15-819/18-879: Logical Analysis of Hybrid Systems

05: Differential Equations

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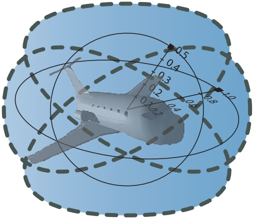

Outline

1. Differential Equations
   - Intuition
   - ODE & IVP
   - Examples
   - Peano Existence
   - Picard-Lindelöf Uniqueness

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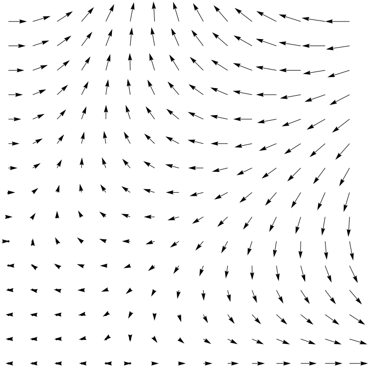
Relate continuously changing quantity and its rate of change (derivative)



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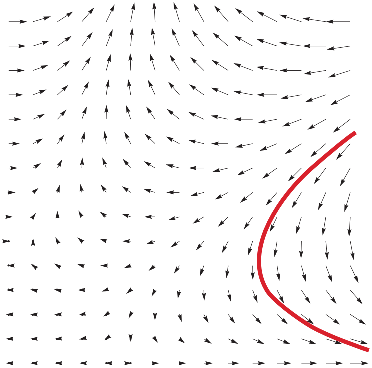

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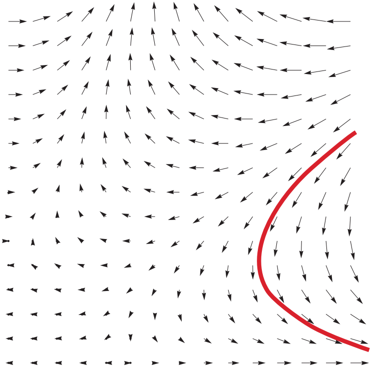

$$\left[\begin{array}{l} y'(t) = f(t, y) \\ y(t_0) = y_0 \end{array}\right]$$

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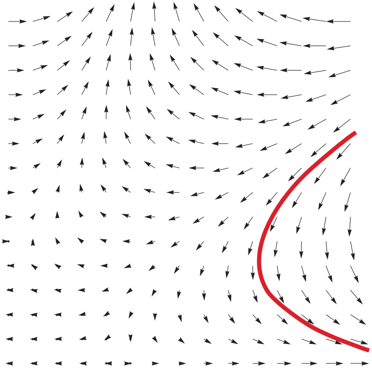
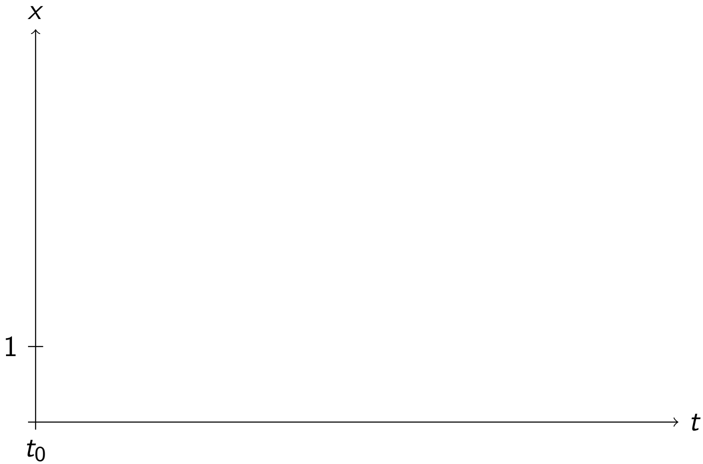


$$\left[\begin{array}{l} y'(t) = f(t,y) \\ y(t_0) = y_0 \end{array}\right]$$

in which direction y evolves as time t progresses
where y starts at time t_0

Intuition for Differential Equations



$$\begin{bmatrix} x'(t) = \frac{1}{4}x(t) \\ x(t_0) = 1 \end{bmatrix}$$

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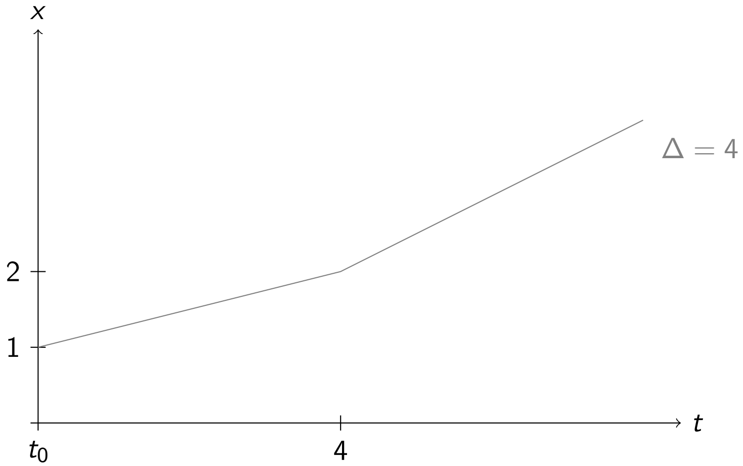

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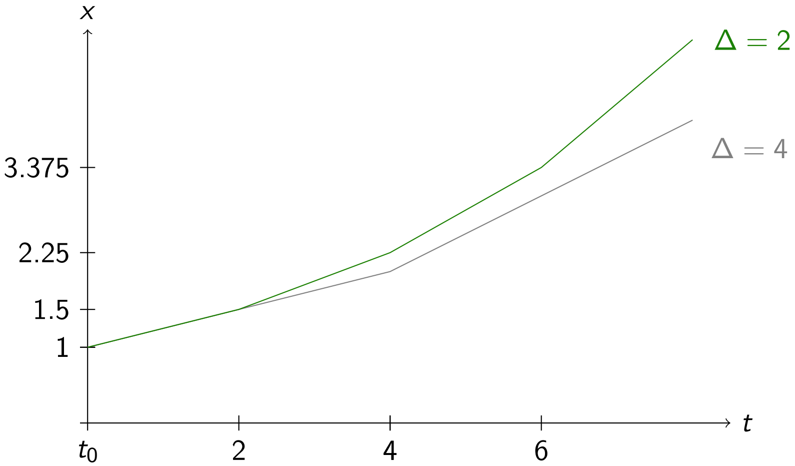

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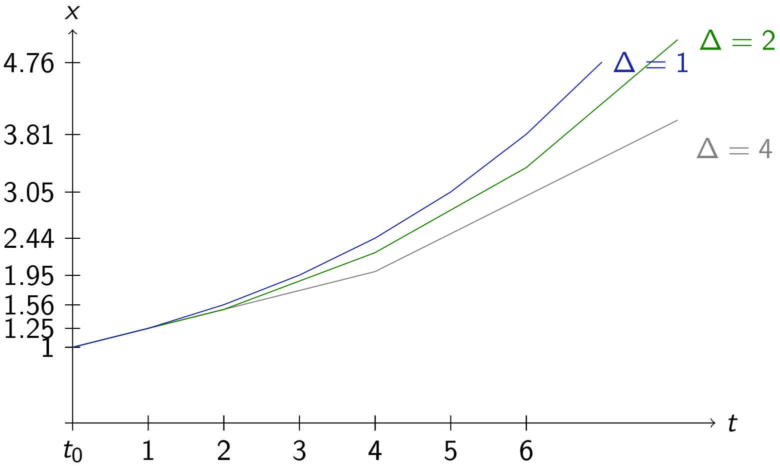

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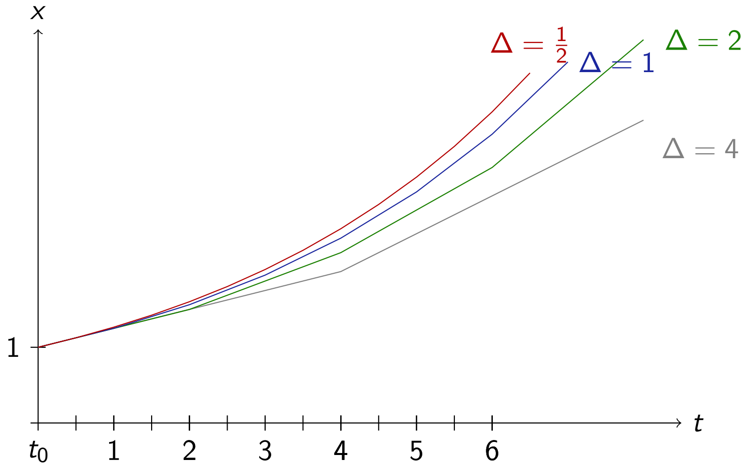

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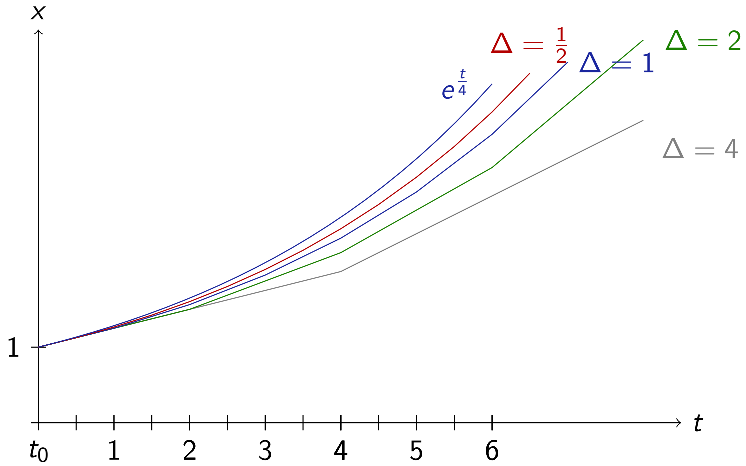

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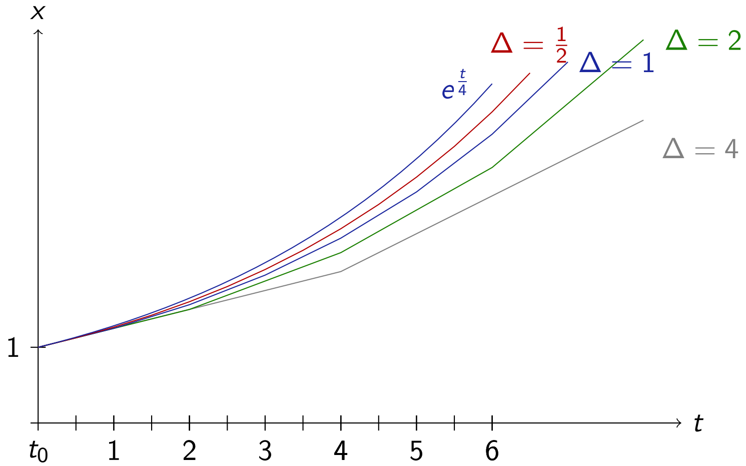

$$\begin{bmatrix} x'(t) = \frac{1}{4}x(t) \\ x(t_0) = 1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} x(t+\Delta) := x(t) + \frac{1}{4}x(t)\Delta \\ x(t_0) := 1 \end{bmatrix}$$

Differential Equations & Initial-Value Problems

Definition (Ordinary Differential Equation, ODE)

$f : D \to \mathbb{R}^n$ on domain $D \subseteq \mathbb{R} \times \mathbb{R}^n$ (i.e., open connected). Then $Y : I \to \mathbb{R}^n$ is *solution* of IVP

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If $f \in C(D, \mathbb{R}^n)$, then $Y \in C^1(I, \mathbb{R}^n)$.

ODE Examples

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Proof.

$$y'(t) = \frac{\mathrm{d}\frac{1}{1-t}}{\mathrm{d}t} = \frac{0 - \frac{\mathrm{d}(1-t)}{\mathrm{d}t}}{(1-t)^2} = \frac{1}{(1-t)^2} = y(t)^2$$

$$y(0) = \frac{1}{1-0} = 1$$

□

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What is a solution of the following IVP?

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ODE Examples

ODE	Solution
$x' = 1, x(0) = x_0$	$x(t) = x_0 + t$

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$x' = \frac{1}{x}, x(0) = 1$	$x(t) = \sqrt{1 + 2t} \dots$

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$x' = x^2 + x^4$	???

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$x' = x^2 + x^4$	???
$x'(t) = e^{t^2}$	non-elementary

▸ ATC ▸ HA

Existence: Peano

Theorem (Existence theorem of Peano'1890)

$f \in C(D, \mathbb{R}^n)$ on open, connected domain $D \subseteq \mathbb{R} \times \mathbb{R}^n$ with $(x_0, y_0) \in D$. Then, IVP has a solution. Further, every solution can be continued arbitrarily close to the border of D.

Example (Solvable)

$$\left[\begin{array}{r} y' = \sqrt{|y|} \\ y(0) = 0 \end{array}\right]$$

$$\left[\begin{array}{r} y'(x) = 3x^2 y - \frac{1}{y}\sin x \cos y \\ y(0) = 1 \end{array}\right]$$

Existence: Peano

Example (Solvable but not uniquely)

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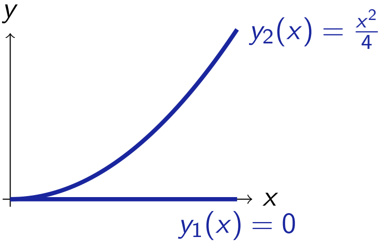

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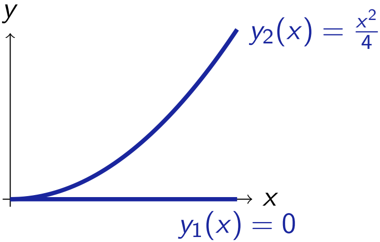


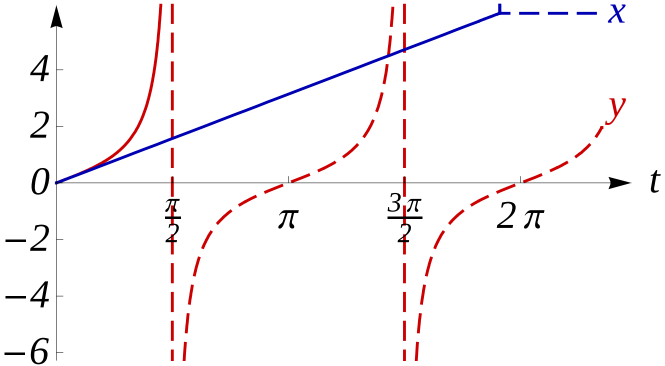
Example (Solvable but not uniquely)

$$\begin{bmatrix} y' = \sqrt[3]{y} \\ y(0) = 0 \end{bmatrix} \rightsquigarrow \quad y(t) = \left(\frac{2}{3}t\right)^{\frac{3}{2}} \quad \text{or} \quad y(t) = 0$$

Existence: Peano

Example (Continuable but limited)

$$\begin{bmatrix} y' = 1 + y^2 \\ y(0) = 0 \end{bmatrix}$$



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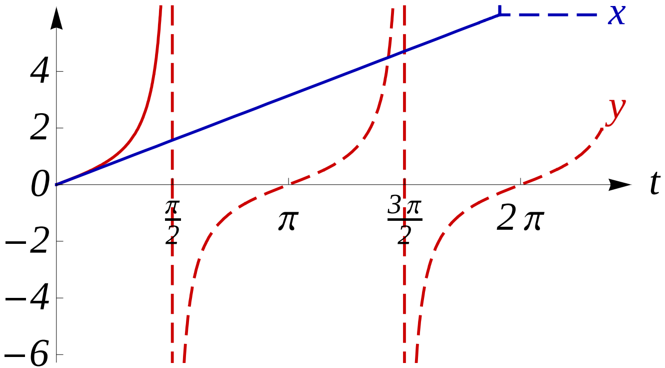

Lipschitz-Continuity

Definition (Lipschitz-continuous)

$f : D \to \mathbb{R}^n$ with $D \subseteq \mathbb{R} \times \mathbb{R}^n$ is *Lipschitz-continuous* for y iff there is an $L \in \mathbb{R}$ such that for all $(x, y), (x, \bar{y}) \in D$:

$$\|f(x, y) - f(x, \bar{y})\| \leq L\|y - \bar{y}\|$$

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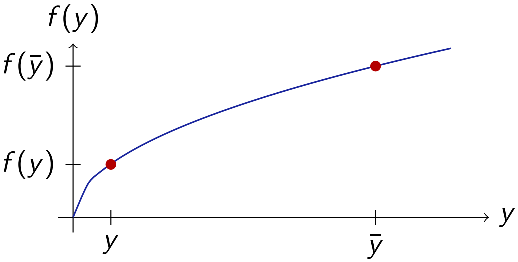

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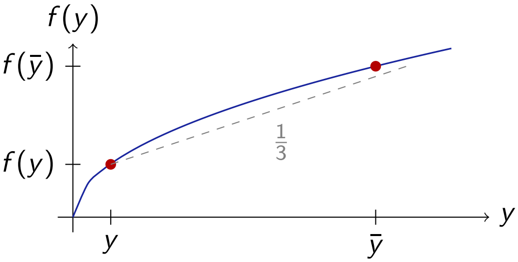

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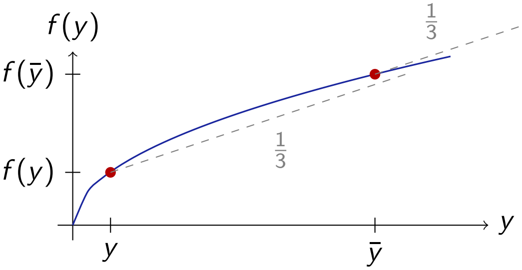

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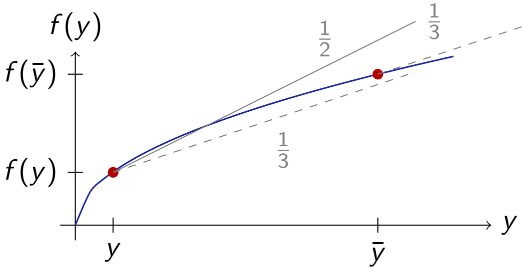

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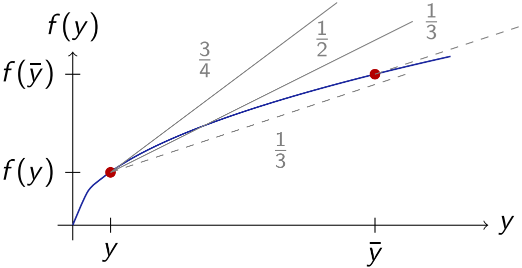

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$f \in C^1(D, \mathbb{R}^n)$ then locally Lipschitz-continuous, as f' locally bounded.

Theorem (Uniqueness theorem of Picard-Lindelöf'1894)

In addition to Peano premisses, let f be locally Lipschitz-continuous for y (e.g. $f \in C^1(D, \mathbb{R}^n)$). Then, there is a unique solution of IVP.

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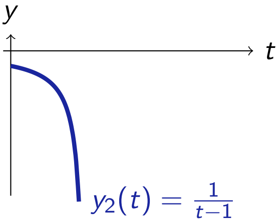
Proposition (Global uniqueness theorem of Picard-Lindelöf)

$f \in C([0,a] \times \mathbb{R}^n, \mathbb{R}^n)$ Lipschitz-continuous for y. Then, there is a unique solution of IVP on $[0,a]$.

Example (Unique solution but not global)

$$\left[\begin{array}{r} y' = -y^2 \\ y(0) = -1 \end{array}\right]$$



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