that an edge where no guard has been specified means that the guard is "true". An edge where no reset has been specified means that the reset is $x := x, y := y, z := z$ and so on. For each of the automata below, please answer the following questions:

1. Is the automaton safe? That is:
   For the first automaton: does $0 \leq x$ always hold if it holds initially? Does $x \leq 12$ always hold? Explain why.
   For the second automaton: does $0 \leq x$ always hold if it starts at $x = s = t = 0, b = 10$? Explain why.

2. Is the automaton suitable for implementation? Explain why.
3. Is the automaton a well-designed automaton? List all problems and say if they are serious problems, not necessarily serious but red flags that violate sane design principles, or just minor issues.





Exercise 4 Hybrid Dynamic Systems (20p)

In class, we have discussed the general notion of dynamical systems $\varphi : T \times \mathcal{X} \to \mathcal{X}$. In particular, we have looked into the two specializations of discrete dynamical systems (where $T = \mathbb{N}$) and of continuous dynamical systems (where $T = \mathbb{R}_{\geq 0}$ and $\mathcal{X} = \mathbb{R}^n$).

Later, we have seen hybrid automata, which is one model for hybrid systems. How do these hybrid systems relate to (general) dynamical systems? Hybrid systems have both elements of discrete dynamical systems and of continuous dynamical systems. What should be the appropriate definition of hybrid dynamical systems?

1. Propose a time domain T for hybrid dynamical systems.
2. Propose a state space $\mathcal{X}$ for hybrid dynamical systems.
3. Propose a definition of hybrid dynamical systems (HDS) as a specialization of the general notion of dynamical systems.
4. Does your notion of hybrid dynamical systems satisfy the requirements of general dynamical systems? Prove or disprove.
5. Define an embedding $\iota : HA \to HDS$ of hybrid automata into your notion of hybrid dynamical systems that translates a hybrid automaton into a hybrid dynamical system (that is meant to do the same thing as the automaton).
6. The set of executions of a hybrid automaton A have been defined in class in terms of a labelled transition system (LTS). Is your definition of embeddings and hybrid dynamical systems compatible with the definitions of hybrid automaton executions? In other words, do the executions that the LTS of a hybrid automaton A allows coincide with the paths that its hybrid dynamical system $\iota(A)$ allows? Please explain whether that is the case.
7. Briefly discuss advantages and disadvantages of your design choices.