# Hybrid Systems Verification and Robotics

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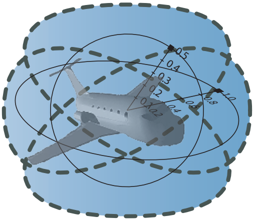
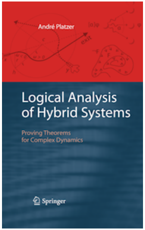
http://symbolaris.com/





# Outline

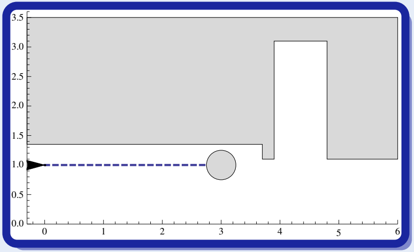
## Can you trust a computer to control physics?

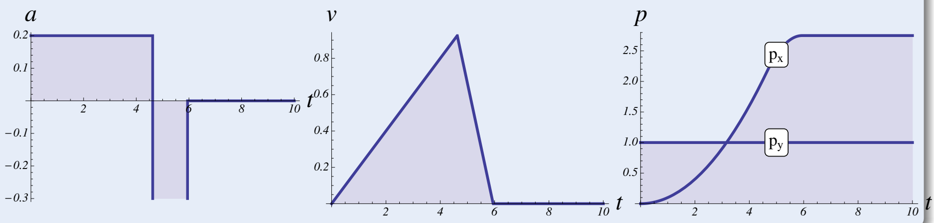
# Outline

## Challenge (Hybrid Systems)

Fixed rule describing state evolution with both

- Discrete dynamics (control decisions)
- Continuous dynamics (differential equations)



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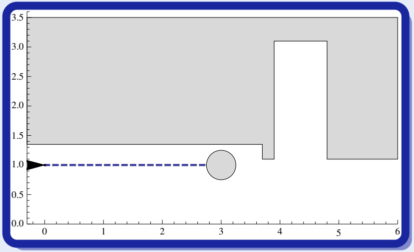


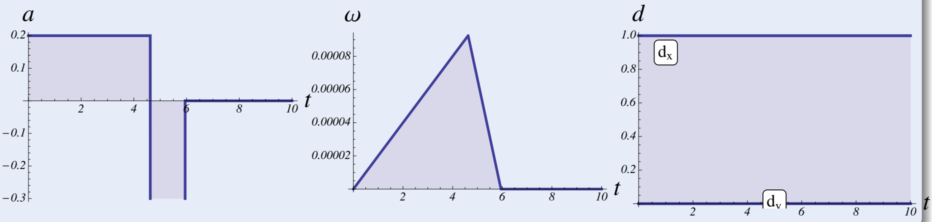

# Hybrid Systems Analysis

## Challenge (Hybrid Systems)

Fixed rule describing state evolution with both

- Discrete dynamics (control decisions)
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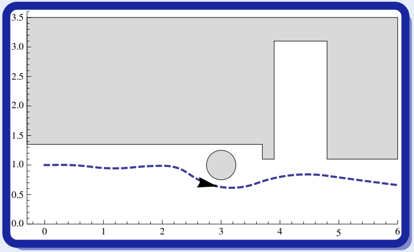


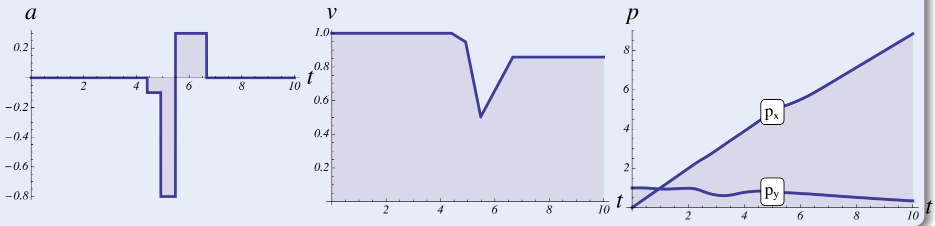

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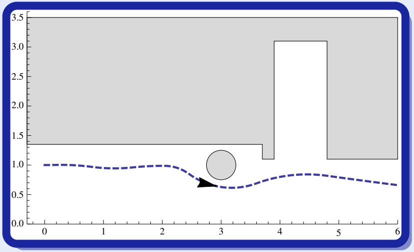


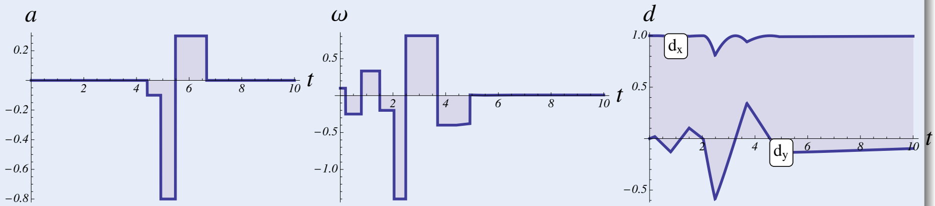

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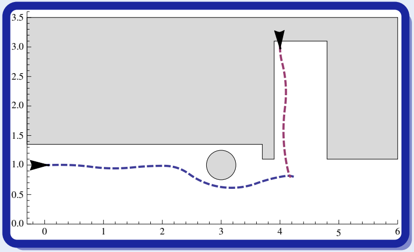

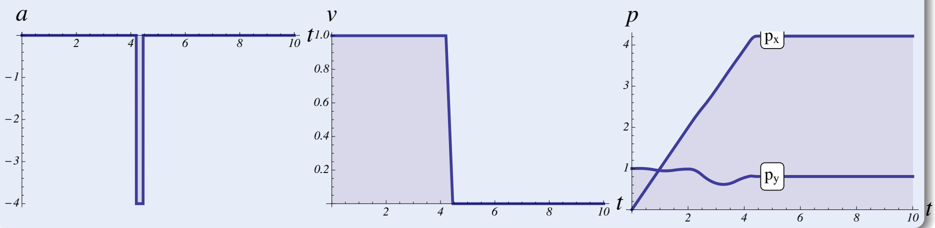

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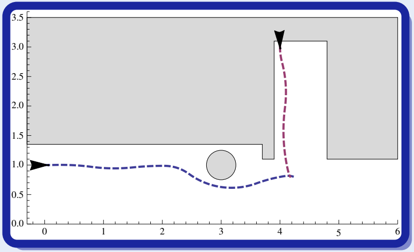

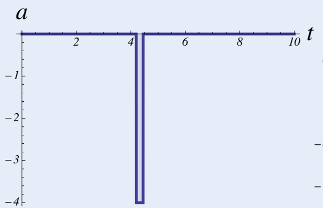


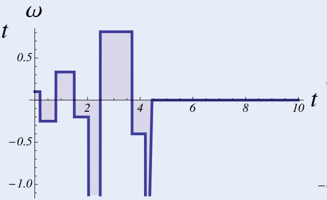


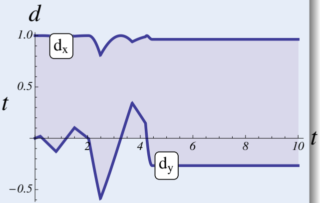

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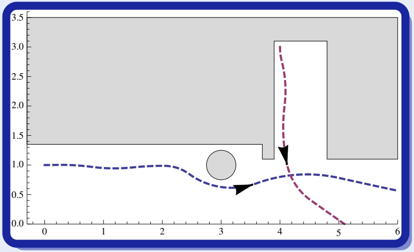

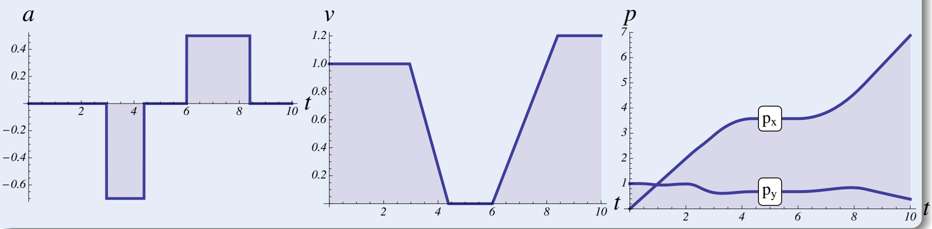

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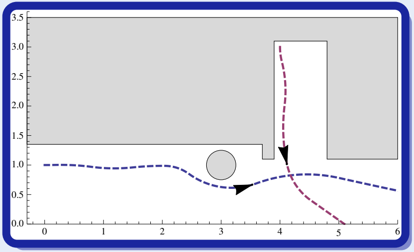
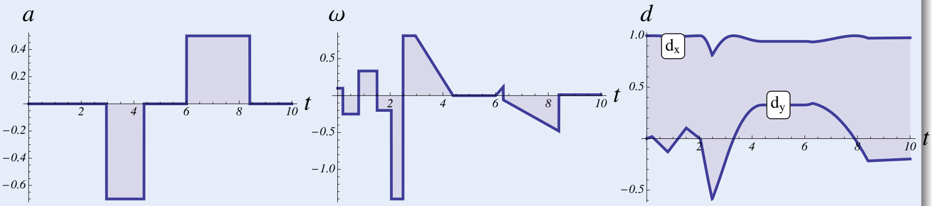


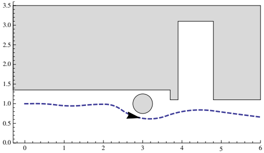


# Outline

# Logic for Hybrid Systems

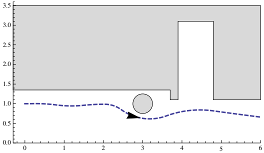
**differential dynamic logic**

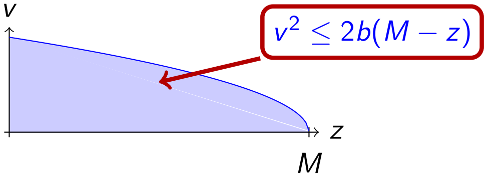
$$d\mathcal{L} = \qquad DL + HP$$



**differential dynamic logic**

$d\mathcal{L} = \mathrm{FOL}_{\mathbb{R}}$



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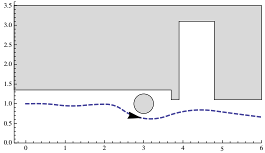


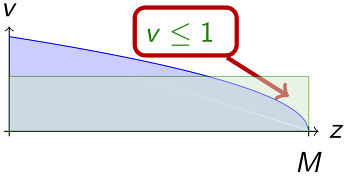

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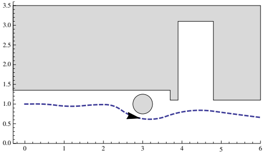


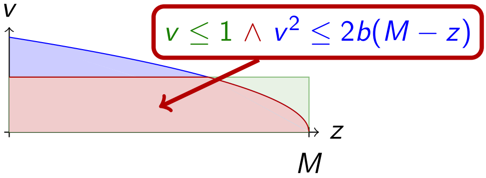

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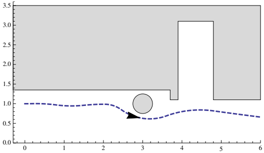
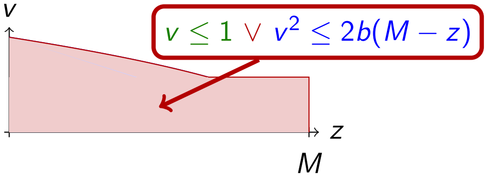
$\mathsf{d}\mathcal{L} = \mathrm{FOL}_{\mathbb{R}}$

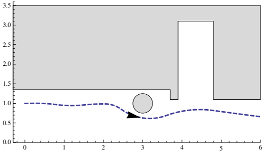


# Logic for Hybrid Systems

**differential dynamic logic**

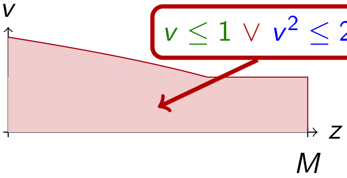
$\mathsf{d}\mathcal{L} = \mathrm{FOL}_{\mathbb{R}}$

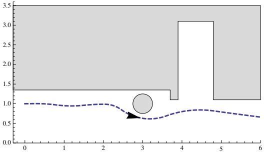



$\forall M\, \exists SB \ \ldots$

$\forall t{\geq}0 \ \ldots$

$v \leq 1 \lor v^2 \leq 2b(M - z)$

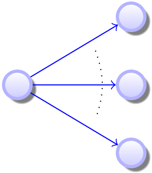





**differential dynamic logic**

$d\mathcal{L} = \mathrm{FOL}_{\mathbb{R}} +$

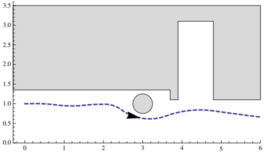
$v^2 \leq 2b$

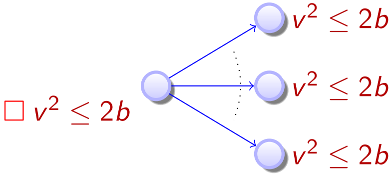


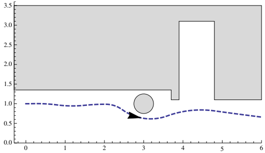
# Logic for Hybrid Systems

**differential dynamic logic**

$$d\mathcal{L} = \text{FOL}_{\mathbb{R}} + \text{ML}$$



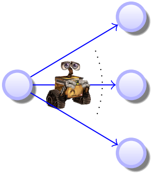


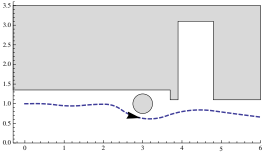



**differential dynamic logic**

$d\mathcal{L} = \mathrm{FOL}_{\mathbb{R}} + \mathrm{DL}$

[  ] $v^2 \leq 2b$

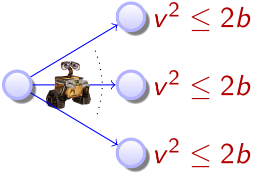





**differential dynamic logic**

$d\mathcal{L} = \mathrm{FOL}_{\mathbb{R}} + \mathrm{DL} + \mathrm{HP}$

$[z'' = a]\, v^2 \leq 2b$



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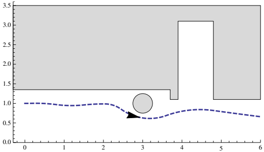
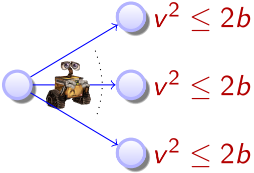
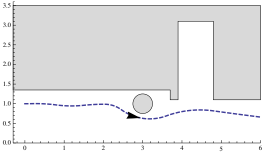



**differential dynamic logic**

$$d\mathcal{L} = \mathrm{FOL}_{\mathbb{R}} + \mathrm{DL} + \mathrm{HP}$$

$$[\mathtt{if}(z > SB)\; a := -b;\; z'' = a]\; v^2 \le 2b$$

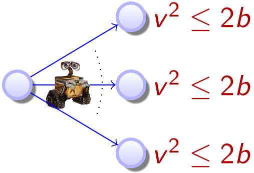





**differential dynamic logic**

$$d\mathcal{L} = \text{FOL}_{\mathbb{R}} + \text{DL} + \text{HP}$$

$$[\underbrace{\texttt{if}(z > SB)\, a := -b;\ z'' = a}_{\text{hybrid program}}]\, v^2 \le 2b$$



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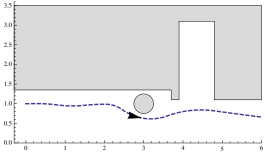
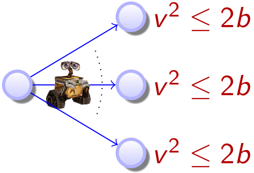



**differential dynamic logic**

$$\mathsf{d}\mathcal{L} = \mathrm{FOL}_{\mathbb{R}} + \mathrm{DL} + \mathrm{HP}$$

$$\mathcal{C} \rightarrow [\underbrace{\texttt{if}(z > SB)\, a := -b;\ z'' = a}_{\text{hybrid program}}]\, v^2 \leq 2b$$



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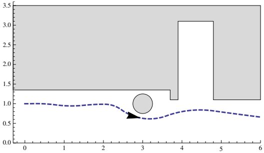


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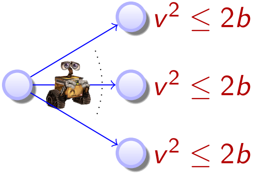


Initial condition

**differential dynamic logic**

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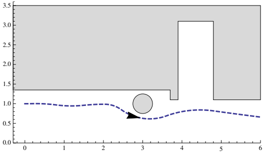


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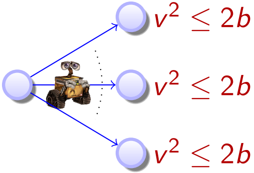
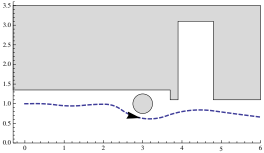



Initial condition

System dynamics



**differential dynamic logic**

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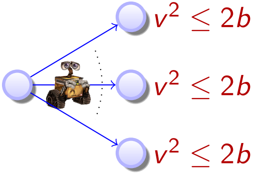



Initial condition

System dynamics

Post condition

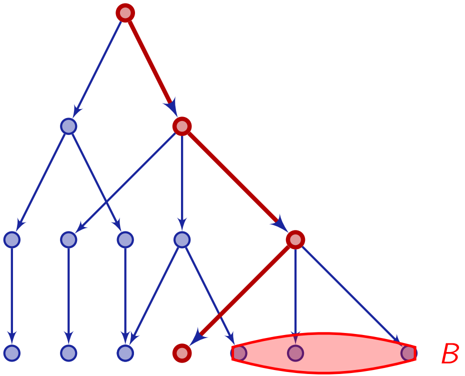
# Outline

Follow all transitions of the system
from a set of states
$\approx$ set-valued simulation

# Model Checking in a Nutshell

**Definition (Model Checking Problem)**

Given initial states $Q_0 \subseteq Q$ and bad states $B \subseteq Q$ for a transition system, check whether there is a trace from some $q_0 \in Q_0$ to some $q_b \in B$.



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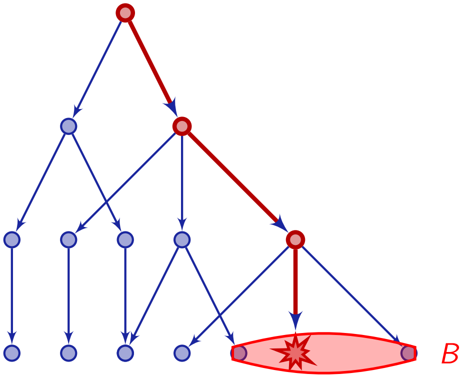
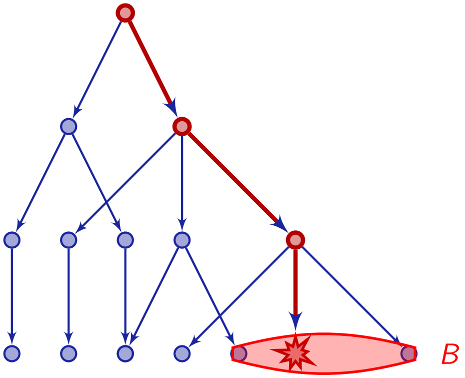


# Model Checking in a Nutshell

## Definition (Image Computation)

$$Post_A(Y) \ := \ \{q^+ \in Q \ : \ q \xrightarrow{a} q^+ \text{ for some } q \in Y, a \in A\}$$



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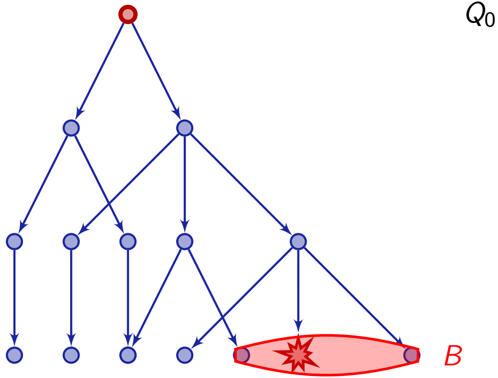

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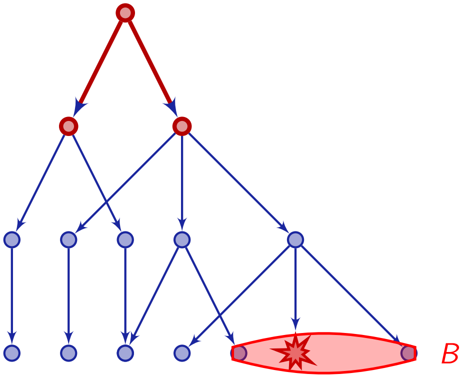


$Q_0$

$Post_A(Q_0)$

$Q_1 = Post_A(Q_0)$

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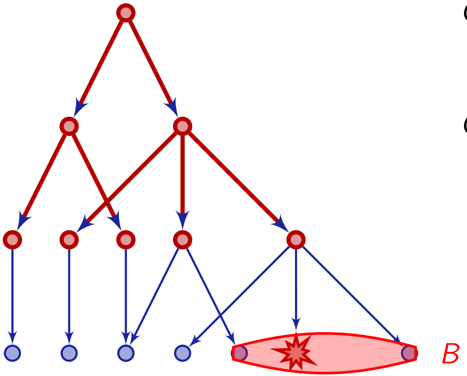


$Q_0$

$Post_A(Q_0)$

$Q_1 = Post_A(Q_0)$

$Post_A(Q_1)$

$Q_2 = Post_A^2(Q_0)$

# Model Checking in a Nutshell

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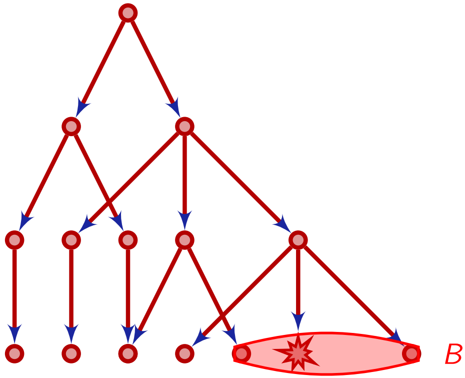



$Q_0$

$\downarrow Post_A(Q_0)$

$Q_1 = Post_A(Q_0)$

$\downarrow Post_A(Q_1)$

$Q_2 = Post_A^2(Q_0)$

$\downarrow Post_A(Q_2)$

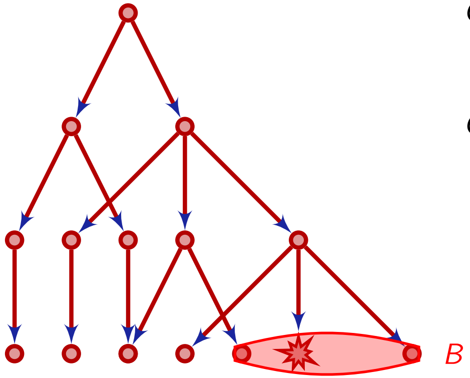
$Q_3 = Post_A^3(Q_0)$

# Model Checking in a Nutshell

## Definition (Image Computation)

$$Post_A(Y) := \{q^+ \in Q : q \xrightarrow{a} q^+ \text{ for some } q \in Y, a \in A\}$$

$$Post^*_A(Y) := \bigcup_{n \in \mathbb{N}} Post^n_A(Y) = \mu Z.(Y \cup Z \cup Post_A(Z))$$




$Q_0$

$Post_A(Q_0)$

$Q_1 = Post_A(Q_0)$

$Post_A(Q_1)$

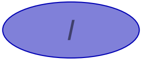
$Q_2 = Post^2_A(Q_0)$

$Post_A(Q_2)$

$Q_3 = Post^3_A(Q_0)$

## Uncountably state spaces require extra care

# Image Computation in Hybrid Systems




- Analyse image computation problem in hybrid systems
- Approximation refinement techniques and their limits
- Representation of regions in state space
- Numerical versus symbolic algorithms
  $1.421 \in \mathbb{Q}$ versus $x^2 + 2xy$ term computations

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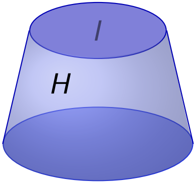


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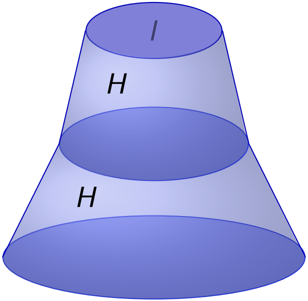


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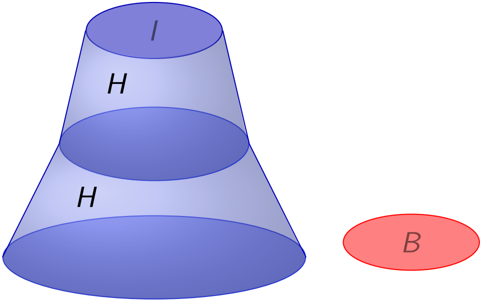


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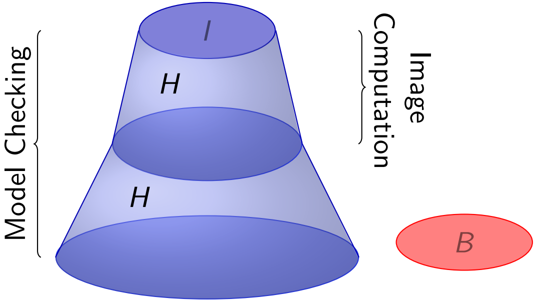


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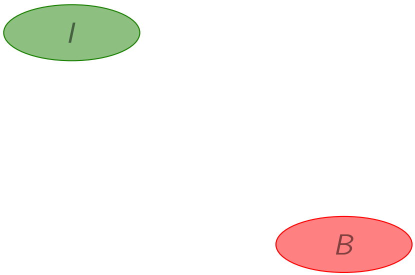
# AMC: Approximation Refinement Model Checking

AMC($B$ reachable from $I$ in $H$):

1. $A :=$ approx($H$) uniformly
2. blur by uniform approximation error $+\epsilon$
3. check($B$ reachable from $I$ in $A + \epsilon$)
4. $B$ not reachable $\Rightarrow$ $H$ safe



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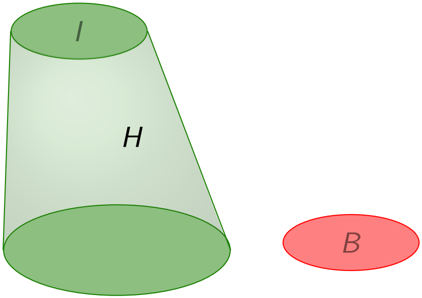

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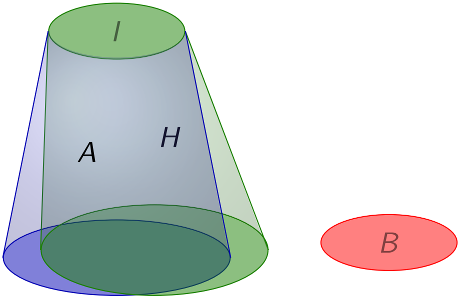

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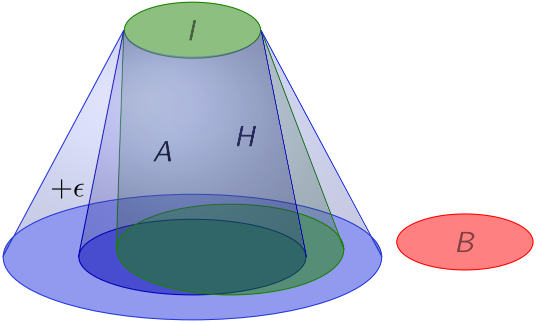

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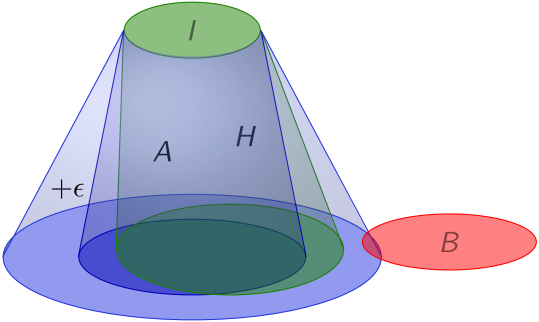

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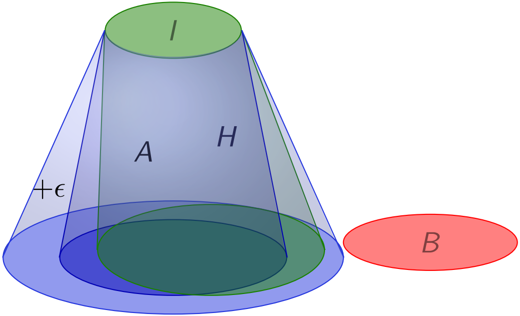

# AMC: Exact Image Computation

AMC($B$ reachable from $I$ in $H$):

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**Proposition (Semialgebraic images)** (HSCC'07)

*check and blur can be implemented for*

- *$I$ and $B$ semialgebraic (propositional combinations of $p \geq 0$)*
- *$A$ with polynomial flows over $\mathbb{R}$*
- *$+$Piecewise definitions*
- *$+$Rational extensions (e.g. multivariate rational splines)*

# AMC: Image Approximation

AMC($B$ reachable from $I$ in $H$):

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**Proposition (*Existence* of approximations) (HSCC'07)**

*approx exists for all uniform errors $\epsilon > 0$ when*

- *using polynomials to build $A$*
- *Flows $\varphi \in C(D, \mathbb{R}^n)$ of $H$*
- *$D \subset \mathbb{R} \times \mathbb{R}^n$ compact closure of an open set*

Approximation can solve problems without effective exact solution

Existence of solutions may be computationally insufficient

# Summary: Model Checking

- Image computation in hybrid systems model checking HSCC'07
  1. approx uniformly
  2. blur by uniform error
  3. check for $B$

| flows | approx / image computation |
|---|---|
| continuous | uniform approx exists, but... |
| smooth | undecidable by evaluation |
| bounded by $b$ | decidable |
| bound probabilities | probabilistically decidable |
| ODE $\ell$-Lipschitz | decidable |

- Combine numerical algorithms with symbolic analysis
- Roundabout maneuver unsafe

# Outline

## Verify using many simple symbolic proof steps

# Differential Dynamic Logic: Axiomatization

$$[:=] \quad [x := \theta][(x)]\phi x \leftrightarrow [(x)]\phi\theta$$

$$[?] \quad [?H]\phi \leftrightarrow (H \rightarrow \phi)$$

$$['] \quad [x' = f(x)]\phi \leftrightarrow \forall t{\geq}0\,[x := y(t)]\phi \qquad (y'(t) = f(y))$$

$$[\cup] \quad [\alpha \cup \beta]\phi \leftrightarrow [\alpha]\phi \wedge [\beta]\phi$$

$$[;] \quad [\alpha; \beta]\phi \leftrightarrow [\alpha][\beta]\phi$$

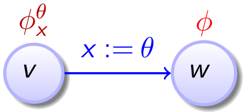
$$[^*] \quad [\alpha^*]\phi \leftrightarrow \phi \wedge [\alpha][\alpha^*]\phi$$

$$\mathsf{K} \quad [\alpha](\phi \rightarrow \psi) \rightarrow ([\alpha]\phi \rightarrow [\alpha]\psi)$$

$$\mathsf{I} \quad [\alpha^*](\phi \rightarrow [\alpha]\phi) \rightarrow (\phi \rightarrow [\alpha^*]\phi)$$

$$\mathsf{C} \quad [\alpha^*]\forall v{>}0\,(\varphi(v) \rightarrow \langle\alpha\rangle\varphi(v-1)) \rightarrow \forall v\,(\varphi(v) \rightarrow \langle\alpha^*\rangle\exists v{\leq}0\,\varphi(v))$$

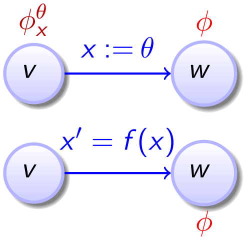
$$\frac{\phi_x^\theta}{[x := \theta]\phi}$$



# Proofs for Hybrid Systems

$$\frac{\phi_x^\theta}{[x := \theta]\phi}$$

$$\frac{\forall t{\geq}0\, [x := y_x(t)]\phi}{[x' = f(x)]\phi}$$



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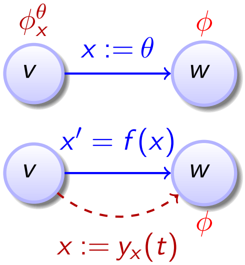
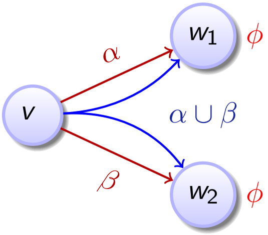


compositional semantics $\Rightarrow$ compositional rules!

# Proofs for Hybrid Systems

$$\frac{[\alpha]\phi \land [\beta]\phi}{[\alpha \cup \beta]\phi}$$



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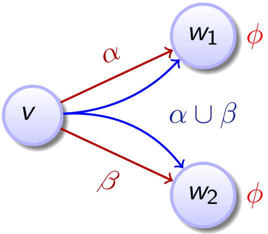
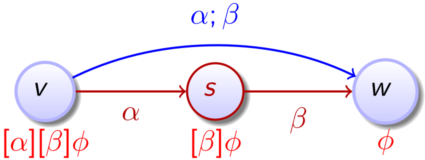



$$\frac{[\alpha][\beta]\phi}{[\alpha; \beta]\phi}$$



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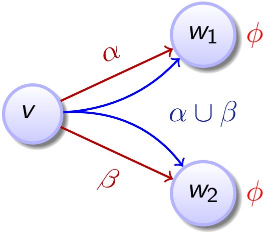
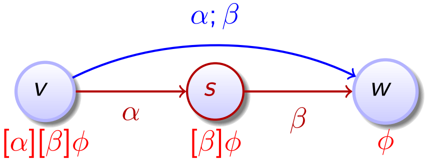
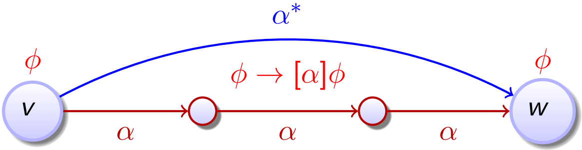


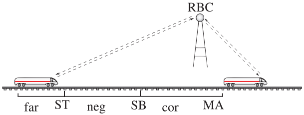

$$\frac{[\alpha][\beta]\phi}{[\alpha; \beta]\phi}$$




$$\frac{\phi \quad (\phi \rightarrow [\alpha]\phi)}{[\alpha^*]\phi}$$



# Deduction Modulo (Free Variables for Automation)




$$v \geq 0, z < m \rightarrow \exists t \geq 0 \, \langle z := -\frac{b}{2}t^2 + vt + z \rangle z > m$$

$$v \geq 0, z < m \rightarrow \langle z' = v, v' = -b \rangle z > m$$

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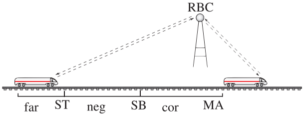
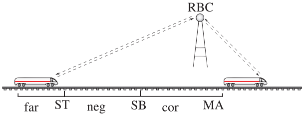



$$\dfrac{v \geq 0, z < m \to T \geq 0 \qquad \dfrac{v \geq 0, z < m \to -\frac{b}{2}T^2 + vT + z > m}{v \geq 0, z < m \to \langle z := -\frac{b}{2}T^2 + vT + z\rangle z > m}}{\dfrac{v \geq 0, z < m \to T \geq 0 \land \langle z := -\frac{b}{2}T^2 + vT + z\rangle z > m}{\dfrac{v \geq 0, z < m \to \exists t{\geq}0\, \langle z := -\frac{b}{2}t^2 + vt + z\rangle z > m}{\dfrac{v \geq 0, z < m \to \langle z' = v, v' = -b\rangle z > m}{v \geq 0 \land z < m \to \langle z' = v, v' = -b\rangle z > m}}}}$$

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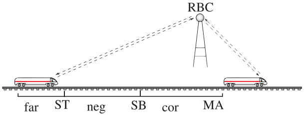


$$v \geq 0, z < m \rightarrow \mathrm{QE}\left(\exists T\,(\ldots\, T{\geq}0 \land -\tfrac{b}{2}T^2 + vT + z > m)\right)$$

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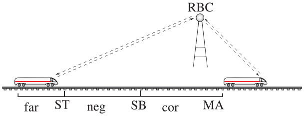


$$v \geq 0, z < m \to v^2 > 2b(m - z)$$

$$v \geq 0, z < m \to -\frac{b}{2}T^2 + vT + z > m$$

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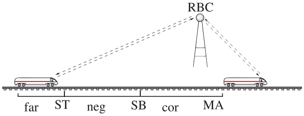
- For requantification, not for unification

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# Complete Proof Theory of Hybrid Systems

**Theorem (Continuous Relative Completeness) (J.Autom.Reas. 2008)**

d$\mathcal{L}$ *calculus is a sound & complete axiomatization of hybrid systems relative to differential equations.*

▸ Proof 15pp

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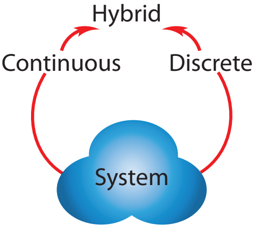

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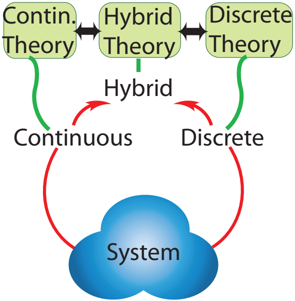

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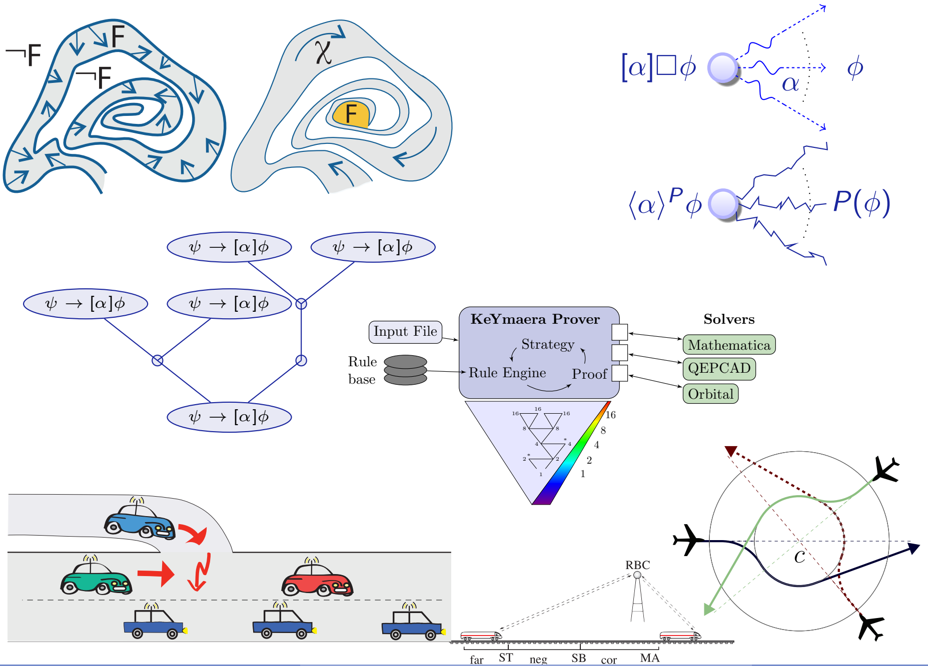
**Corollary (Relative Decidability)**

*Verification & synthesis decidable relative to differential equations.*

**Corollary (Relative Extension)**

*All research on differential equations extends to hybrid systems.*

# Outline



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F
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$[\alpha]\Box\phi$
$\alpha$
$\phi$
$\langle\alpha\rangle^P\phi$
$P(\phi)$
$\psi \to [\alpha]\phi$
$\psi \to [\alpha]\phi$
$\psi \to [\alpha]\phi$
$\psi \to [\alpha]\phi$
$\psi \to [\alpha]\phi$
KeYmaera Prover
Input File
Rule base
Strategy
Rule Engine
Proof
Solvers
Mathematica
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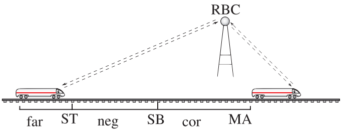
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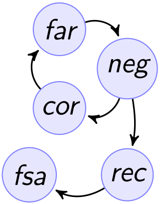
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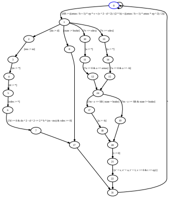

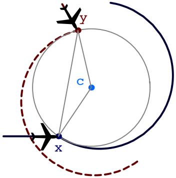
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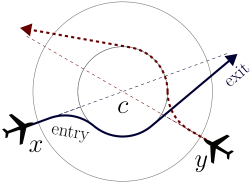
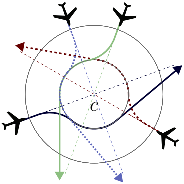

c
entry
exit
x
y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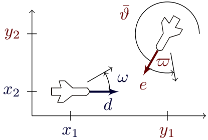
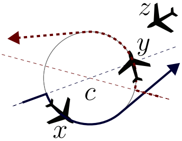

$y_2$
$x_2$
$x_1$
$y_1$
$\bar{\vartheta}$
$\omega$
$\varpi$
$e$
$d$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c
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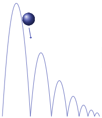
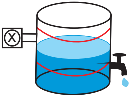
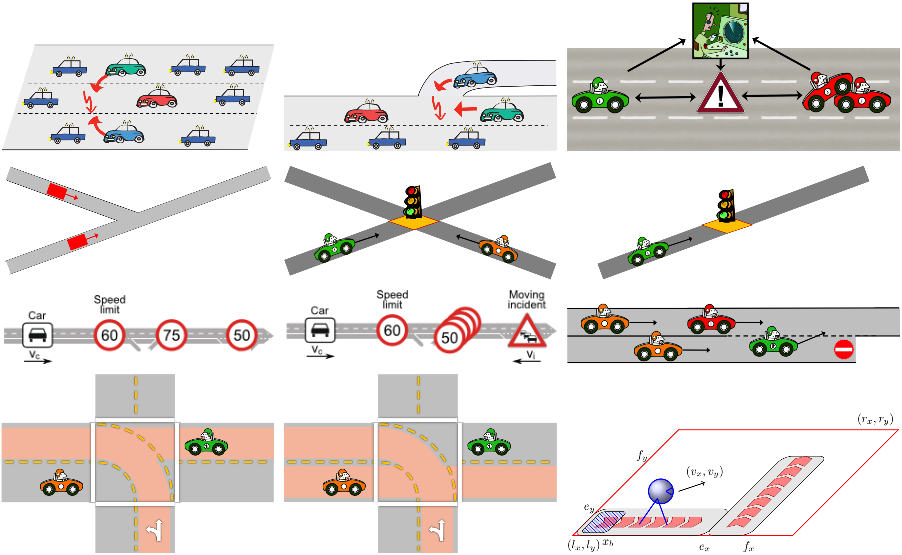
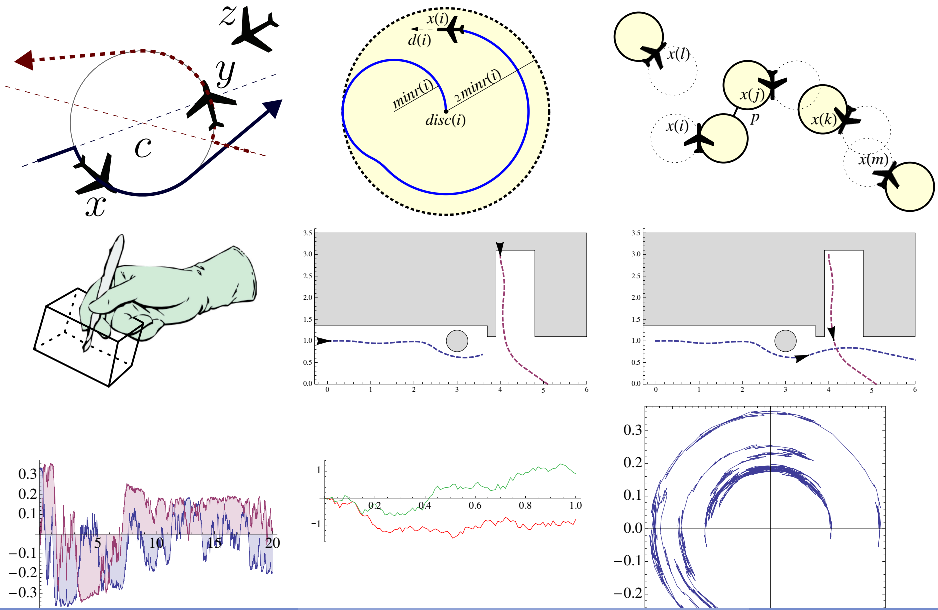




# Successful Hybrid Systems Proofs

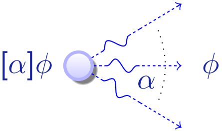


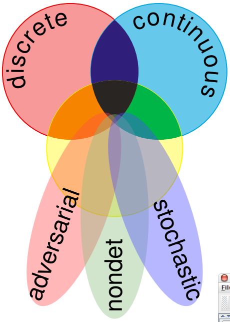

z
y
c
x
x(i)
d(i)
minr(i)
2minr(i)
disc(i)
x(l)
x(j)
p
x(k)
x(i)
x(m)

# Outline

# Hybrid Systems Verification and Robotics

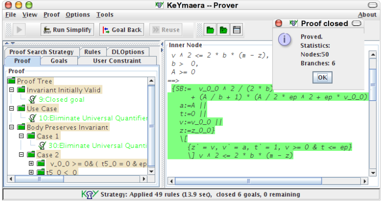
- Hybrid system models
- Discrete dynamics
- Continuous dynamics
- Correctness properties
- Safety, liveness ...

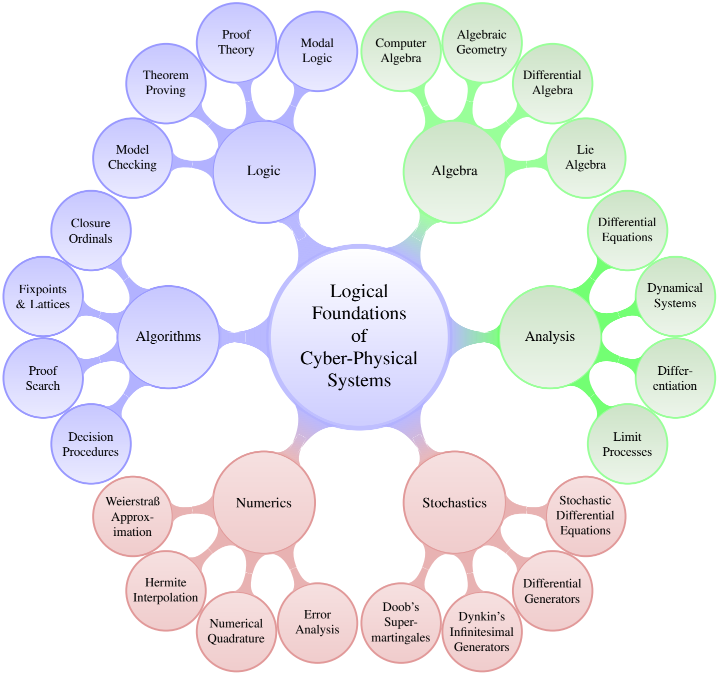


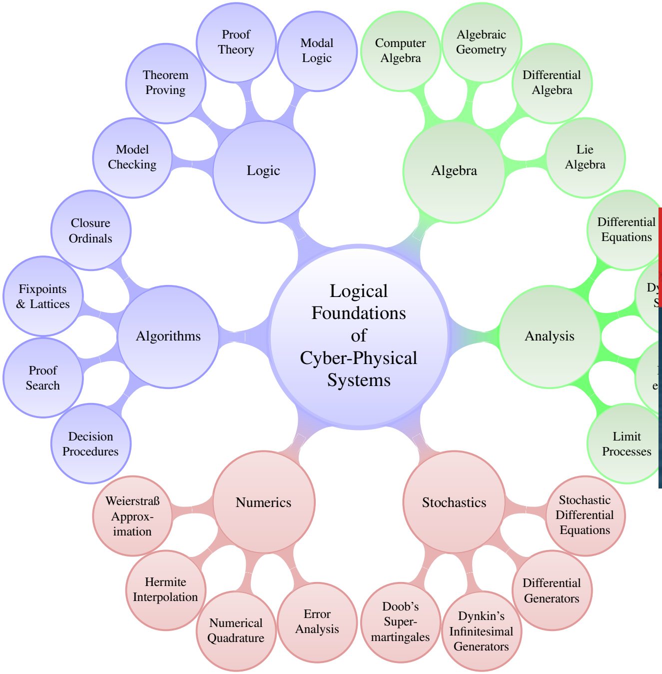




- Model checking
- Logic & proofs
- Cyber-physical systems
- Differential invariants

KeYmaera




Logical Foundations of Cyber-Physical Systems
Logic
Proof Theory
Modal Logic
Theorem Proving
Model Checking
Algebra
Computer Algebra
Algebraic Geometry
Differential Algebra
Lie Algebra
Algorithms
Closure Ordinals
Fixpoints & Lattices
Proof Search
Decision Procedures
Analysis
Differential Equations
Dynamical Systems
Differ-entiation
Limit Processes
Numerics
Weierstraß Approx-imation
Hermite Interpolation
Numerical Quadrature
Error Analysis
Stochastics
Stochastic Differential Equations
Differential Generators
Dynkin's Infinitesimal Generators
Doob's Super-martingales



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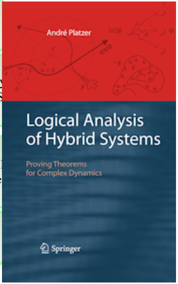

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