Characterizing Algebraic Invariants by Differential Radical Invariants

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TACAS, Grenoble France
April 9th, 2014

Context: Hybrid Systems Model

`Sensing:  read data from sensors`

Context: Hybrid Systems Model

Sensing: read data from sensors
Control: actuate

Context: Hybrid Systems Model

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Sensing:  read data from sensors
Control:  actuate
Plant:  evolve
```

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(
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)*
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Context: Hybrid Systems Model

Init

$\longrightarrow$

(

Sensing: read data from sensors

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$)^*$

Safety

Context: Hybrid Systems Model

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[

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$)^*$

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Safety

Evolution

- Continuous time
- Ordinary Differential Equations (ODE)

Related work: Handling the continuous part

- Solutions are hard to compute symbolically
- No closed form solution exists in general
- Alternatives
  - Local approximations (Taylor series) [Lanotte et al. 2005]
  - Inductive (differential) Invariants [Maths, ThPhy 1870-] [Control 1900-] [FM 2001-]
- Limitations
  - Linear differential equations [Tiwari et al. 2003-]
  - Restrictive subclasses of Invariants [Sankaranarayanan et al 2006-, Matringe et al. 2009-, Platzer 2010]
  - Expensive procedure [Liu et al. 2011]

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  - Linear differential equations [Tiwari et al. 2003-] **Polynomial**
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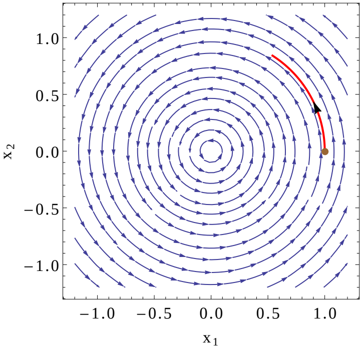
  - Expensive procedure [Liu et al. 2011] — **Efficient**

Algebraic Invariant Equations

$$(\dot{x}_1, \dot{x}_2) = (-x_2, x_1)$$



The solution for $\mathbf{x}_0 = (1, 0)$ for $t = [0, 1]$

Algebraic Invariant Equations

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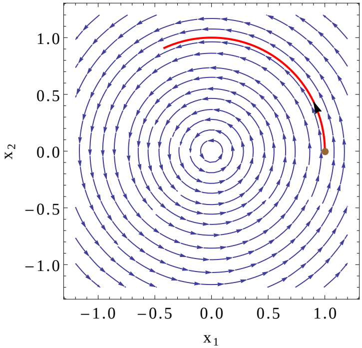

The solution for $\mathbf{x}_0 = (1, 0)$ for $t = [0, 2]$

Algebraic Invariant Equations

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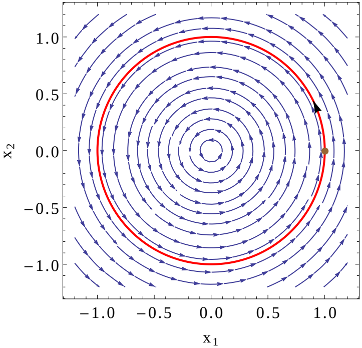
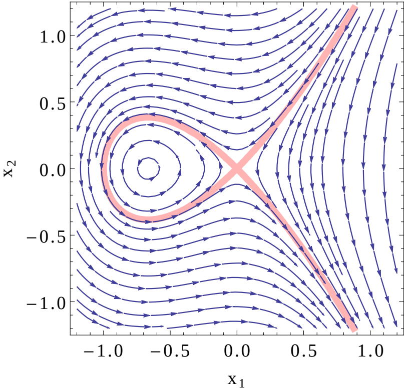


Algebraic Invariant Equation

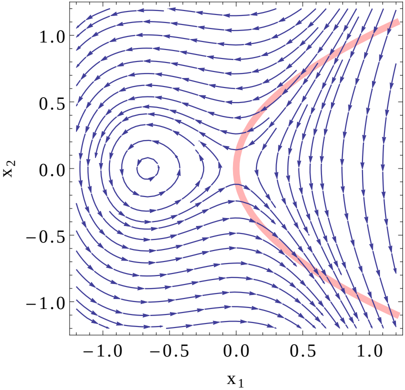
The solution for $\mathbf{x}_0 = (1, 0)$ respects $\boxed{x_1(t)^2 + x_2(t)^2 - 1 = 0}$ $\forall t$

Problem I. *Checking the invariance of Algebraic Equations*

Given $\dot{\mathbf{x}} = \mathbf{p}$, and $\mathbf{x}_0$ such that $h(\mathbf{x}_0) = 0$, is $h(\mathbf{x}(t)) = 0$ for all t ?



$h(x_1, x_2) = x_1^2 + x_1^3 - x_2^2$



$h(x_1, x_2) = x_1 - x_2^2$

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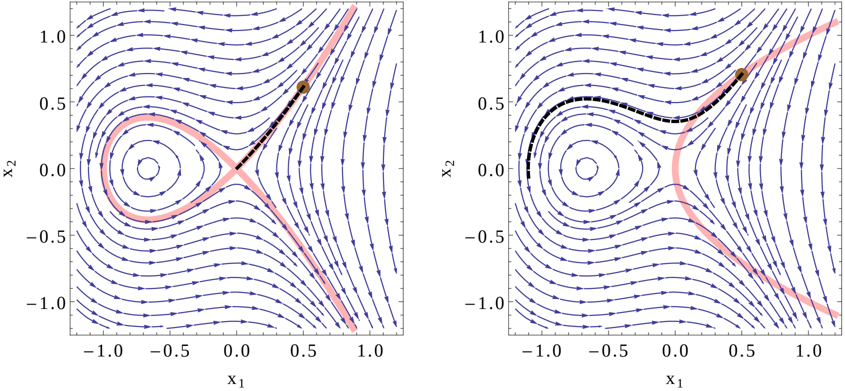

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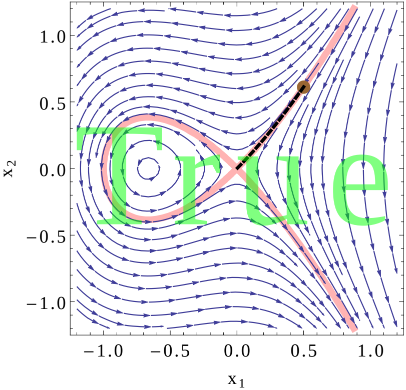

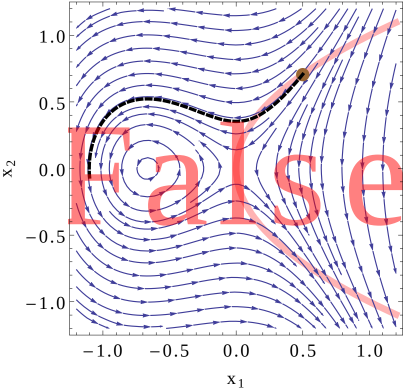

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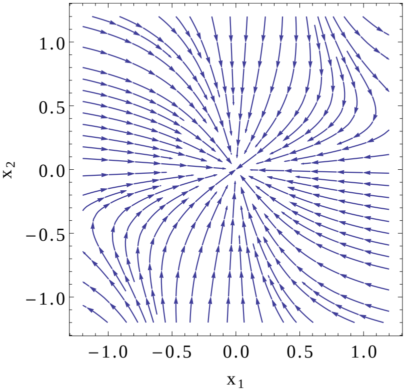
Problem II. *Generate Algebraic Invariant Equations*

Given $\dot{\mathbf{x}} = \mathbf{p}$, how to generate h such that $h(\mathbf{x}(t)) = 0$?



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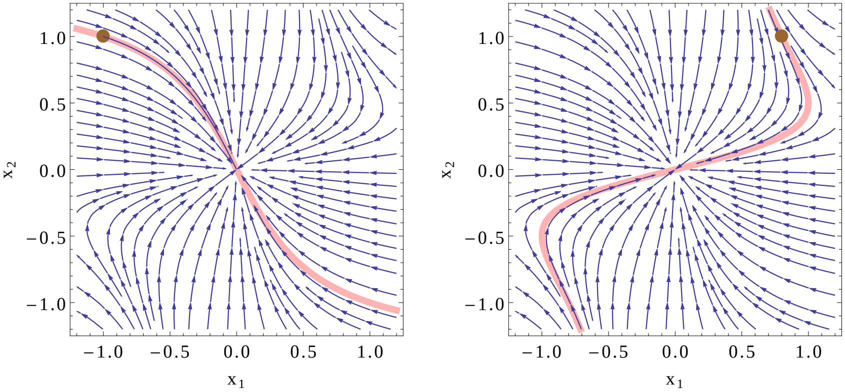

$$h_{(x_1(0),x_2(0))}(x_1, x_2) = (x_2(0) - x_1(0)x_2(0)^2)x_1 - x_1(0)(x_2 - x_1 x_2^2)$$

Outline

Checking Invariance of Candidates

Already existing proof rules

I. Checking the invariance of Algebraic Equations

Given $\dot{\mathbf{x}} = \mathbf{p}$, and $\mathbf{x}_0$ such that $h(\mathbf{x}_0) = 0$, is $h(\mathbf{x}(t)) = 0$ for all t ?

$$\frac{\mathfrak{D}(h) = \lambda h \quad (\lambda \in \mathbb{R}[\mathbf{x}]) \qquad \mathfrak{D}(h) = 0}{(h = 0) \rightarrow [\dot{\mathbf{x}} = \mathbf{p}](h = 0)}$$

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$$h = 0 \longrightarrow \mathfrak{D}(h) = 0 \quad \text{(unsound)}$$

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- $h = x_1^2 + x_2^2$
- $\mathfrak{D}(h) = 2x_1\mathfrak{D}(x_1) + 2x_2\mathfrak{D}(x_2) = 4x_1x_2$ (Chain Rule)
- $\mathfrak{D}^{(2)}(h) = 4(x_1^2 + x_2^2)$

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$$\mathfrak{D}^{(2)}(h) = 4h$$

No $\lambda \in \mathbb{R}[x]$ s.t. $4x_1x_2 = \lambda(x_1^2 + x_2^2)$

$$\frac{\mathfrak{D}(h) = 4x_1x_2 \neq 0}{(h = 0) \rightarrow [\dot{\mathbf{x}} = \mathbf{p}](h = 0)}$$

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still unsound! (counterexample: $h = x_1$)

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Checking Invariance of Candidates

Differential Radical Invariants [paper, Theorem 2]

$$\mathfrak{D}^{(N)}(h) = \sum_{i=0}^{N-1} \lambda_i \mathfrak{D}^{(i)}(h)\ (\lambda_i \in \mathbb{R}[\mathbf{x}]) \ \wedge\ h = 0 \to \bigwedge_{i=1}^{N-1} \mathfrak{D}^{(i)}(h) = 0$$

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$$\mathfrak{D}^{(3)}(h) = \sum_{i=0}^{2} \lambda_i \mathfrak{D}^{(i)}(h)\ (\lambda_i \in \mathbb{R}[\mathbf{x}]) \ \wedge\ h = 0 \to \bigwedge_{i=1}^{2} \mathfrak{D}^{(i)}(h) = 0$$

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- order N is **finite**
- **Necessary** and **sufficient** condition
- **Decidable**
  - Existence of λ_i: Gröbner Basis
  - $h = 0 \to \mathfrak{D}^{(i)}(h) = 0$: Quantifier Elimination

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$$\frac{\begin{array}{c}\mathfrak{D}^{(N)}(h) = \sum_{i=0}^{N-1} \lambda_i \mathfrak{D}^{(i)}(h)\ (\lambda_i \in \mathbb{R}[\mathbf{x}])\ \wedge\ h = 0 \to \bigwedge_{i=1}^{N-1} \mathfrak{D}^{(i)}(h) = 0 \\ \dots \\ \mathfrak{D}^{(3)}(h) = \sum_{i=0}^{2} \lambda_i \mathfrak{D}^{(i)}(h)\ (\lambda_i \in \mathbb{R}[\mathbf{x}])\ \wedge\ h = 0 \to \bigwedge_{i=1}^{2} \mathfrak{D}^{(i)}(h) = 0 \\ \mathfrak{D}^{(2)}(h) = \lambda_0 h + \lambda_1 \mathfrak{D}(h)\ (\lambda_i \in \mathbb{R}[\mathbf{x}])\ \wedge\ h = 0 \to \mathfrak{D}(h) = 0 \\ \mathfrak{D}(h) = \lambda h\ (\lambda \in \mathbb{R}[\mathbf{x}])\end{array}}{(h = 0) \to [\dot{\mathbf{x}} = \mathbf{p}](h = 0)}$$

- order N is **finite**
- **Necessary** and **sufficient** condition
- **Decidable**
  - Existence of λ_i: Gröbner Basis
  - $h = 0 \to \mathfrak{D}^{(i)}(h) = 0$: Quantifier Elimination

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Differential Radical Characterization [paper, Theorem 1]

$$\frac{\mathfrak{D}^{(N)}(h) = \sum_{i=0}^{N-1} \lambda_i \mathfrak{D}^{(i)}(h)\ (\lambda_i \in \mathbb{R}[\mathbf{x}]) \ \wedge \ h = 0 \rightarrow \bigwedge_{i=1}^{N-1} \mathfrak{D}^{(i)}(h) = 0}{(h = 0) \rightarrow [\dot{\mathbf{x}} = \mathbf{p}](h = 0)}$$

Algebraic Framework

$\mathcal{O}(\mathbf{x}_0)$		$I(\mathcal{O}(\mathbf{x}_0))$		$V(I(\mathcal{O}(\mathbf{x}_0)))$		$\mathcal{O}(\mathbf{x}_0)$
Reachable Set	$\xrightarrow{I}$	**Vanishing Ideal**	$\xrightarrow{V}$	**Closure** **smallest variety**	$\supseteq$	**Reachable Set**

Vanishing Ideal $I(\mathcal{O}(\mathbf{x}_0))$ all polynomials that vanish on $\mathcal{O}(\mathbf{x}_0)$

Closure $V(I(\mathcal{O}(\mathbf{x}_0)))$ common roots of all polynomials in $I(\mathcal{O}(\mathbf{x}_0))$

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Algebraic Framework

$$\underset{\text{Reachable Set}}{\mathcal{O}(\mathbf{x}_0)} \xrightarrow{I} \underset{\text{Vanishing Ideal}}{I(\mathcal{O}(\mathbf{x}_0))} \xrightarrow{V} \underset{\text{Closure — smallest variety}}{V(I(\mathcal{O}(\mathbf{x}_0)))} \supseteq \underset{\text{Reachable Set}}{\mathcal{O}(\mathbf{x}_0)}$$

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Outline

Generation of Invariant Algebraic Sets

Necessary and sufficient condition [paper, Theorem 3]

II. Generate Algebraic Invariant Equations

Given $\dot{\mathbf{x}} = \mathbf{p}$, how to generate h such that $h(\mathbf{x}(t)) = 0$?

Theorem

$S \in \mathbb{R}^n$ is an invariant algebraic set **if and only if**

$$S = \text{ Set of roots of the system } \begin{cases} h &= 0 \\ & \vdots \\ \mathfrak{D}^{(N-1)}(h) &= 0 \end{cases}$$

for some polynomial h with **order N**, that is

$$\mathfrak{D}^{(N)}(h) = \sum_{i=0}^{N-1} \lambda_i \mathfrak{D}^{(i)}(h)$$

Generation of Invariant Algebraic Sets

First integrals vs. Local invariant regions [paper, Theorem 4]

Suppose we found h and N such that

$$\mathfrak{D}^{(N)}(h) = \sum_{i=0}^{N-1} \lambda_i \mathfrak{D}^{(i)}(h)$$

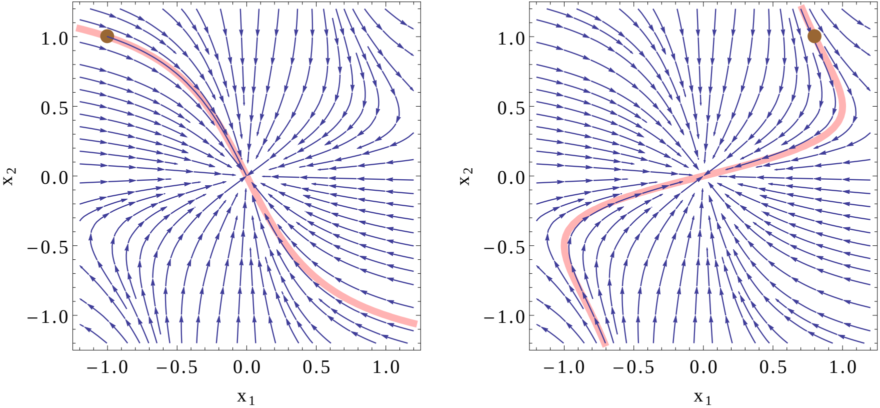
Case 1: First Integral

For all $\mathbf{x}_0 \in \mathbb{R}^n$, $h(\mathbf{x}_0) = 0 \wedge \cdots \wedge \mathfrak{D}^{(N-1)}(h)(\mathbf{x}_0) = 0$

Case 2: Local Invariant Regions (e.g. limiting cycle, equilibria)

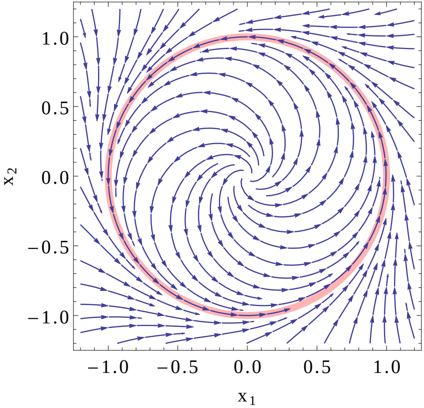
Restrict $\mathbf{x}_0$ such that $h(\mathbf{x}_0) = 0 \wedge \cdots \wedge \mathfrak{D}^{(N-1)}(h)(\mathbf{x}_0) = 0$

Example: First Integrals

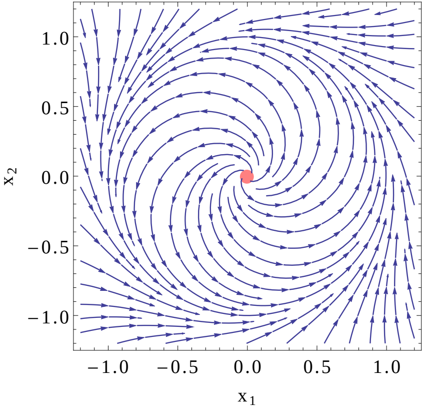


$$h_{(x_1(0),x_2(0))}(x_1, x_2) = (x_2(0) - x_1(0)x_2(0)^2)x_1 - x_1(0)(x_2 - x_1 x_2^2)$$

Example: Local invariant regions



$$h(x_1, x_2) = x_1^2 + x_2^2 - 1$$



$$h(x_1, x_2) = x_1^2 + x_2^2$$

But ...

How to **generate** h and N such that

$$\mathfrak{D}^{(N)}(h) = \sum_{i=0}^{N-1} \lambda_i \mathfrak{D}^{(i)}(h)$$

Matrix Representation: Intuition

Suppose we have a 2-dimensional ODE $(\dot{x}_1, \dot{x}_2) = (x_1, x_2)$

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Study the **null space** (kernel) of $M(\beta)$

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- Increases the chances of finding **first integrals**
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$$h = x_2(0)x_1 - x_1(0)x_2$$

Toward a Generation Procedure ?

We started with a parametrized polynomial h of degree 1 and $N = 1 \ldots$

If no invariants:

- Increase order N versus increase the polynomial degree of h ?
- Any bound on N ?
- Any bound on the degree of h ?

Case Study: Longitudinal Dynamics of an Airplane

6th Order Longitudinal Equations

$$
\begin{aligned}
\dot{u} &= \frac{X}{m} - g\sin(\theta) - qw && u : \text{axial velocity} \\
\dot{w} &= \frac{Z}{m} + g\cos(\theta) + qu && w : \text{vertical velocity} \\
\dot{x} &= \cos(\theta)u + \sin(\theta)w && x : \text{range} \\
\dot{z} &= -\sin(\theta)u + \cos(\theta)w && z : \text{altitude} \\
\dot{q} &= \frac{M}{I_{yy}} && q : \text{pitch rate} \\
\dot{\theta} &= q && \theta : \text{pitch angle}
\end{aligned}
$$

Case Study: Generated Invariants

Automatically Generated Invariant Functions

$$\frac{Mz}{I_{yy}} + g\theta + \left(\frac{X}{m} - qw\right)\cos(\theta) + \left(\frac{Z}{m} + qu\right)\sin(\theta)$$

$$\frac{Mx}{I_{yy}} - \left(\frac{Z}{m} + qu\right)\cos(\theta) + \left(\frac{X}{m} - qw\right)\sin(\theta)$$

$$-q^2 + \frac{2M\theta}{I_{yy}}$$

Conclusion

Checking

- **Invariance** of Algebraic Sets is **Decidable**
- **DRI** Necessary and Sufficient **Proof Rule**

Generation

- **Generation** Problem $\sim$ Symbolic **Linear Algebra**
- Equivalent to the Min Rank Problem: **NP-hard**

- **Higher-order Derivatives** are crucial
- Real Algebraic Geometry $\leftrightharpoons$ Logic $\leftrightharpoons$ Verification

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