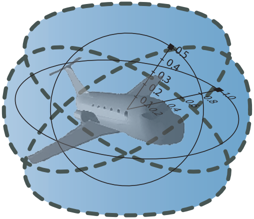
15-819/18-879: Hybrid Systems Analysis & Theorem Proving

03: Numerical versus Symbolic Analysis

André Platzer

aplatzer@cs.cmu.edu
Carnegie Mellon University, Pittsburgh, PA



Outline

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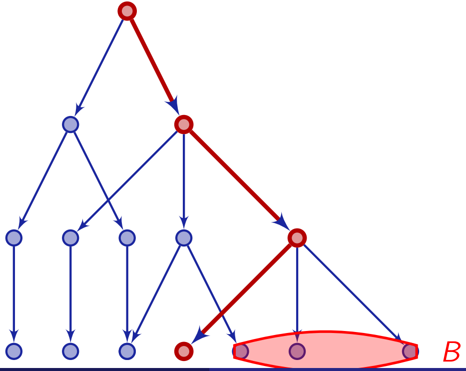
Model Checking in a Nutshell

Definition (Model Checking Problem)

Given initial states $Q_0 \subseteq Q$ and bad states $B \subseteq Q$ for a transition system, check whether there is a trace from some $q_0 \in Q_0$ to some $q_b \in B$.



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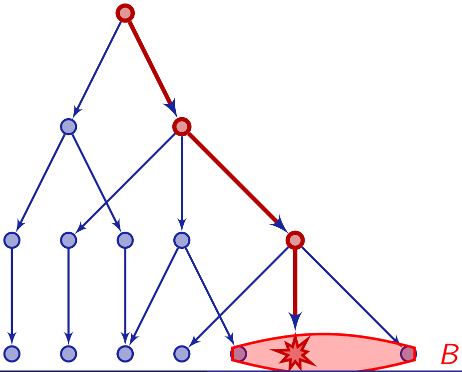
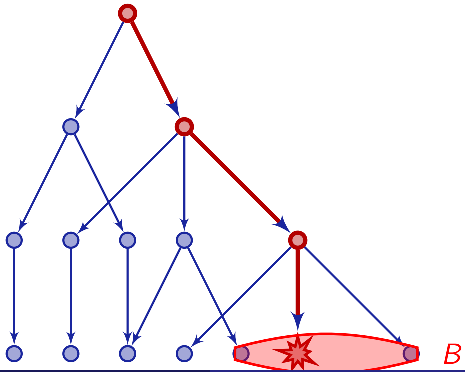


Model Checking in a Nutshell

Definition (Image Computation)

$$Post_A(Y) \;:=\; \{q^+ \in Q \;:\; q \xrightarrow{a} q^+ \text{ for some } q \in Y, a \in A\}$$



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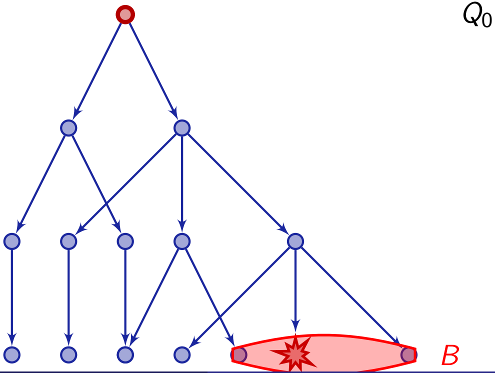

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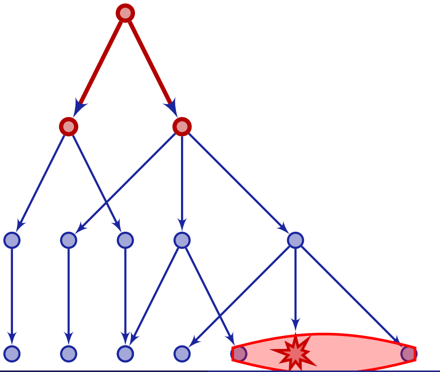

Q_0

$Post_A(Q_0)$

$Q_1 = Post_A(Q_0)$

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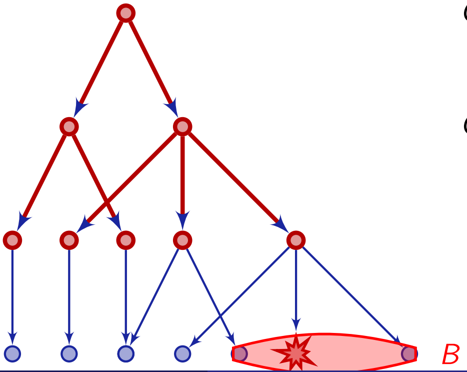

$$\begin{array}{l} Q_0 \\ \quad\downarrow Post_A(Q_0) \\ Q_1 = Post_A(Q_0) \\ \quad\downarrow Post_A(Q_1) \\ Q_2 = Post_A^2(Q_0) \end{array}$$

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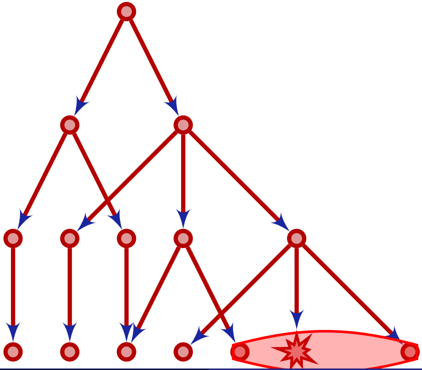
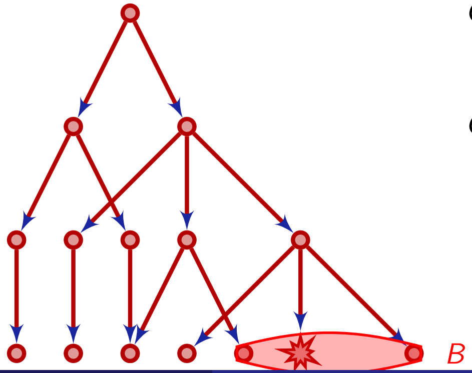


$$Q_0 \xrightarrow{Post_A(Q_0)} Q_1 = Post_A(Q_0) \xrightarrow{Post_A(Q_1)} Q_2 = Post_A^2(Q_0) \xrightarrow{Post_A(Q_2)} Q_3 = Post_A^3(Q_0)$$

Model Checking in a Nutshell

Definition (Image Computation)

$$
\begin{aligned}
Post_A(Y) &:= \{q^+ \in Q \;:\; q \xrightarrow{a} q^+ \text{ for some } q \in Y, a \in A\} \\
Post_A^*(Y) &:= \mu Z.(Y \cup Z \cup Post_A(Z))
\end{aligned}
$$



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Can we use this for hybrid systems?

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Proposition (Decision)

For finite-state systems, this naïve MC algorithm gives a (slow) decision procedure.

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For (computable) countably infinite-state systems, naïve MC gives a (slow) semidecision procedure.

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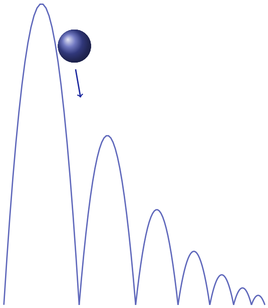
Hybrid systems have uncountable state spaces

(Uncountably) infinite state spaces require extra care

PHAVer Tool Example



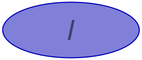
PHAVer Tool Example

```
g:=1; // constant for gravity
automaton bouncing_ball
contr_var: x, v;
synclabs: jump;
loc state:
 while x>=0 & x<=10 & v<=10 & v>=-10 wait {x'==v&v'==-g};
 // transitions
 when x==0&v<0 sync jump do {v'==-v*0.5&x'==x} goto state;
 when x==0&-0.1<v&v<0.1 sync jump do {v'==v&x'==x} goto frz
loc frz:
    while x>=0 & x<=10 & v<=10 & v>=-10 wait {x'==0&v'==0};
initially: state & x==2 & v==0;
end

reg=bouncing_ball.reachable;
// reg=bouncing_ball.is_reachable(frz);
reg.print("out_reach",2); // output reachable set
```

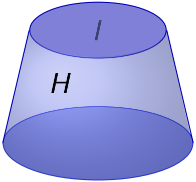
What analysis is doable at all?

Image Computation in Hybrid Systems



- Analyse image computation problem in hybrid systems
- Approximation refinement techniques and their limits
- Numerical versus symbolic algorithms
  $1.421 \in \mathbb{Q}$ versus $x^2 + 2xy$ term computations

Image Computation in Hybrid Systems



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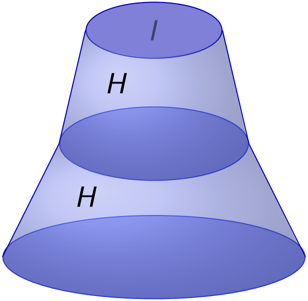

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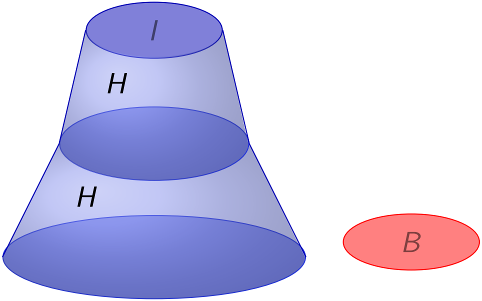

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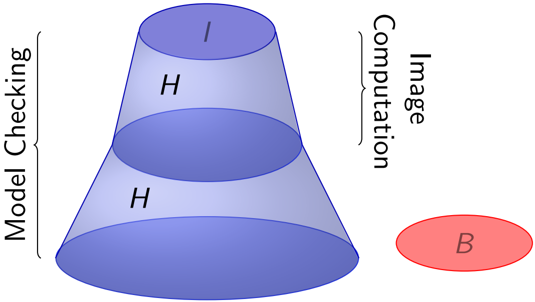

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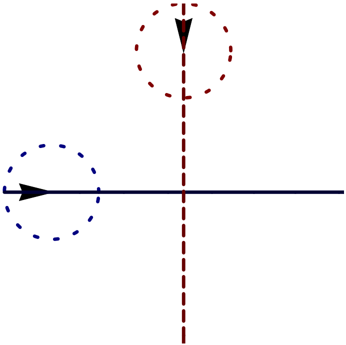
Air Traffic Management



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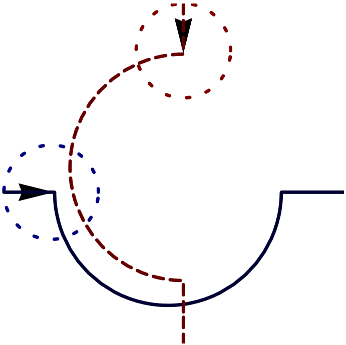

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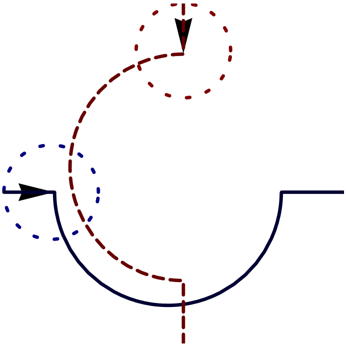
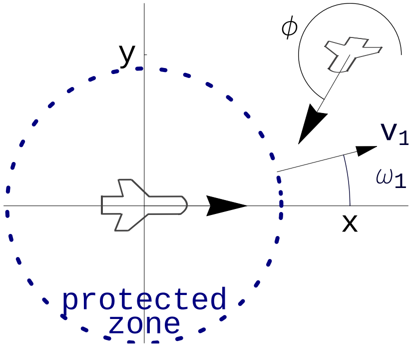
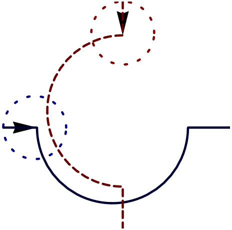
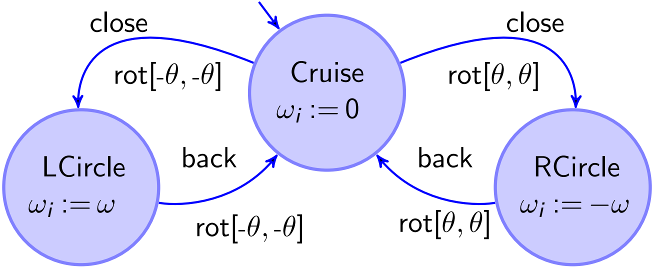




ATM: Roundabout Maneuver Automaton



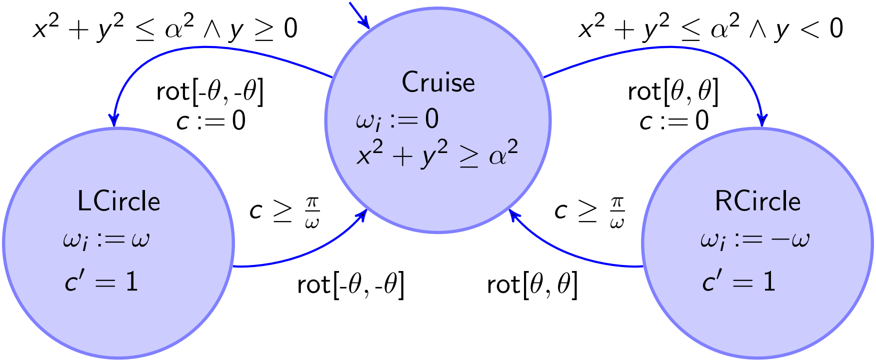
$$\begin{bmatrix} x' & = -v_1 & +v_2\cos\phi & +\omega_1 y \\ y' & = & v_2\sin\phi & -\omega_1 x \\ \phi' & = & \omega_2 & -\omega_1 \end{bmatrix}$$



Details

ATM: Roundabout Maneuver Automaton

$$\begin{bmatrix} x' = -v_1 + v_2 \cos\phi + \omega_1 y \\ y' = v_2 \sin\phi - \omega_1 x \\ \phi' = \omega_2 - \omega_1 \end{bmatrix}$$



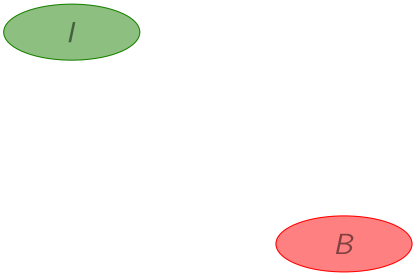
Return

Outline

AMC: Approximation Refinement Model Checking

AMC(B reachable from I in H):

1. $A :=$ approx(H) uniformly
2. blur by uniform approximation error $+\epsilon$
3. check(B reachable from I in $A + \epsilon$)
4. B not reachable $\Rightarrow$ H safe



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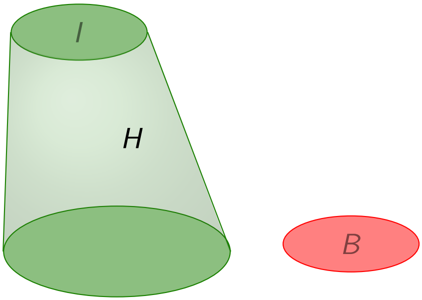

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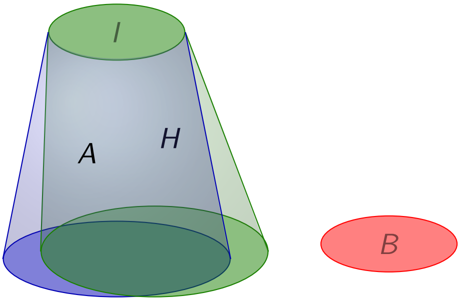

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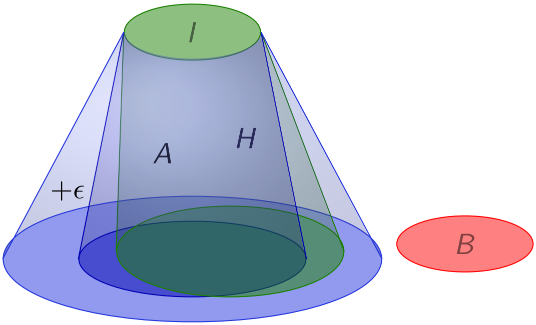

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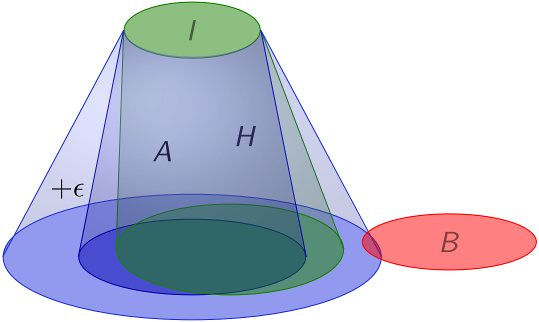

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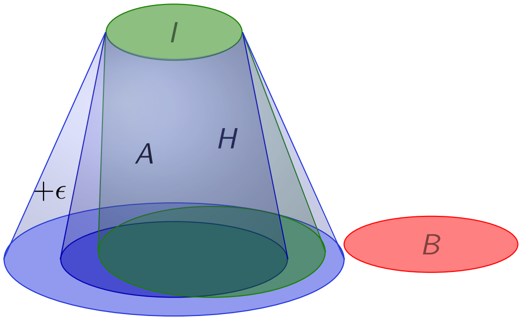

AMC: Exact Image Computation

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Proposition

check and blur can be implemented for

- *I and B semialgebraic (propositional combinations of $p \geq 0$)*
- *A with polynomial flows over $\mathbb{R}$*
- *+Piecewise definitions*
- *+Rational extensions (e.g. multivariate rational splines)*

AMC: Image Approximation

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Proposition

approx exists for all uniform errors $\epsilon > 0$ when

- *using polynomials to build A*
- *Flows $\varphi \in C(D, \mathbb{R}^n)$ of H*
- *$D \subset \mathbb{R} \times \mathbb{R}^n$ compact closure of an open set*

Approximation can solve problems without effective exact solution

Weierstraßian Flow Approximation

Proposition

approx exists for all uniform errors $\varepsilon > 0$:

- $\varphi \in C(D, \mathbb{R}^n)$ *on compact closure* $D \subset \mathbb{R} \times \mathbb{R}^n$ *of an open set*

$\Rightarrow$ $\forall \varepsilon > 0 \ \exists p \in \mathbb{R}[t, x_1, \ldots, x_n]^n \, \forall Y \subseteq \mathbb{R}^n$

$$Post_{\varphi|_D}(Y) \ \subseteq \ \mathcal{U}_\varepsilon(Post_{p|_D}(Y))$$

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$$Post_{\varphi|_D}(Y) \ \subseteq \ \mathcal{U}_\varepsilon(Post_{p|_D}(Y))$$

Where $\mathcal{U}_\varepsilon(Y)$ is the ε ball around set Y:

$$\mathcal{U}_\varepsilon(Y) := \{x \ : \ \|x - y\| < \varepsilon \text{ for some } y \in Y\}$$

Weierstraßian Flow Approximation

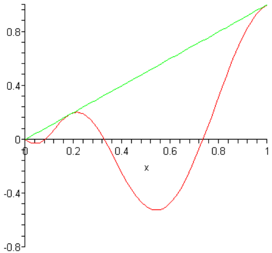
Theorem (Stone-Weierstraß Approximation)

Polynomials uniformly approximate cont. functions on compact domains:

- $\varphi \in C(D, \mathbb{R}^n)$ *on compact domain* $D \subset \mathbb{R} \times \mathbb{R}^n$

$\Rightarrow$ $\forall \varepsilon > 0 \ \exists p \in \mathbb{R}[t, x_1, \dots, x_n]^n \, \forall (t, x) \in D$

$$\|\varphi(t; x) - p(t, x)\| < \varepsilon$$



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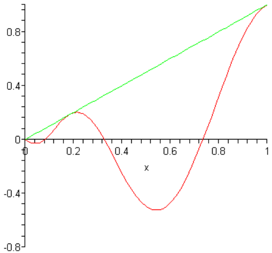

Existence of solutions may be computationally insufficient

Exact Image Computation: Polynomials

Proposition

check and blur can be implemented for

- I, D, B *definable in* $\mathrm{FOL}_{\mathbb{R}}$, *i.e., semialgebraic*
- A *with polynomial flows over* $\mathbb{R}$

Exact Image Computation: Polynomials

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check and *blur* can be implemented for

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Proof.

Inductive consequence of $\mathcal{U}_{\varepsilon}(Post_{p|_D}(Y))$ being definable in $\mathrm{FOL}_{\mathbb{R}}$, thus being decidable: Let Y, D be defined by $\mathrm{FOL}_{\mathbb{R}}$ formulas F_Y, F_D.

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1. "$z \in Post_{p|_D}(Y)$" is definable as:

$$\exists x\, \exists t{\geq}0\, (F_Y(x) \wedge \forall 0{\leq}s{\leq}t\, F_D(s, p(s,x)) \wedge z = p(t,x))$$

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2. "$z \in \mathcal{U}_\varepsilon(Y)$" is definable in $\mathrm{FOL}_{\mathbb{R}}$, thus decidable:

$$\exists y\, (F_Y y \land \sum_{i=1}^{n} (y_i - z_i)^2 < \varepsilon^2)$$

Exact Image Computation: Piecewise Polynomials

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$s : D \longrightarrow \mathbb{R}$ consists of polynomial pieces $p_i : D_i \longrightarrow \mathbb{R}$ for disjoint definable D_i with $D = D_1 \cup \ldots \cup D_n$. Then, we define $\mathcal{U}_\varepsilon(Post_{s|_D}(Y))$:

Exact Image Computation: Piecewise Polynomials

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1. "$z = s(x)$" is definable:

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Exact Image Computation: Piecewise Polynomials

Proposition

check and *blur* can be implemented for

- I, D, B definable in $\mathrm{FOL}_{\mathbb{R}}$, i.e., semialgebraic
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$$Post_{s|_D}(Y) = \bigcup_{i=1}^{n} Post_{p_i|_{D_i}}(Y) \quad \text{and} \quad \mathcal{U}_\varepsilon(X \cup Y) = \mathcal{U}_\varepsilon(X) \cup \mathcal{U}_\varepsilon(Y)$$

Exact Image Computation: Rational Functions

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Tarski's theorem extends to rational functions.

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\begin{aligned}
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p(x)/q(x) > 0 &\equiv (p(x) > 0 \land q(x) > 0) \lor (p(x) < 0 \land q(x) < 0)
\end{aligned}
$$

□

Logical foundation for effective image computation operations

Outline

Bounded Flow Approximation

Proposition (Effective Weierstraß approximation)

- *Flows* $\varphi \in C^1(D, \mathbb{R}^n)$ *arbitrarily effective, D effective*
- *Bounds* $b := \max_{x \in D} \|\varphi'(x)\|$

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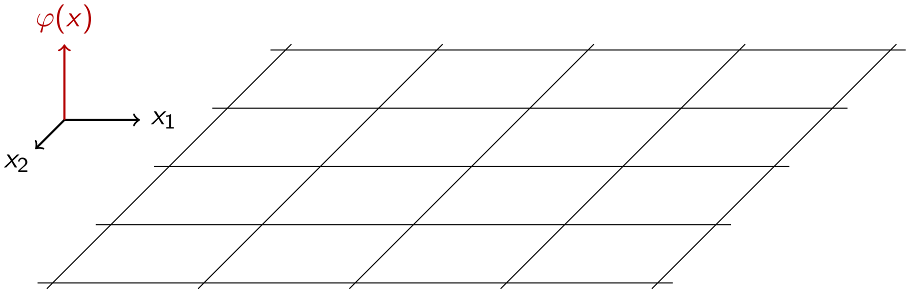

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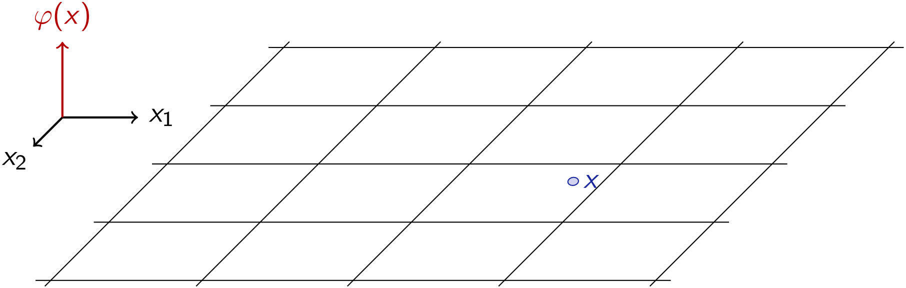

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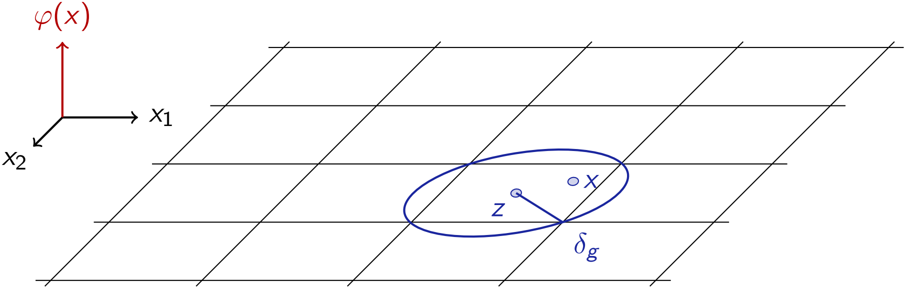

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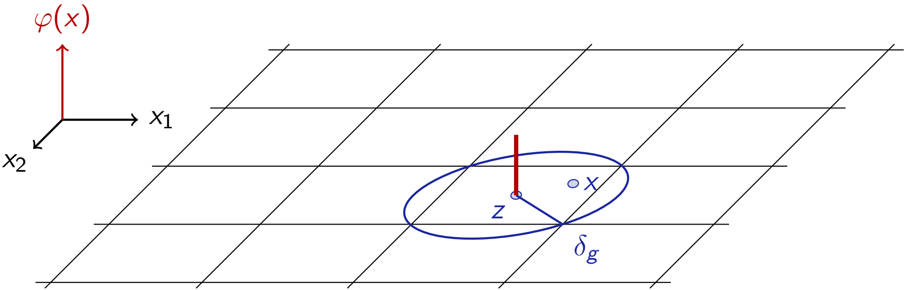

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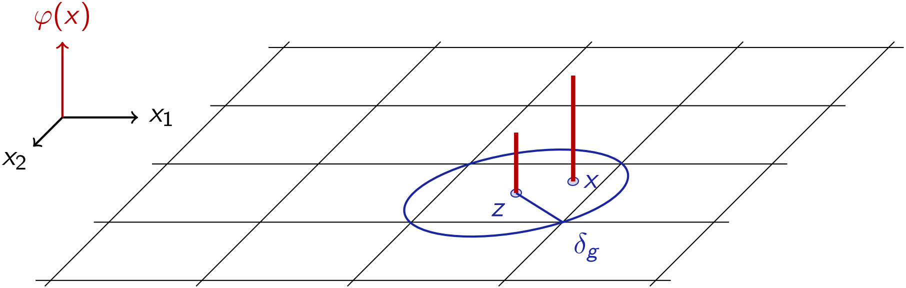

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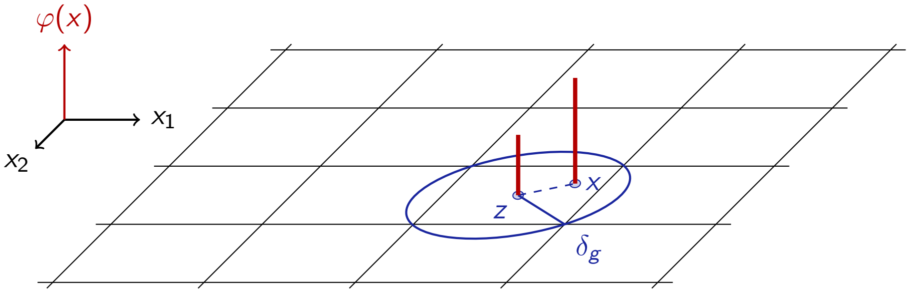

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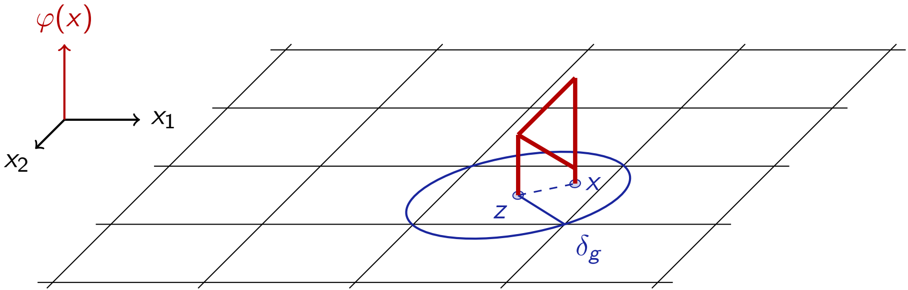

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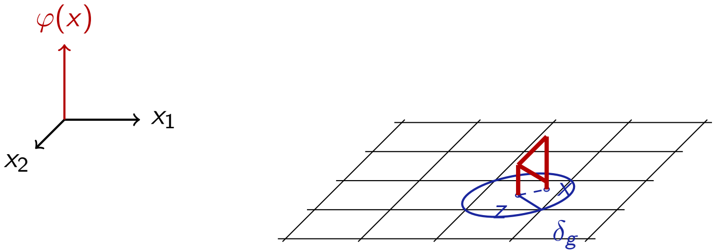

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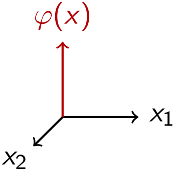

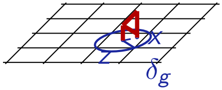

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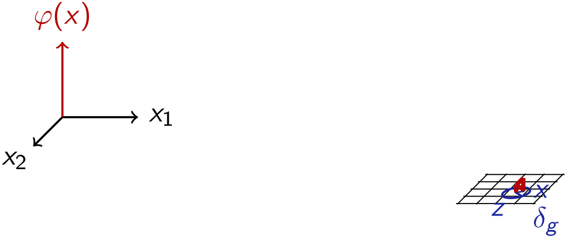

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- Approximate by step functions $f_{\delta_c}(z)$ on $\pm\delta_g/2$ hypercube around z.

Bounded Flow Approximation

Proposition (Effective Weierstraß approximation)

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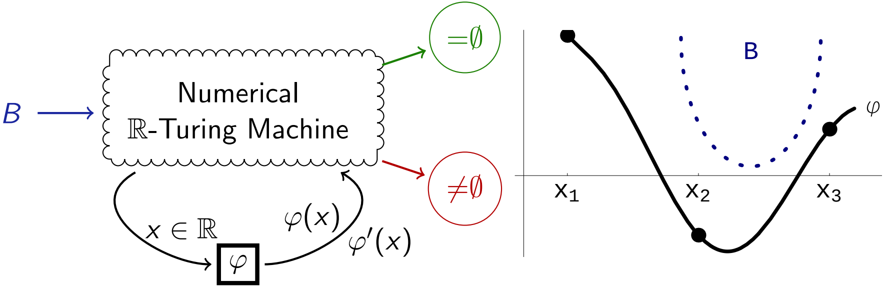
Only need to find the bound b . . .

Finding bounds is easier than verification?

Continuous Image Computation



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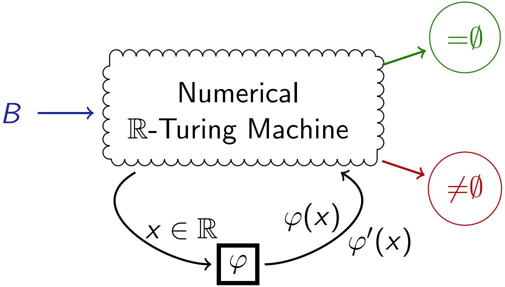

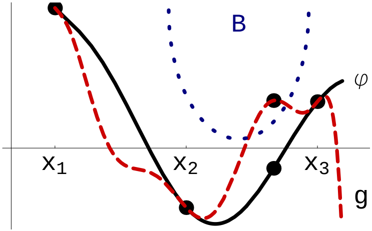

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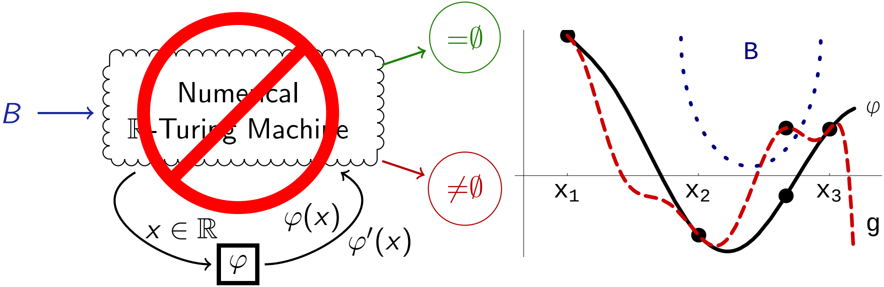

Proposition (Image computation undecidable for...)

- *arbitrarily effective flow* $\varphi \in C^k(D \subseteq \mathbb{R}^n, \mathbb{R}^m)$; *D, B effective*
- *tolerate error* $\epsilon > 0$ *in decisions*

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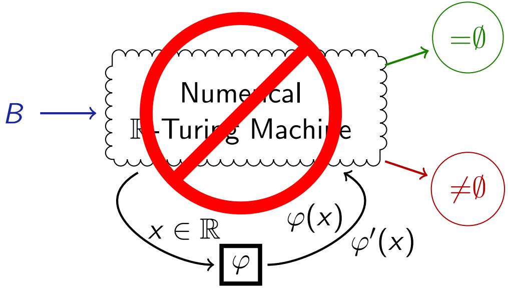

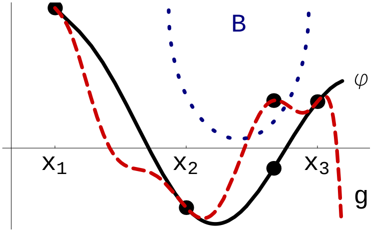

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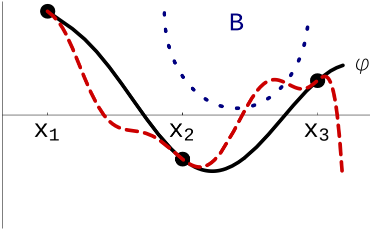
- φ *smooth polynomial function with* $\mathbb{Q}$*-coefficients*

Probabilistic Model Checking



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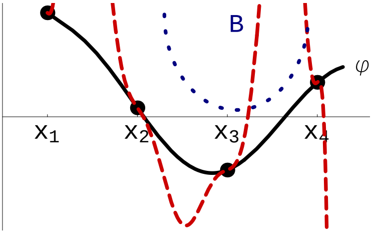

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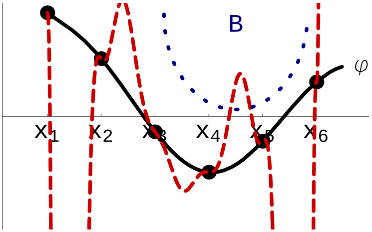

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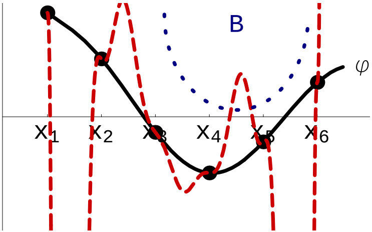
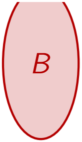


Proposition

- $P(\|\varphi'\|_\infty > b) \to 0$ *as* $b \to \infty$
- φ *evaluated on finite subset* $X = \{x_i\}$ *of open or compact* D

$\Rightarrow$ $P(\mathit{decision\ correct}) \to 1$ *as* $\|d(\cdot, X)\|_\infty \to 0$

Probabilistic Model Checking: Proof Illustration



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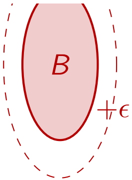

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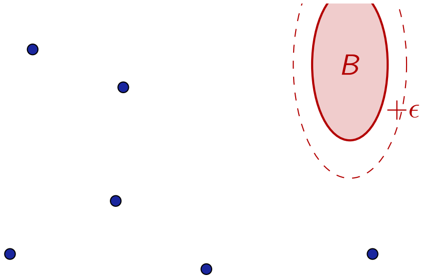

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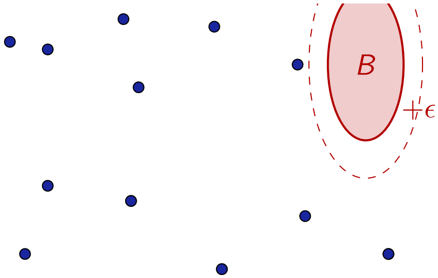

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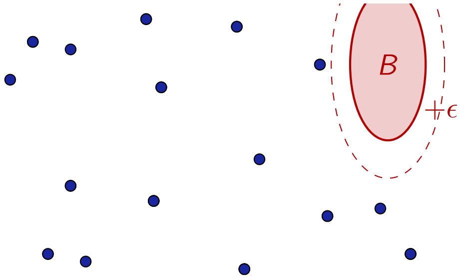

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Proof. (for problem with tolerance $\epsilon > 0$).

- Let $X \subseteq D$ set of points where φ is evaluated and $\nu := \|d(\cdot, X)\|_\infty$.

$\square$

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- Suppose $\exists x \in D \ \varphi(x) \in B$. Let $x_i \in X$ have smallest distance to x.

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- Suppose $\exists x \in D \ \ \varphi(x) \in B$. Let $x_i \in X$ have smallest distance to x.
- Assume $S[x, x_i] \subseteq D$ (use a $\nu > 0$ with $\mathcal{U}_\nu(x) \subseteq D$ as D open).

□

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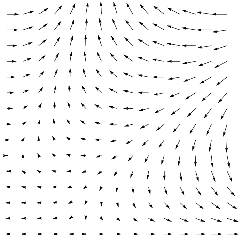
- Because $P(\|\varphi'\|_\infty \geq \frac{\epsilon}{\nu}) \to 0$ for $\nu \to 0$ by premise, as ϵ is a constant independent of ν and $\frac{\epsilon}{\nu} \to \infty$ as $\nu \to 0$.

$\square$

Differential Flow Approximation



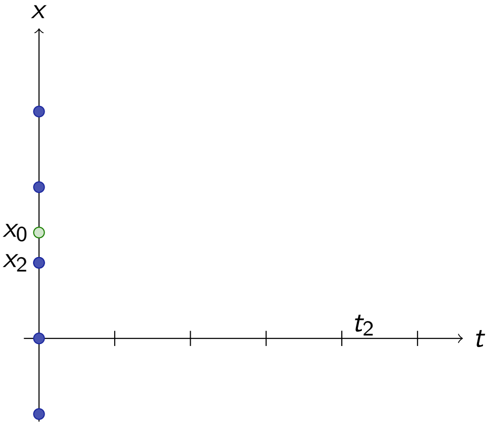
φ solves
$x'(t) = f(t, x)$

Proposition

- *Flow φ is solution of $x'(t) = f(t, x)$*
- $f \in C([a, b] \times \mathbb{R}^n, \mathbb{R}^n)$
- *ℓ-Lipschitz-continuous:* $\|f(t, x_1) - f(t, x_2)\| \leq \ell \|x_1 - x_2\|$

$\Rightarrow$ *Continuous image computation decidable*

Differential Flow Approximation: Proof Illustration



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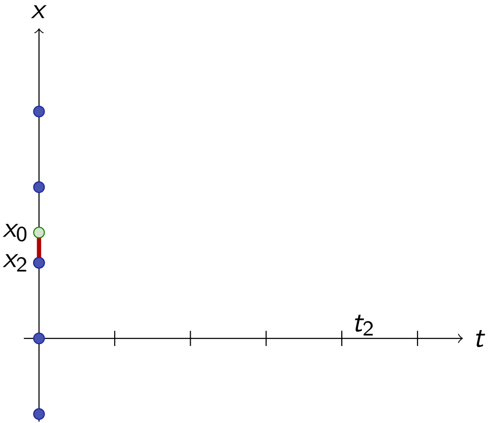

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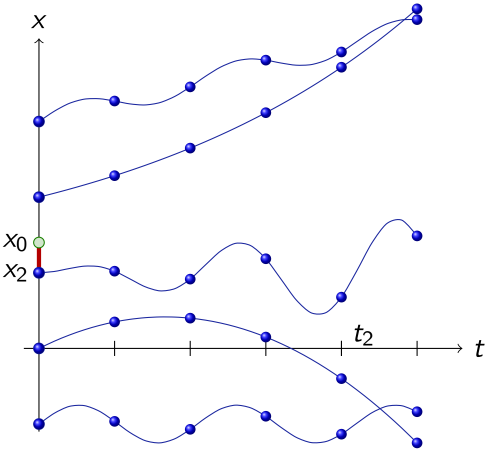

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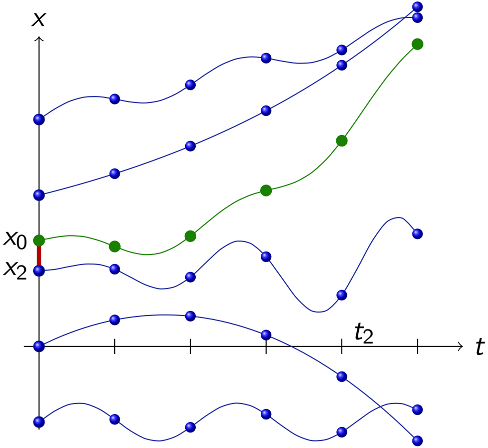

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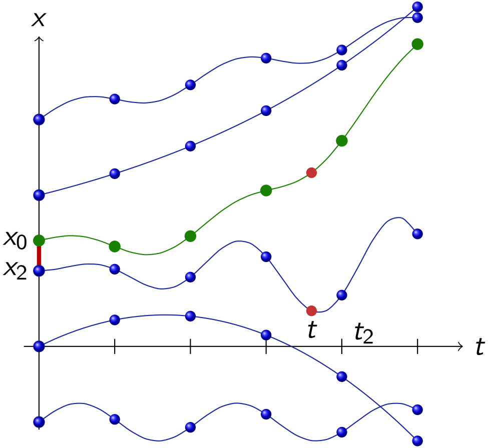

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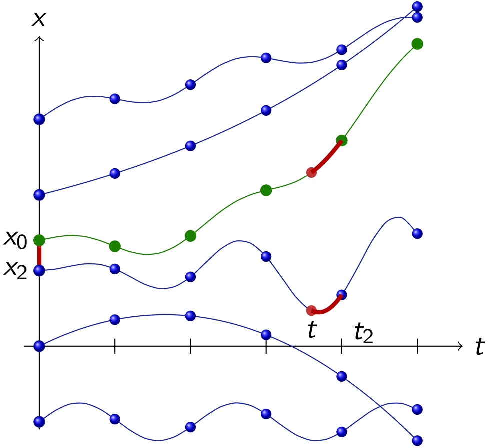

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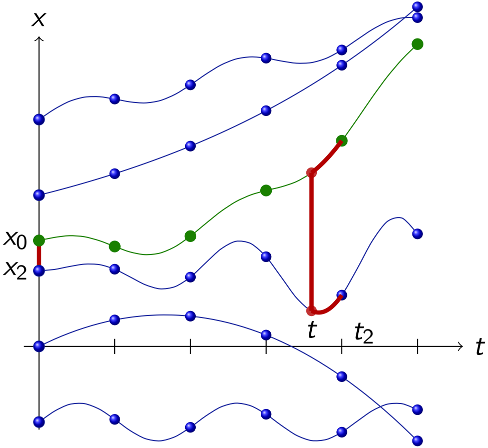

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\|\varphi(t; x_0) - \varphi(t_2; x_2)\| &\leq \|\varphi(t; x_0) - \varphi(t; x_2)\| + \|\varphi(t; x_2) - \varphi(t_2; x_2)\| \\
&\leq e^{\ell|t-t_0|}\|x_0 - x_2\| + \|\varphi'(\xi; x_2)\| \cdot |t - t_2| \\
&= \underbrace{e^{\ell|t-t_0|}}_{\text{bounded}}\|x_0 - x_2\| + \underbrace{\|f(\xi, \varphi(\xi; x_2))\|}_{\text{bounded}} \cdot |t - t_2|
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by corollary of Picard-Lindelöf and MVT with $\xi \in (t, t_2)$.

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Approximation Errors

Exponential terms in approximation error computations are bad

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but tight!

Example

$$x' = \ell x$$

is ℓ-Lipschitz-continuous with unique global solution$\varphi(t; x_0) = x_0 e^{\ell(t-t_0)}$

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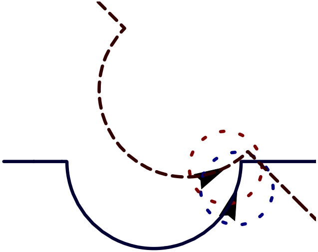
Outline

Experiments with Roundabout ATC

Counterexamples with distances ≈0.0016mi after 3 refinements

in absolute coords



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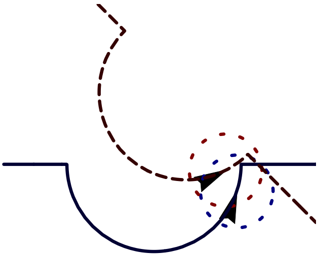
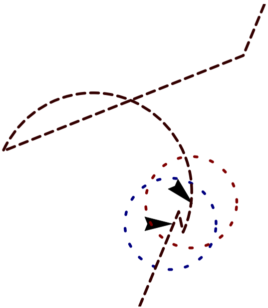


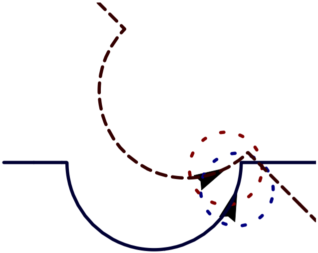
relative coords



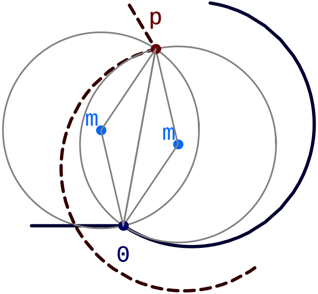
Experiments with Tangential Roundabout ATC

Solution: adaptively choose rotation using tangential construction

classical



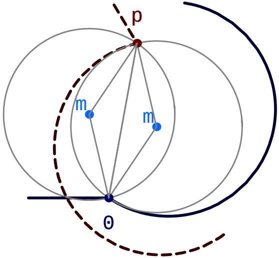
tangential



No more counterexamples found

Experimental Results: Tangential Roundabout ATM

$$\alpha^2 = \|m - 0\|^2$$
$$\alpha^2 = \|m - p\|^2$$
$$\gamma_1 = \angle(m - 0)$$
$$\gamma_2 = \angle(m - p)$$



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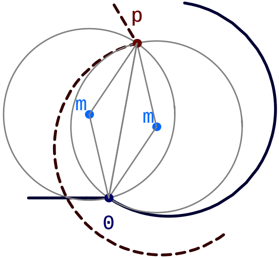

Solutions for θ_j using any $k, \ell \in \{1, 2\}$:

$$\angle\left((-1)^{j+1}\frac{x^3 + xy^2 + (-1)^{j+k}\imath\sqrt{x^2(x^2+y^2)(4\alpha^2 - x^2 - y^2)}}{x(x - \imath y)}\right) + (-1)^{\ell}\frac{\pi}{2}$$

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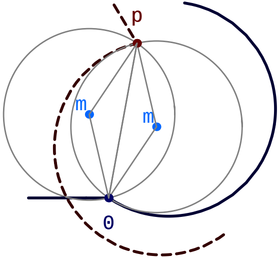

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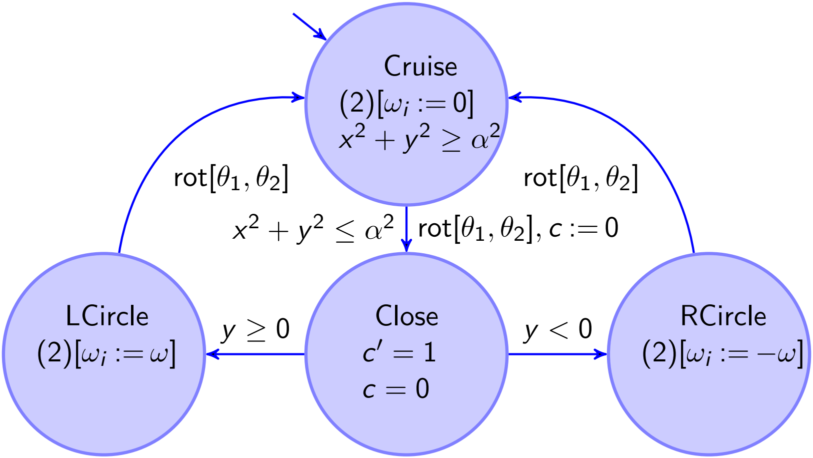
$$\min_{k,\ell} \max(|\theta_1 - 0|, |\theta_2 - \phi|)$$

Tangential Roundabout Maneuver Automaton



Return

Outline

Summary

- Image computation in hybrid systems model checking
  1. approx uniformly
  2. blur by uniform error
  3. check for B

flows	approx / image computation
continuous	uniform approx exists, but...
smooth	undecidable by evaluation
bounded by b	decidable
bound probabilities	probabilistically decidable
ODE ℓ-Lipschitz	decidable

- Combine numerical algorithms with symbolic analysis
- Roundabout maneuver unsafe
- Solution: adaptively choose rotations by tangential construction

Possible Extensions for Projects

- Extend tangential roundabout maneuver
  - Determine speed/thrust bounds
  - Position discrepancies caused by imprecise tracking
  - Verify liveness: aircraft finally on original route
  - Full curve dynamics
- Combine numerical algorithms with symbolic analysis ...
- Improved model checker
- Multivariate rational spline approximation

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