Differential Dynamic Logic and Differential Invariants for Hybrid Systems

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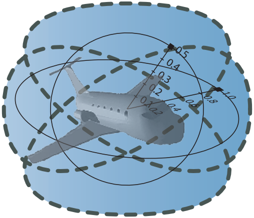
http://symbolaris.com/



How can we design computers that are guaranteed to interact correctly with the physical world?

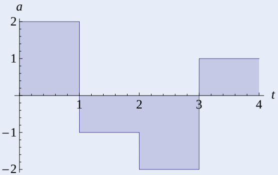
Outline

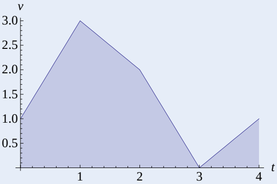
Hybrid Systems Analysis: Car Control

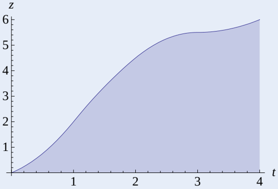
Challenge (Hybrid Systems)

Fixed rule describing state evolution with both

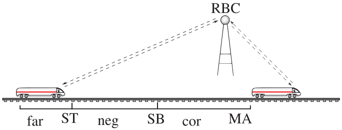
- Continuous dynamics (differential equations)
- Discrete dynamics (control decisions)

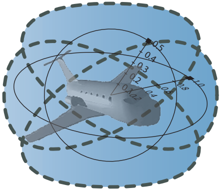


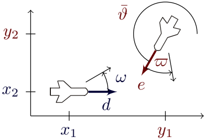


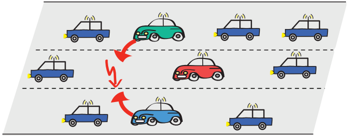


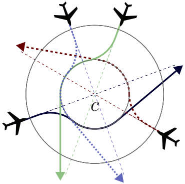
Hybrid Systems Analysis is Important for . . .

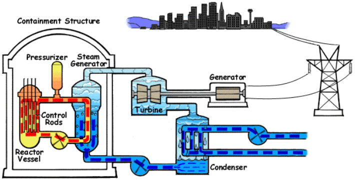


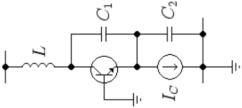












Outline

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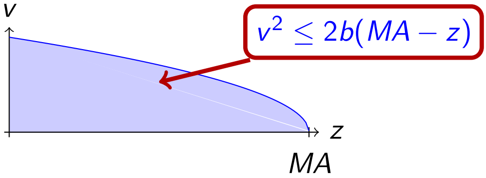
differential dynamic logic

$d\mathcal{L} = \mathrm{FOL}_{\mathbb{R}}$

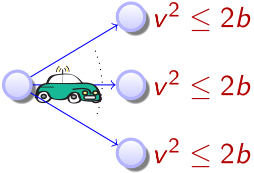


Differential Dynamic Logic for Hybrid Systems

differential dynamic logic

$\mathsf{d}\mathcal{L} = \mathrm{FOL}_{\mathbb{R}} + \mathrm{DL} + \mathrm{HP}$

$$\mathcal{C} \rightarrow [\underbrace{\mathtt{if}(z > SB)\, a := -b;\ z'' = a}_{\text{hybrid program}}]\, v^2 \leq 2b$$



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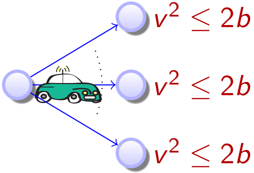
Initial condition — System dynamics — Post condition



Differential Dynamic Logic d$\mathcal{L}$: Syntax

Definition (Hybrid program α)

$$x := \theta \mid ?H \mid x' = f(x) \,\&\, H \mid \alpha \cup \beta \mid \alpha ; \beta \mid \alpha^*$$

Definition (d$\mathcal{L}$ Formula ϕ)

$$\theta_1 \geq \theta_2 \mid \neg\phi \mid \phi \land \psi \mid \forall x\, \phi \mid \exists x\, \phi \mid [\alpha]\phi \mid \langle\alpha\rangle\phi$$

Differential Dynamic Logic d$\mathcal{L}$: Syntax

Discrete Assign | Test Condition | Differential Equation | Nondet. Choice | Seq. Compose | Nondet. Repeat

Definition (Hybrid program α)

$$x := \theta \mid ?H \mid x' = f(x) \,\&\, H \mid \alpha \cup \beta \mid \alpha; \beta \mid \alpha^*$$

Definition (d$\mathcal{L}$ Formula ϕ)

$$\theta_1 \geq \theta_2 \mid \neg\phi \mid \phi \wedge \psi \mid \forall x\, \phi \mid \exists x\, \phi \mid [\alpha]\phi \mid \langle\alpha\rangle\phi$$

All Reals | Some Reals | All Runs | Some Runs

Differential Dynamic Logic d$\mathcal{L}$: Semantics

Definition (Hybrid program α)

$$
\begin{aligned}
\rho(x := \theta) &= \{(v,w) \ : \ w = v \text{ except } [\![x]\!]_w = [\![\theta]\!]_v\} \\
\rho(?H) &= \{(v,v) \ : \ v \models H\} \\
\rho(x' = f(x)) &= \{(\varphi(0), \varphi(r)) \ : \ \varphi \models x' = f(x) \text{ for some duration } r\} \\
\rho(\alpha \cup \beta) &= \rho(\alpha) \cup \rho(\beta) \\
\rho(\alpha;\beta) &= \rho(\beta) \circ \rho(\alpha) \\
\rho(\alpha^*) &= \bigcup_{n \in \mathbb{N}} \rho(\alpha^n)
\end{aligned}
$$

Definition (d$\mathcal{L}$ Formula ϕ)

$v \models \theta_1 \geq \theta_2$ iff $[\![\theta_1]\!]_v \geq [\![\theta_2]\!]_v$

$v \models [\alpha]\phi$ iff $w \models \phi$ for all w with $(v,w) \in \rho(\alpha)$

$v \models \langle\alpha\rangle\phi$ iff $w \models \phi$ for some w with $(v,w) \in \rho(\alpha)$

$v \models \forall x\, \phi$ iff $w \models \phi$ for all w that agree with v except for x

$v \models \exists x\, \phi$ iff $w \models \phi$ for some w that agrees with v except for x

$v \models \phi \land \psi$ iff $v \models \phi$ and $v \models \psi$

Differential Dynamic Logic dℒ: Axiomatization

([:=])	$[x := \theta][(x)]\phi x \leftrightarrow [(x)]\phi\theta$
([?])	$[?H]\phi \leftrightarrow (H \to \phi)$
([′])	$[x' = f(x)]\phi \leftrightarrow \forall t{\geq}0\, [x := y(t)]\phi \qquad (y'(t) = f(y))$
([∪])	$[\alpha \cup \beta]\phi \leftrightarrow [\alpha]\phi \wedge [\beta]\phi$
([;])	$[\alpha; \beta]\phi \leftrightarrow [\alpha][\beta]\phi$
([*])	$[\alpha^*]\phi \leftrightarrow \phi \wedge [\alpha][\alpha^*]\phi$
(K)	$[\alpha](\phi \to \psi) \to ([\alpha]\phi \to [\alpha]\psi)$
(I)	$[\alpha^*](\phi \to [\alpha]\phi) \to (\phi \to [\alpha^*]\phi)$
(C)	$[\alpha^*]\forall v{>}0\, (\varphi(v) \to \langle\alpha\rangle\varphi(v-1)) \to \forall v\, (\varphi(v) \to \langle\alpha^*\rangle\exists v{\leq}0\, \varphi(v))$

Differential Dynamic Logic d$\mathcal{L}$: Axiomatization

(G) $\dfrac{\phi}{[\alpha]\phi}$

(MP) $\dfrac{\phi \to \psi \quad \phi}{\psi}$

($\forall$) $\dfrac{\phi}{\forall x\, \phi}$

Differential Dynamic Logic d$\mathcal{L}$: Axiomatization

(G) $$\frac{\phi}{[\alpha]\phi}$$

(MP) $$\frac{\phi \rightarrow \psi \quad \phi}{\psi}$$

($\forall$) $$\frac{\phi}{\forall x\, \phi}$$

(B) $$\forall x\, [\alpha]\phi \rightarrow [\alpha]\forall x\, \phi \qquad (x \notin \alpha)$$

(V) $$\phi \rightarrow [\alpha]\phi \qquad (FV(\phi) \cap BV(\alpha) = \emptyset)$$

Soundness

Theorem (Soundness)

d$\mathcal{L}$ *calculus is sound, i.e., all provable* d$\mathcal{L}$ *formulas are valid:*

$$\vdash \phi \text{ implies } \vDash \phi$$

What about the converse?

Complete Proof Theory of Hybrid Systems

Theorem (Relative Completeness) (J.Autom.Reas. 2008)

d$\mathcal{L}$ *calculus is a sound & complete axiomatization of hybrid systems relative to differential equations.* Proof 15p

Complete Proof Theory of Hybrid Systems

Theorem (Continuous Relative Completeness) (J.Autom.Reas. 2008)

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▸ Proof 15p

Theorem (Discrete Relative Completeness) (LICS'12)

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▸ Proof +10p

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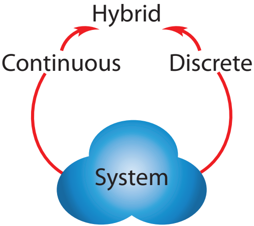

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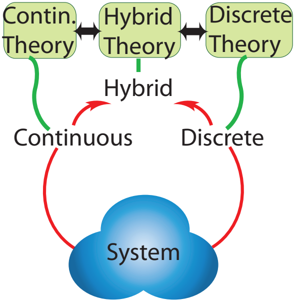
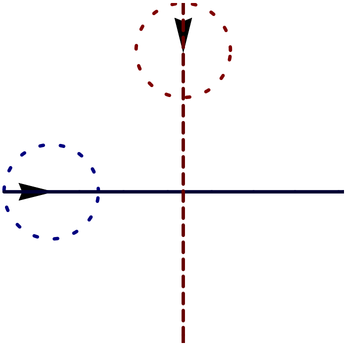


Outline

Air Traffic Control



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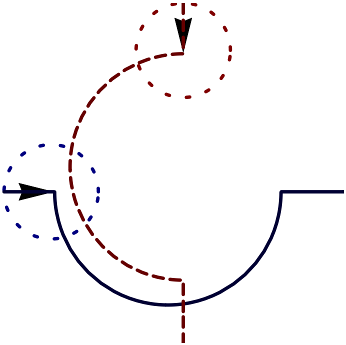

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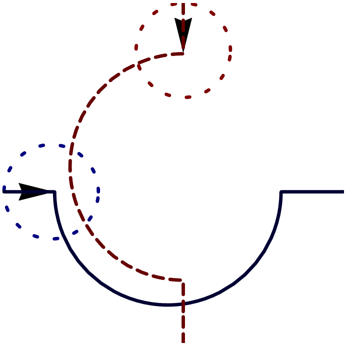
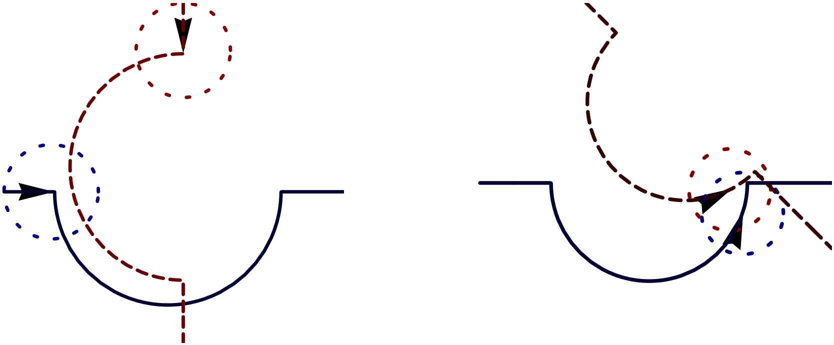


Verification?

looks correct

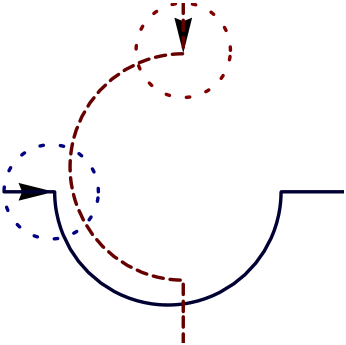
Air Traffic Control

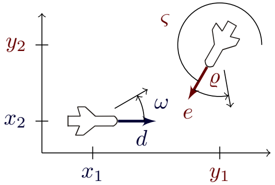


Verification?

looks correct NO!

Air Traffic Control



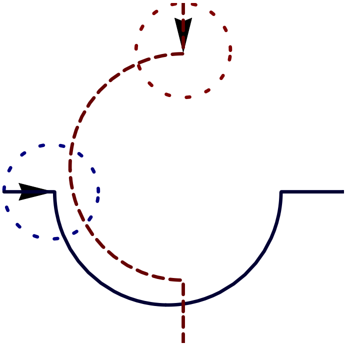


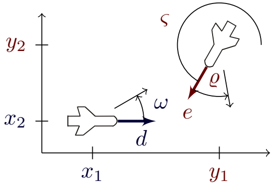
$$\begin{bmatrix} x_1' = -v_1 + v_2 \cos\vartheta + \omega x_2 \\ x_2' = v_2 \sin\vartheta - \omega x_1 \\ \vartheta' = \varpi - \omega \end{bmatrix}$$

Verification?

looks correct NO!

Air Traffic Control



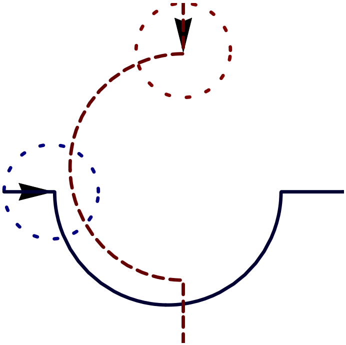


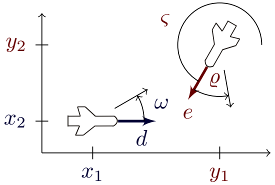
$$\begin{bmatrix} x_1' = -v_1 + v_2 \cos\vartheta + \omega x_2 \\ x_2' = v_2 \sin\vartheta - \omega x_1 \\ \vartheta' = \varpi - \omega \end{bmatrix}$$

Example ("Solving" differential equations)

$$x_1(t) = \frac{1}{\omega\varpi}\big(x_1\omega\varpi \cos t\omega - v_2\omega \cos t\omega \sin\vartheta + v_2\omega \cos t\omega \cos t\varpi \sin\vartheta - v_1\varpi \sin t\omega$$
$$+ x_2\omega\varpi \sin t\omega - v_2\omega \cos\vartheta \cos t\varpi \sin t\omega - v_2\omega\sqrt{1 - \sin\vartheta^2} \sin t\omega$$
$$+ v_2\omega \cos\vartheta \cos t\omega \sin t\varpi + v_2\omega \sin\vartheta \sin t\omega \sin t\varpi\big) \ldots$$

Air Traffic Control





$$\begin{bmatrix} x_1' = -v_1 + v_2 \cos\vartheta + \omega x_2 \\ x_2' = v_2 \sin\vartheta - \omega x_1 \\ \vartheta' = \varpi - \omega \end{bmatrix}$$

Example ("Solving" differential equations)

$$\forall t{\geq}0 \quad \frac{1}{\omega\varpi}\big(x_1\omega\varpi\cos t\omega - v_2\omega\cos t\omega\sin\vartheta + v_2\omega\cos t\omega\cos t\varpi\sin\vartheta - v_1\varpi\sin t\omega$$

$$+ x_2\omega\varpi\sin t\omega - v_2\omega\cos\vartheta\cos t\varpi\sin t\omega - v_2\omega\sqrt{1-\sin\vartheta^2}\sin t\omega$$

$$+ v_2\omega\cos\vartheta\cos t\omega\sin t\varpi + v_2\omega\sin\vartheta\sin t\omega\sin t\varpi\big)\ldots$$

```
\forall R ts2.
  (   0 <= ts2 & ts2 <= t2_0
   ->       (  (om_1)^-1
             * (omb_1)^-1
             * (  om_1 * omb_1 * x1 * Cos(om_1 * ts2)
                + om_1 * v2 * Cos(om_1 * ts2) * (1 + -1 * (Cos(u))^2)^(1 / 2)
                + -1 * omb_1 * v1 * Sin(om_1 * ts2)
                + om_1 * omb_1 * x2 * Sin(om_1 * ts2)
                + om_1 * v2 * Cos(u) * Sin(om_1 * ts2)
                + -1 * om_1 * v2 * Cos(omb_1 * ts2) * Cos(u) * Sin(om_1 * ts2)
                + om_1 * v2 * Cos(om_1 * ts2) * Cos(u) * Sin(omb_1 * ts2)
                + om_1 * v2 * Cos(om_1 * ts2) * Cos(omb_1 * ts2) * Sin(u)
                + om_1 * v2 * Sin(om_1 * ts2) * Sin(omb_1 * ts2) * Sin(u)))
        ^2
       +    (  (om_1)^-1
             * (omb_1)^-1
             * (  -1 * omb_1 * v1 * Cos(om_1 * ts2)
                + om_1 * omb_1 * x2 * Cos(om_1 * ts2)
                + omb_1 * v1 * (Cos(om_1 * ts2))^2
                + om_1 * v2 * Cos(om_1 * ts2) * Cos(u)
                + -1 * om_1 * v2 * Cos(om_1 * ts2) * Cos(omb_1 * ts2) * Cos(u)
                + -1 * om_1 * omb_1 * x1 * Sin(om_1 * ts2)
                +   -1
                  * om_1
                  * v2
                  * (1 + -1 * (Cos(u))^2)^(1 / 2)
                  * Sin(om_1 * ts2)
                + omb_1 * v1 * (Sin(om_1 * ts2))^2
                + -1 * om_1 * v2 * Cos(u) * Sin(om_1 * ts2) * Sin(omb_1 * ts2)
                + -1 * om_1 * v2 * Cos(omb_1 * ts2) * Sin(om_1 * ts2) * Sin(u)
                + om_1 * v2 * Cos(om_1 * ts2) * Sin(omb_1 * ts2) * Sin(u)))
        ^2
      >= (p)^2),
 t2_0 >= 0,
 x1^2 + x2^2 >= (p)^2
==>
```

```
\forall R t7.
  (   t7 >= 0
   ->       (   (om_3)^-1
             * (     om_3
                  * (   (om_1)^-1
                     * (omb_1)^-1
                     * (   om_1 * omb_1 * x1 * Cos(om_1 * t2_0)
                        +    om_1
                          * v2
                          * Cos(om_1 * t2_0)
                          * (1 + -1 * (Cos(u))^2)^(1 / 2)
                        + -1 * omb_1 * v1 * Sin(om_1 * t2_0)
                        + om_1 * omb_1 * x2 * Sin(om_1 * t2_0)
                        + om_1 * v2 * Cos(u) * Sin(om_1 * t2_0)
                        +    -1
                          * om_1
                          * v2
                          * Cos(omb_1 * t2_0)
                          * Cos(u)
                          * Sin(om_1 * t2_0)
                        +    om_1
                          * v2
                          * Cos(om_1 * t2_0)
                          * Cos(u)
                          * Sin(omb_1 * t2_0)
                        +    om_1
                          * v2
                          * Cos(om_1 * t2_0)
                          * Cos(omb_1 * t2_0)
                          * Sin(u)
                        +    om_1
                          * v2
                          * Sin(om_1 * t2_0)
                          * Sin(omb_1 * t2_0)
                          * Sin(u)))
```

```
    * Cos(om_3 * t5)
  +   v2
    * Cos(om_3 * t5)
    *   (  1
         +   -1
           * (Cos(-1 * om_1 * t2_0 + omb_1 * t2_0 + u + Pi / 4))^2)
     ^(1 / 2)
  + -1 * v1 * Sin(om_3 * t5)
  +   om_3
    * (  (om_1)^-1
       * (omb_1)^-1
       * (  -1 * omb_1 * v1 * Cos(om_1 * t2_0)
          + om_1 * omb_1 * x2 * Cos(om_1 * t2_0)
          + omb_1 * v1 * (Cos(om_1 * t2_0))^2
          + om_1 * v2 * Cos(om_1 * t2_0) * Cos(u)
          +   -1
            * om_1
            * v2
            * Cos(om_1 * t2_0)
            * Cos(omb_1 * t2_0)
            * Cos(u)
          + -1 * om_1 * omb_1 * x1 * Sin(om_1 * t2_0)
          +   -1
            * om_1
            * v2
            * (1 + -1 * (Cos(u))^2)^(1 / 2)
            * Sin(om_1 * t2_0)
          + omb_1 * v1 * (Sin(om_1 * t2_0))^2
          +   -1
            * om_1
            * v2
            * Cos(u)
            * Sin(om_1 * t2_0)
            * Sin(omb_1 * t2_0)
```

```
                +   -1
                  * om_1
                  * v2
                  * Cos(omb_1 * t2_0)
                  * Sin(om_1 * t2_0)
                  * Sin(u)
                +   om_1
                  * v2
                  * Cos(om_1 * t2_0)
                  * Sin(omb_1 * t2_0)
                  * Sin(u)))
          * Sin(om_3 * t5)
        +   v2
          * Cos(-1 * om_1 * t2_0 + omb_1 * t2_0 + u + Pi / 4)
          * Sin(om_3 * t5)
        +   v2
          * (Cos(om_3 * t5))^2
          * Sin(-1 * om_1 * t2_0 + omb_1 * t2_0 + u + Pi / 4)
        +   v2
          * (Sin(om_3 * t5))^2
          * Sin(-1 * om_1 * t2_0 + omb_1 * t2_0 + u + Pi / 4)))
 ^2
+   (  (om_3)^-1
     * (  -1 * v1 * Cos(om_3 * t5)
        +   om_3
          * (  (om_1)^-1
             * (omb_1)^-1
             * (  -1 * omb_1 * v1 * Cos(om_1 * t2_0)
                + om_1 * omb_1 * x2 * Cos(om_1 * t2_0)
                + omb_1 * v1 * (Cos(om_1 * t2_0))^2
                + om_1 * v2 * Cos(om_1 * t2_0) * Cos(u)
                +   -1
                  * om_1
                  * v2
                  * Cos(om_1 * t2_0)
                  * Cos(omb_1 * t2_0)
```

```
        + -1 * om_1 * omb_1 * x1 * Sin(om_1 * t2_0)
        +   -1
          * om_1
          * v2
          * (1 + -1 * (Cos(u))^2)^(1 / 2)
          * Sin(om_1 * t2_0)
        + omb_1 * v1 * (Sin(om_1 * t2_0))^2
        +   -1
          * om_1
          * v2
          * Cos(u)
          * Sin(om_1 * t2_0)
          * Sin(omb_1 * t2_0)
        +   -1
          * om_1
          * v2
          * Cos(omb_1 * t2_0)
          * Sin(om_1 * t2_0)
          * Sin(u)
        +   om_1
          * v2
          * Cos(om_1 * t2_0)
          * Sin(omb_1 * t2_0)
          * Sin(u)))
  * Cos(om_3 * t5)
+ v1 * (Cos(om_3 * t5))^2
+   v2
  * Cos(om_3 * t5)
  * Cos(-1 * om_1 * t2_0 + omb_1 * t2_0 + u + Pi / 4)
+   -1
  * v2
  * (Cos(om_3 * t5))^2
  * Cos(-1 * om_1 * t2_0 + omb_1 * t2_0 + u + Pi / 4)
```

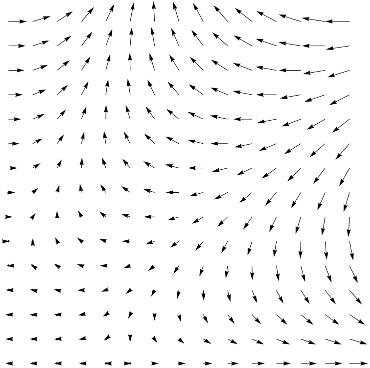
```
+   -1
  * om_3
  * (  (om_1)^-1
     * (omb_1)^-1
     * (  om_1 * omb_1 * x1 * Cos(om_1 * t2_0)
        +   om_1
          * v2
          * Cos(om_1 * t2_0)
          * (1 + -1 * (Cos(u))^2)^(1 / 2)
        + -1 * omb_1 * v1 * Sin(om_1 * t2_0)
        + om_1 * omb_1 * x2 * Sin(om_1 * t2_0)
        + om_1 * v2 * Cos(u) * Sin(om_1 * t2_0)
        +   -1
          * om_1
          * v2
          * Cos(omb_1 * t2_0)
          * Cos(u)
          * Sin(om_1 * t2_0)
        +   om_1
          * v2
          * Cos(om_1 * t2_0)
          * Cos(u)
          * Sin(omb_1 * t2_0)
        +   om_1
          * v2
          * Cos(om_1 * t2_0)
          * Cos(omb_1 * t2_0)
          * Sin(u)
        +   om_1
          * v2
          * Sin(om_1 * t2_0)
          * Sin(omb_1 * t2_0)
          * Sin(u)))
  * Sin(om_3 * t5)
```

```
        +   -1
          * v2
          *   (  1
                +   -1
                  * (Cos(-1 * om_1 * t2_0 + omb_1 * t2_0 + u + Pi / 4))^2)
           ^(1 / 2)
          * Sin(om_3 * t5)
        + v1 * (Sin(om_3 * t5))^2
        +   -1
          * v2
          * Cos(-1 * om_1 * t2_0 + omb_1 * t2_0 + u + Pi / 4)
          * (Sin(om_3 * t5))^2))
  ^2
>= (p)^2)
```

This is just one branch to prove

"Definition" (Differential Invariant)

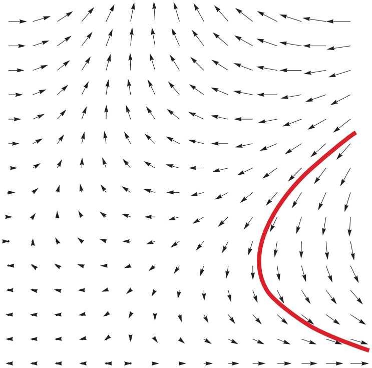
"Formula that remains true in the direction of the dynamics"



Differential Invariants for Differential Equations

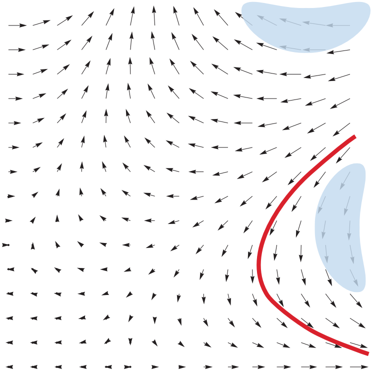
"Definition" (Differential Invariant)

"Formula that remains true in the direction of the dynamics"



"Definition" (Differential Invariant)

"Formula that remains true in the direction of the dynamics"



Differential Induction: Local Dynamics w/o Solutions

Definition (Differential Invariant) (J.Log.Comput. 2010)

F closed under total differentiation with respect to differential constraints

Differential Induction: Local Dynamics w/o Solutions

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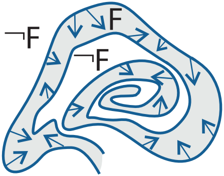


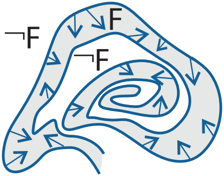
$$\frac{(\chi \to F')}{\chi \to F \to [x' = \theta \,\&\, \chi]F}$$

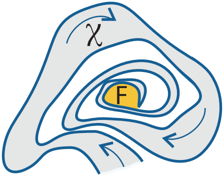
$$\frac{F \to [\alpha]F}{F \to [\alpha^*]F}$$

Differential Induction: Local Dynamics w/o Solutions

Definition (Differential Invariant) (J.Log.Comput. 2010)

F closed under total differentiation with respect to differential constraints





$$\frac{(\chi \to F')}{\chi \to F \to [x' = \theta \,\&\, \chi]F}$$

Definition (Differential Invariant) (J.Log.Comput. 2010)

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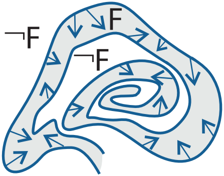
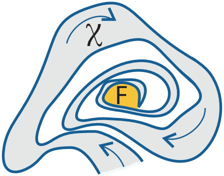




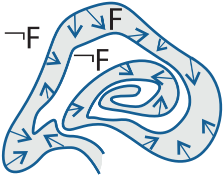
$$\frac{(\chi \rightarrow F')}{\chi \rightarrow F \rightarrow [x' = \theta \,\&\, \chi]F}$$

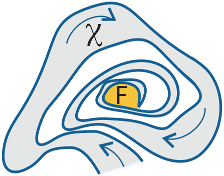
$$\frac{(\neg F \land \chi \rightarrow F'_{\gg})}{[x' = \theta \,\&\, \neg F]\chi \rightarrow \langle x' = \theta \,\&\, \chi\rangle F}$$

Differential Induction: Local Dynamics w/o Solutions

Definition (Differential Invariant) (J.Log.Comput. 2010)

F closed under total differentiation with respect to differential constraints





$$\frac{(\chi \rightarrow F')}{\chi \rightarrow F \rightarrow [x' = \theta \,\&\, \chi]F}$$

$$\frac{(\neg F \wedge \chi \rightarrow F'_{\gg})}{[x' = \theta \,\&\, \neg F]\chi \rightarrow \langle x' = \theta \,\&\, \chi\rangle F}$$

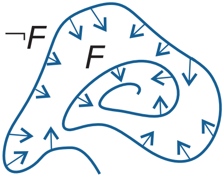
Total differential F' of *formulas*?

Equational Differential Invariants

Equational Differential Invariants

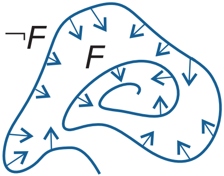
$$\frac{}{\rightarrow [x' = \theta \,\&\, H] p = 0}$$

Equational Differential Invariants



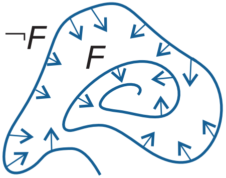
$$\frac{}{(H \rightarrow p = 0) \rightarrow [x' = \theta \,\&\, H] p = 0}$$

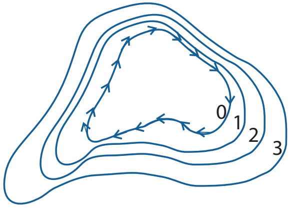
Equational Differential Invariants



$$\frac{H \rightarrow p' = 0}{(H \rightarrow p = 0) \rightarrow [x' = \theta \,\&\, H] p = 0}$$

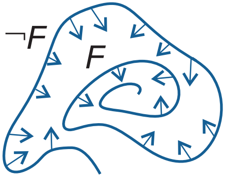
Equational Differential Invariants

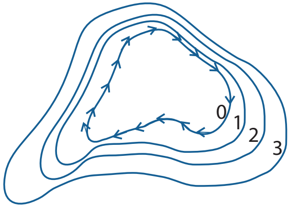




$$\frac{H \to p' = 0}{(H \to p = 0) \to [x' = \theta \,\&\, H]p = 0}$$

Equational Differential Invariants

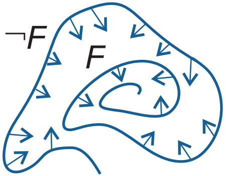


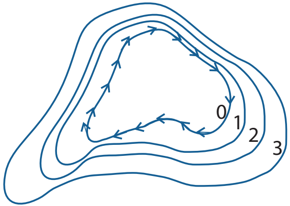


$$\frac{H \rightarrow p' = 0}{(H \rightarrow p = 0) \rightarrow [x' = \theta \,\&\, H] p = 0}$$

$$\frac{}{p = c \rightarrow [x' = f(x) \,\&\, H] p = c}$$

Equational Differential Invariants

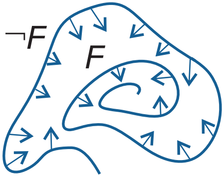


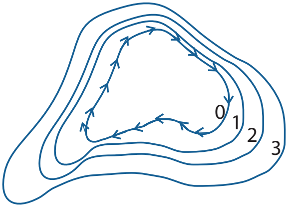


$$\frac{H \rightarrow p' = 0}{(H \rightarrow p = 0) \rightarrow [x' = \theta \,\&\, H]p = 0}$$

$$\frac{H \rightarrow p' = 0}{p = c \rightarrow [x' = f(x) \,\&\, H]p = c}$$

Equational Differential Invariants





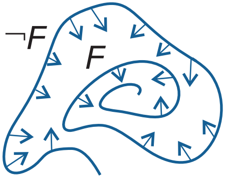
$$\frac{H \to p' = 0}{(H \to p = 0) \to [x' = \theta \,\&\, H]p = 0}$$

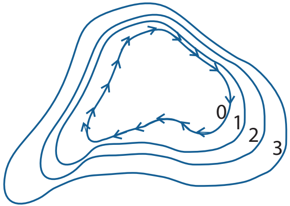
$$\frac{H \to p' = 0}{\forall c\, \big(p = c \to [x' = f(x) \,\&\, H]p = c\big)}$$

Equational Differential Invariants

Theorem (Lie)

$$\frac{H \to p' = 0}{\forall c\, \big(p = c \to [x' = f(x) \,\&\, H] p = c\big)}$$ *equivalence if H open*





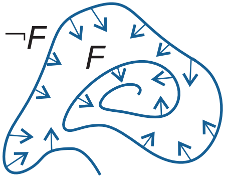
$$\frac{H \to p' = 0}{(H \to p = 0) \to [x' = \theta \,\&\, H] p = 0}$$

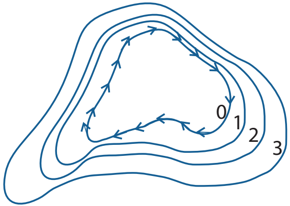
$$\frac{H \to p' = 0}{\forall c\, \big(p = c \to [x' = f(x) \,\&\, H] p = c\big)}$$

Equational Differential Invariants

Theorem (Lie)

$$\frac{H \to p' = 0}{\forall c\, (p = c \to [x' = f(x) \,\&\, H] p = c)} \quad \textit{equivalence if } H \textit{ open}$$





$$\frac{H \to p' = 0}{(H \to p = 0) \to [x' = \theta \,\&\, H] p = 0}$$

$$\frac{H \to p' = 0}{\forall c\, (p = c \to [x' = f(x) \,\&\, H] p = c)}$$

Corollary (Decidable invariant polynomials)

Decidable whether polynomial p invariant function of $x' = f(x)$ on open H

Lie Generates Invariants

Corollary (Decidable invariant polynomials)

Decidable whether polynomial p invariant function of $x' = f(x)$ on open H

Lie Generates Invariants

Corollary (Decidable invariant polynomials)

Decidable whether polynomial p *invariant function of* $x' = f(x)$ *on open* H

Corollary (Invariant polynomials with $\mathbb{R} \cap \overline{\mathbb{Q}}$ coefficients r.e.)

Invariant polynomial function $p \in (\mathbb{R} \cap \overline{\mathbb{Q}})[x]$ *of* $x' = f(x)$ *on open* H *r.e.*

Lie Generates Invariants

Corollary (Decidable invariant polynomials)

Decidable whether polynomial p invariant function of $x' = f(x)$ on open H

Corollary (Invariant polynomials with $\mathbb{R} \cap \overline{\mathbb{Q}}$ coefficients r.e.)

Invariant polynomial function $p \in (\mathbb{R} \cap \overline{\mathbb{Q}})[x]$ of $x' = f(x)$ on open H r.e.

Proof (Direct Method).

1. for $p \stackrel{\text{def}}{=} a_2 x^2 + a_1 x + a_0$
2. with $a_2 = 4, a_1 = -1, a_0 = 5$
3. prove $\forall x\, (H \rightarrow p' = 0)$

□

Lie Generates Invariants

Corollary (Decidable invariant polynomials)

Decidable whether polynomial p invariant function of $x' = f(x)$ on open H

Corollary (Invariant polynomials with $\mathbb{R} \cap \overline{\mathbb{Q}}$ coefficients r.e.)

Invariant polynomial function $p \in (\mathbb{R} \cap \overline{\mathbb{Q}})[x]$ of $x' = f(x)$ on open H r.e.

Proof (Direct Method).

1. for $p \stackrel{\text{def}}{=} a_2x^2 + a_1x + a_0$
2. with $a_2 = 4, a_1 = -1, a_0 = 6$
3. prove $\forall x\,(H \rightarrow p' = 0)$

$\square$

Lie Generates Invariants

Corollary (Decidable invariant polynomials)

Decidable whether polynomial p invariant function of $x' = f(x)$ on open H

Corollary (Invariant polynomials with $\mathbb{R} \cap \overline{\mathbb{Q}}$ coefficients r.e.)

Invariant polynomial function $p \in (\mathbb{R} \cap \overline{\mathbb{Q}})[x]$ of $x' = f(x)$ on open H r.e.

Proof (Direct Method).

1. for $p \stackrel{\text{def}}{=} a_2x^2 + a_1x + a_0$
2. with $a_2 = 4, a_1 = -1, a_0 = 7$
3. prove $\forall x\, (H \rightarrow p' = 0)$

□

Lie Generates Invariants

Corollary (Decidable invariant polynomials)

Decidable whether polynomial p invariant function of $x' = f(x)$ on open H

Corollary (Invariant polynomials with $\mathbb{R} \cap \overline{\mathbb{Q}}$ coefficients r.e.)

Invariant polynomial function $p \in (\mathbb{R} \cap \overline{\mathbb{Q}})[x]$ of $x' = f(x)$ on open H r.e.

Proof (Direct Method).

1. for $p \stackrel{\text{def}}{=} a_2x^2 + a_1x + a_0$
2. with $a_2 = 4, a_1 = -2, a_0 = 5$
3. prove $\forall x\,(H \rightarrow p' = 0)$

□

Lie Generates Invariants

Corollary (Decidable invariant polynomials)

Decidable whether polynomial p invariant function of $x' = f(x)$ on open H

Corollary (Invariant polynomials with $\mathbb{R} \cap \overline{\mathbb{Q}}$ coefficients r.e.)

Invariant polynomial function $p \in (\mathbb{R} \cap \overline{\mathbb{Q}})[x]$ of $x' = f(x)$ on open H r.e.

Proof (Direct Method).

1. for $p \stackrel{\text{def}}{=} a_2x^2 + a_1x + a_0$
2. with $a_2 = -4, a_1 = 2, a_0 = 8$
3. prove $\forall x\,(H \rightarrow p' = 0)$
3. Problem: enumerating all polynomials takes a while . . .

□

Lie Generates Invariants

Corollary (Decidable invariant polynomials)

Decidable whether polynomial p invariant function of $x' = f(x)$ on open H

Corollary (Invariant polynomials with $\mathbb{R} \cap \overline{\mathbb{Q}}$ coefficients r.e.)

Invariant polynomial function $p \in (\mathbb{R} \cap \overline{\mathbb{Q}})[x]$ of $x' = f(x)$ on open H r.e.

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3. prove $\forall x\,(H \rightarrow p' = 0)$
3. Instead: $\exists a\, \forall x\,(H \rightarrow p' = 0)$

□

Lie Generates Invariants

Corollary (Decidable invariant polynomials)

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Corollary (Invariant polynomials with $\mathbb{R} \cap \overline{\mathbb{Q}}$ coefficients r.e.)

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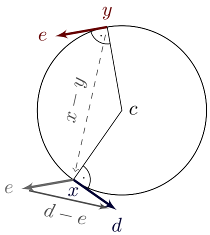
Proof (Direct Method).

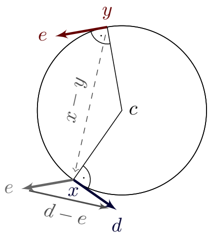
1. for $p \stackrel{\text{def}}{=} a_2 x^2 + a_1 x + a_0$
2. with $a_2 = -4, a_1 = 2, a_0 = 8$
3. prove $\forall x\,(H \rightarrow p' = 0)$
3. Instead: $\exists a\,\forall x\,(H \rightarrow p' = 0)$
4. Still enumerate polynomial degrees ...

□

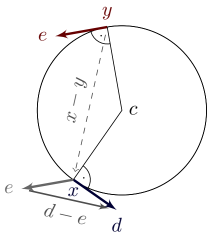
$$\frac{}{x^2 + y^2 = 1 \land e = x \rightarrow [x' = -y, y' = e, e' = -y](x^2 + y^2 = 1 \land e = x)}$$



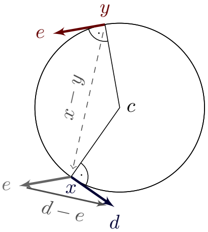
$$\frac{\overline{\qquad -y\frac{\partial(x^2+y^2)}{\partial x} + e\frac{\partial(x^2+y^2)}{\partial y} = 0 \land -y\frac{\partial e}{\partial e} = -y\frac{\partial x}{\partial x}}}{x^2+y^2=1 \land e=x \rightarrow [x'=-y, y'=e, e'=-y](x^2+y^2=1 \land e=x)}$$



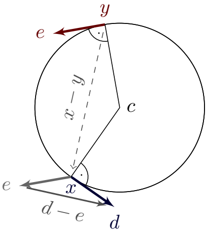
$$\dfrac{\dfrac{(-y)2x + e2y = 0 \land -y = -y}{-y\frac{\partial(x^2+y^2)}{\partial x} + e\frac{\partial(x^2+y^2)}{\partial y} = 0 \land -y\frac{\partial e}{\partial e} = -y\frac{\partial x}{\partial x}}}{x^2+y^2=1 \land e = x \rightarrow [x' = -y, y' = e, e' = -y](x^2+y^2=1 \land e = x)}$$



$$\dfrac{\dfrac{\dfrac{-2xy + 2ey = 0}{(-y)2x + e2y = 0 \land -y = -y}}{-y\frac{\partial(x^2+y^2)}{\partial x} + e\frac{\partial(x^2+y^2)}{\partial y} = 0 \land -y\frac{\partial e}{\partial e} = -y\frac{\partial x}{\partial x}}}{x^2 + y^2 = 1 \land e = x \rightarrow [x' = -y, y' = e, e' = -y](x^2 + y^2 = 1 \land e = x)}$$



$$\cfrac{\cfrac{\cfrac{\cfrac{\text{not valid}}{-2xy + 2ey = 0}}{(-y)2x + e2y = 0 \land -y = -y}}{-y\frac{\partial(x^2+y^2)}{\partial x} + e\frac{\partial(x^2+y^2)}{\partial y} = 0 \land -y\frac{\partial e}{\partial e} = -y\frac{\partial x}{\partial x}}}{x^2 + y^2 = 1 \land e = x \rightarrow [x' = -y, y' = e, e' = -y](x^2 + y^2 = 1 \land e = x)}$$



Ex: Deconstructed Aircraft (I) — Directly

$$\text{not valid}$$

$$-2xy + 2ey = 0$$

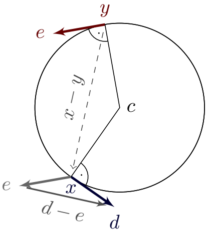
$$(-y)2x + e2y = 0 \land -y = -y$$

$$-y\frac{\partial(x^2+y^2)}{\partial x} + e\frac{\partial(x^2+y^2)}{\partial y} = 0 \land -y\frac{\partial e}{\partial e} = -y\frac{\partial x}{\partial x}$$

$x^2 + y^2 = 1 \land e$ … $y^2 = 1 \land e = x)$

Not Provable?

Wait! It's true. Why not proved?



Ex: Deconstructed Aircraft (I)

not valid

$$-2xy + 2ey = 0$$

$$(-y)2x + e2y = 0 \land -y = -y$$

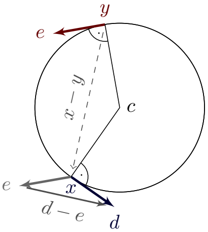
$$-y\frac{\partial(x^2+y^2)}{\partial x} + e\frac{\partial(x^2+y^2)}{\partial y} = 0 \land -y\frac{\partial e}{\partial e} = -y\frac{\partial x}{\partial x}$$

$x^2 + y^2 = 1 \land e$ … $y^2 = 1 \land e = x)$

Not Provable?

Wait! It's true. Why not proved?

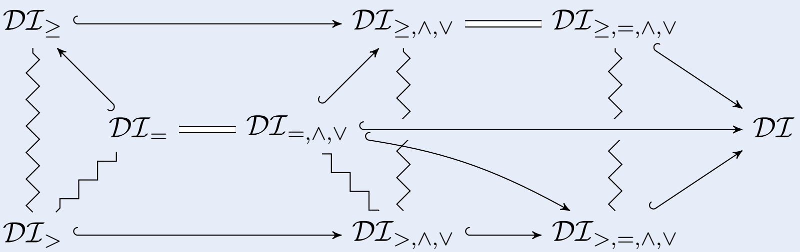
not single equation



Theorem (Closure properties of differential invariants) (LMCS 2012)

Closed under conjunction, differentiation, and propositional equivalences.

Theorem (Differential Invariance Chart) (LMCS 2012)



$$\overline{\qquad\qquad\qquad\qquad} \\ \ldots \rightarrow [x' = -y, y' = e, e' = -y](x^2 + y^2 - 1)^2 + (e - x)^2 = 0$$

Reduce to single equation, try again

Ex: Deconstructed Aircraft (II)

$$
\dfrac{\dfrac{\dfrac{\dfrac{\text{not valid}}{2(x^2+y^2-1)(-2yx+2ey)=0}}{2(x^2+y^2-1)(-y2x+e2y)+2(e-x)(-y-(-y))=0}}{(-y\frac{\partial}{\partial x}+e\frac{\partial}{\partial y}-y\frac{\partial}{\partial e})\big((x^2+y^2-1)^2+(e-x)^2\big)=0}}{\ldots\rightarrow[x'=-y,y'=e,e'=-y](x^2+y^2-1)^2+(e-x)^2=0}
$$

Reduce to single equation, try again

not valid

$2(x^2 + y^2 - 1)(-2yx + 2ey) = 0$

$2(x^2 + y^2 - 1)(-y2x + e2y) + 2(e - x)(-y - (-y)) = 0$

$(-y\frac{\partial}{\partial x} + e\frac{\partial}{\partial y} - y\frac{\partial}{\partial e})((x^2 + y^2 - 1)^2 + (e - x)^2) = 0$

$\ldots \rightarrow [x' = \ldots (e - x)^2 = 0$

Not Provable?

Wait! It's true. Why not proved?

not valid

$$2(x^2+y^2-1)(-2yx+2ey)=0$$

$$2(x^2+y^2-1)(-y2x+e2y)+2(e-x)(-y-(-y))=0$$

$$\left(-y\frac{\partial}{\partial x}+e\frac{\partial}{\partial y}-y\frac{\partial}{\partial e}\right)\left((x^2+y^2-1)^2+(e-x)^2\right)=0$$

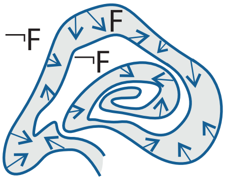
$$\ldots \rightarrow [x'=-y, y'=e, e'=-y](x^2+y^2-1)^2+(e-x)^2=0$$

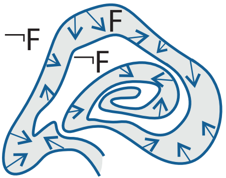
Reduce to single equation, try again

Could Prove?

If only we could assume invariant F during its proof ...

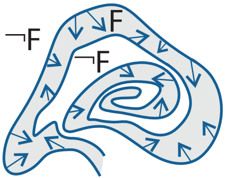
Assuming Differential Invariance



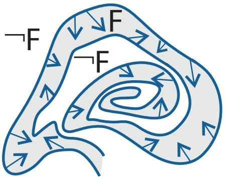


$$\frac{(H \to F')}{(H \to F) \to [x' = \theta \,\&\, H]F}$$

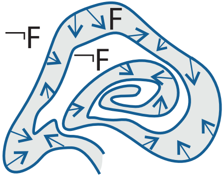
Assuming Differential Invariance



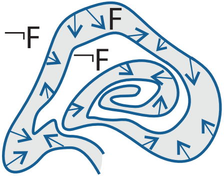
$$\frac{(H \to F')}{(H \to F) \to [x' = \theta \,\&\, H]F}$$



$$\frac{(F \land H \to F')}{(H \to F) \to [x' = \theta \,\&\, H]F}$$



$$\frac{(H \to F')}{(H \to F) \to [x' = \theta \,\&\, H]F}$$

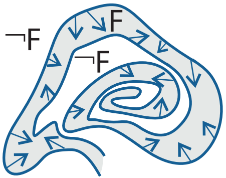


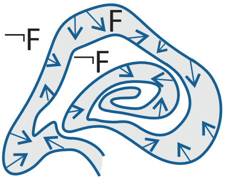
$$\frac{(F \land H \to F')}{(H \to F) \to [x' = \theta \,\&\, H]F}$$

Example (Restrictions)

$$\frac{}{x^2 - 6x + 9 = 0 \to [x' = y, y' = -x]x^2 - 6x + 9 = 0}$$

Assuming Differential Invariance





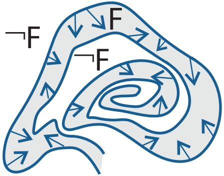
$$\frac{(H \rightarrow F')}{(H \rightarrow F) \rightarrow [x' = \theta \,\&\, H]F}$$

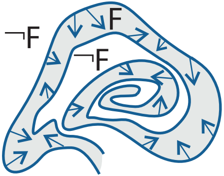
$$\frac{(F \land H \rightarrow F')}{(H \rightarrow F) \rightarrow [x' = \theta \,\&\, H]F}$$

Example (Restrictions)

$$\frac{\overline{x^2 - 6x + 9 = 0 \rightarrow y\frac{\partial(x^2-6x+9)}{\partial x} - x\frac{\partial(x^2-6x+9)}{\partial y} = 0}}{x^2 - 6x + 9 = 0 \rightarrow [x' = y, y' = -x]x^2 - 6x + 9 = 0}$$

Assuming Differential Invariance

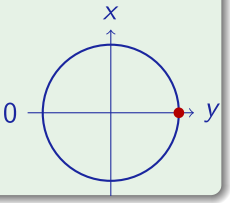




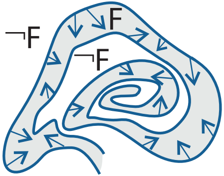
$$\frac{(H \to F')}{(H \to F) \to [x' = \theta \,\&\, H]F} \qquad \frac{(F \land H \to F')}{(H \to F) \to [x' = \theta \,\&\, H]F}$$

Example (Restrictions)

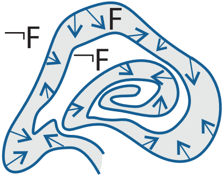
$$\cfrac{\cfrac{x^2 - 6x + 9 = 0 \to y2x - 6y = 0}{x^2 - 6x + 9 = 0 \to y\frac{\partial(x^2-6x+9)}{\partial x} - x\frac{\partial(x^2-6x+9)}{\partial y} = 0}}{x^2 - 6x + 9 = 0 \to [x' = y, y' = -x]x^2 - 6x + 9 = 0}$$



Assuming Differential Invariance



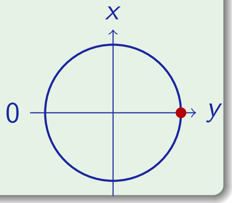
$$\frac{(H \rightarrow F')}{(H \rightarrow F) \rightarrow [x' = \theta \,\&\, H]F}$$



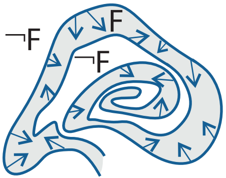
$$\frac{(F \land H \rightarrow F')}{(H \rightarrow F) \rightarrow [x' = \theta \,\&\, H]F}$$

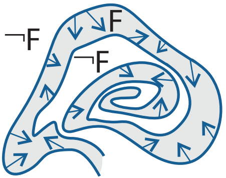
Example (Restrictions are unsound!)

$$\frac{\dfrac{x^2 - 6x + 9 = 0 \rightarrow y2x - 6y = 0}{x^2 - 6x + 9 = 0 \rightarrow y\frac{\partial(x^2-6x+9)}{\partial x} - x\frac{\partial(x^2-6x+9)}{\partial y} = 0}}{x^2 - 6x + 9 = 0 \rightarrow [x' = y, y' = -x]x^2 - 6x + 9 = 0}$$



Assuming Differential Invariance





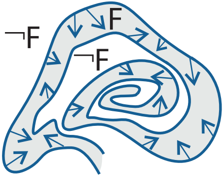
$$\frac{(H \to F')}{(H \to F) \to [x' = \theta \,\&\, H]F}$$

$$\frac{(F \land H \to F')}{(H \to F) \to [x' = \theta \,\&\, H]F}$$

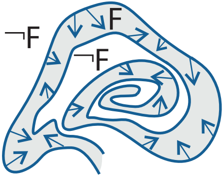
Example (Restrictions)

$$\frac{(x^2 \leq 0 \to 2x \cdot 1 \leq 0)}{x^2 \leq 0 \to [x' = 1]x^2 \leq 0}$$

Assuming Differential Invariance



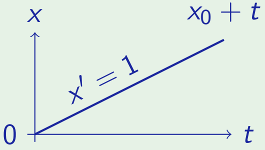
$$\frac{(H \to F')}{(H \to F) \to [x' = \theta \,\&\, H]F}$$



$$\frac{(F \wedge H \to F')}{(H \to F) \to [x' = \theta \,\&\, H]F}$$

Example (Restrictions are unsound!)

$$\frac{(x^2 \leq 0 \to 2x \cdot 1 \leq 0)}{x^2 \leq 0 \to [x' = 1]x^2 \leq 0}$$



$$x^2 + y^2 = 1 \land e = x \rightarrow [x' = -y, y' = e, e' = -y](x^2 + y^2 = 1 \land e = x)$$

$$\frac{}{\ldots \rightarrow [x' = -y, y' = e, e' = -y \,\&\, e = x](x^2 + y^2 = 1 \land e = x)}$$

$$\frac{\overline{e = x \rightarrow [x' = -y, y' = e, e' = -y]e = x} \quad \triangleright}{x^2 + y^2 = 1 \land e = x \rightarrow [x' = -y, y' = e, e' = -y](x^2 + y^2 = 1 \land e = x)}$$

$$\overline{\ldots \rightarrow [x' = -y, y' = e, e' = -y \,\&\, e = x](x^2 + y^2 = 1 \land e = x)}$$

$$\cfrac{\cfrac{\overline{-y\frac{\partial e}{\partial e} = -y\frac{\partial x}{\partial x}}}{e = x \rightarrow [x' = -y, y' = e, e' = -y]e = x \quad \triangleright}}{x^2 + y^2 = 1 \land e = x \rightarrow [x' = -y, y' = e, e' = -y](x^2 + y^2 = 1 \land e = x)}$$

$$\overline{\ldots \to [x' = -y, y' = e, e' = -y \,\&\, e = x](x^2 + y^2 = 1 \land e = x)}$$

$$\cfrac{\cfrac{\cfrac{\cfrac{}{-y = -y}}{-y\frac{\partial e}{\partial e} = -y\frac{\partial x}{\partial x}}}{e = x \to [x' = -y, y' = e, e' = -y]e = x \quad \triangleright}}{x^2 + y^2 = 1 \land e = x \to [x' = -y, y' = e, e' = -y](x^2 + y^2 = 1 \land e = x)}$$

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$$\dfrac{\dfrac{\dfrac{*}{-y = -y}}{-y\frac{\partial e}{\partial e} = -y\frac{\partial x}{\partial x}}}{\dfrac{e = x \rightarrow [x' = -y, y' = e, e' = -y]e = x \qquad \triangleright}{x^2 + y^2 = 1 \land e = x \rightarrow [x' = -y, y' = e, e' = -y](x^2 + y^2 = 1 \land e = x)}}$$

$$\dfrac{}{\dfrac{e = x \to -y\frac{\partial(x^2+y^2)}{\partial x} + e\frac{\partial(x^2+y^2)}{\partial y} = 0}{\ldots \to [x' = -y, y' = e, e' = -y \,\&\, e = x](x^2 + y^2 = 1 \land e = x)}}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{*}{-y = -y}}{-y\frac{\partial e}{\partial e} = -y\frac{\partial x}{\partial x}}}{e = x \to [x' = -y, y' = e, e' = -y]e = x \quad \triangleright}}{x^2 + y^2 = 1 \land e = x \to [x' = -y, y' = e, e' = -y](x^2 + y^2 = 1 \land e = x)}$$

$$\dfrac{\dfrac{\overline{e = x \rightarrow (-y)2x + e2y = 0}}{e = x \rightarrow -y\frac{\partial(x^2+y^2)}{\partial x} + e\frac{\partial(x^2+y^2)}{\partial y} = 0}}{\ldots \rightarrow [x' = -y, y' = e, e' = -y \,\&\, e = x](x^2 + y^2 = 1 \land e = x)}$$

$$\dfrac{\dfrac{\dfrac{*}{-y = -y}}{-y\frac{\partial e}{\partial e} = -y\frac{\partial x}{\partial x}}}{\dfrac{e = x \rightarrow [x' = -y, y' = e, e' = -y]e = x \qquad \triangleright}{x^2 + y^2 = 1 \land e = x \rightarrow [x' = -y, y' = e, e' = -y](x^2 + y^2 = 1 \land e = x)}}$$

$$\dfrac{\dfrac{\dfrac{\overline{e = x \to -2yx + 2xy = 0}}{e = x \to (-y)2x + e2y = 0}}{e = x \to -y\frac{\partial(x^2+y^2)}{\partial x} + e\frac{\partial(x^2+y^2)}{\partial y} = 0}}{\ldots \to [x' = -y, y' = e, e' = -y \,\&\, e = x](x^2 + y^2 = 1 \land e = x)}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{*}{-y = -y}}{-y\frac{\partial e}{\partial e} = -y\frac{\partial x}{\partial x}}}{e = x \to [x' = -y, y' = e, e' = -y]e = x \quad \triangleright}}{x^2 + y^2 = 1 \land e = x \to [x' = -y, y' = e, e' = -y](x^2 + y^2 = 1 \land e = x)}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{*}{e = x \to -2yx + 2xy = 0}}{e = x \to (-y)2x + e2y = 0}}{e = x \to -y\frac{\partial(x^2+y^2)}{\partial x} + e\frac{\partial(x^2+y^2)}{\partial y} = 0}}{\ldots \to [x' = -y, y' = e, e' = -y \,\&\, e = x](x^2 + y^2 = 1 \land e = x)}$$

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$$*$$

$$e = x \to -2yx + 2xy = 0$$

$$e = x \to (-y)2x + e2y = 0$$

$$e = x \to -y\frac{\partial(x^2+y^2)}{\partial x} + e\frac{\partial(x^2+y^2)}{\partial y} = 0$$

$$\ldots \to \ldots y^2 = 1 \land e = x)$$

Successful Proof

Lie & differential cuts separate aircraft

$$*$$

$$-y = -y$$

$$-y\frac{\partial e}{\partial e} = -y\frac{\partial x}{\partial x}$$

$$e = x \to [x' = -y, y' = e, e' = -y]e = x \quad \triangleright$$

$$x^2 + y^2 = 1 \land e = x \to [x' = -y, y' = e, e' = -y](x^2 + y^2 = 1 \land e = x)$$

Ex: Deconstructed Aircraft (IV)

$$\frac{}{e^2 + y^2 = 1 \land e = x \rightarrow [x' = -y, y' = e, e' = -y](e^2 + y^2 = 1 \land e = x)}$$

Ex: Deconstructed Aircraft (IV) Smart

$$\frac{-y\frac{\partial(e^2+y^2)}{\partial e} + e\frac{\partial(e^2+y^2)}{\partial y} = 0 \land -y\frac{\partial e}{\partial e} = -y\frac{\partial x}{\partial x}}{e^2 + y^2 = 1 \land e = x \rightarrow [x' = -y, y' = e, e' = -y](e^2 + y^2 = 1 \land e = x)}$$

$$\dfrac{\dfrac{-y2e + e2y = 0 \land -y = -y}{-y\frac{\partial(e^2+y^2)}{\partial e} + e\frac{\partial(e^2+y^2)}{\partial y} = 0 \land -y\frac{\partial e}{\partial e} = -y\frac{\partial x}{\partial x}}}{e^2 + y^2 = 1 \land e = x \rightarrow [x' = -y, y' = e, e' = -y](e^2 + y^2 = 1 \land e = x)}$$

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$$\dfrac{\dfrac{\dfrac{*}{-y2e + e2y = 0 \land -y = -y}}{-y\frac{\partial(e^2+y^2)}{\partial e} + e\frac{\partial(e^2+y^2)}{\partial y} = 0 \land -y\frac{\partial e}{\partial e} = -y\frac{\partial x}{\partial x}}}{e^2 + y^2 = 1 \land e = x \rightarrow [x' = -y, y' = e, e' = -y](e^2 + y^2 = 1 \land e = x)}$$

Direct Proof

Smart invariant also separates aircraft?!

Differential Cuts

$$\frac{\phi \rightarrow [x' = \theta \,\&\, H]C \qquad \phi \rightarrow [x' = \theta \,\&\, (H \land C)]\phi}{\phi \rightarrow [x' = \theta \,\&\, H]\phi}$$

$$\frac{}{x^3 \geq -1 \land y^5 \geq 0 \rightarrow [x' = (x-3)^4 + y^5, y' = y^2]x^3 \geq -1}$$

Ex: Differential Cuts

$$\dfrac{}{x^3 \geq -1 \land y^5 \geq 0 \rightarrow [x' = (x-3)^4 + y^5, y' = y^2] x^3 \geq -1}$$

$$\dfrac{}{y^5 \geq 0 \rightarrow [x' = (x-3)^4 + y^5, y' = y^2] y^5 \geq 0}$$

Ex: Differential Cuts

$$\frac{}{x^3 \geq -1 \land y^5 \geq 0 \rightarrow [x' = (x-3)^4 + y^5, y' = y^2] x^3 \geq -1}$$

$$\frac{\frac{}{5y^4 y' \geq 0}}{y^5 \geq 0 \rightarrow [x' = (x-3)^4 + y^5, y' = y^2] y^5 \geq 0}$$

Ex: Differential Cuts

$$\frac{}{x^3 \geq -1 \land y^5 \geq 0 \rightarrow [x' = (x-3)^4 + y^5, y' = y^2] x^3 \geq -1}$$

$$\cfrac{\cfrac{\cfrac{}{5y^4 y^2 \geq 0}}{5y^4 y' \geq 0}}{y^5 \geq 0 \rightarrow [x' = (x-3)^4 + y^5, y' = y^2] y^5 \geq 0}$$

$$\overline{x^3 \geq -1 \land y^5 \geq 0 \rightarrow [x' = (x-3)^4 + y^5, y' = y^2] x^3 \geq -1}$$

$$\dfrac{\dfrac{\dfrac{*}{5y^4 y^2 \geq 0}}{5y^4 y' \geq 0}}{y^5 \geq 0 \rightarrow [x' = (x-3)^4 + y^5, y' = y^2] y^5 \geq 0}$$

Ex: Differential Cuts

$$\dfrac{x^3 \geq -1 \to [x' = (x-3)^4 + y^5, y' = y^2 \,\&\, y^5 \geq 0]x^3 \geq -1 \quad \triangleright}{x^3 \geq -1 \land y^5 \geq 0 \to [x' = (x-3)^4 + y^5, y' = y^2]x^3 \geq -1}$$

$$\dfrac{\dfrac{\dfrac{*}{5y^4 y^2 \geq 0}}{5y^4 y' \geq 0}}{y^5 \geq 0 \to [x' = (x-3)^4 + y^5, y' = y^2]y^5 \geq 0}$$

Ex: Differential Cuts

$$\dfrac{\dfrac{y^5 \geq 0 \rightarrow 2x^2 x' \geq 0}{x^3 \geq -1 \rightarrow [x' = (x-3)^4 + y^5, y' = y^2 \,\&\, y^5 \geq 0]x^3 \geq -1} \triangleright}{x^3 \geq -1 \land y^5 \geq 0 \rightarrow [x' = (x-3)^4 + y^5, y' = y^2]x^3 \geq -1}$$

$$\dfrac{\dfrac{\dfrac{*}{5y^4 y^2 \geq 0}}{5y^4 y' \geq 0}}{y^5 \geq 0 \rightarrow [x' = (x-3)^4 + y^5, y' = y^2]y^5 \geq 0}$$

Ex: Differential Cuts

$$\dfrac{\dfrac{\dfrac{y^5 \geq 0 \to 2x^2((x-3)^4 + y^5) \geq 0}{y^5 \geq 0 \to 2x^2 x' \geq 0}}{x^3 \geq -1 \to [x' = (x-3)^4 + y^5, y' = y^2 \,\&\, y^5 \geq 0] x^3 \geq -1} \triangleright}{x^3 \geq -1 \land y^5 \geq 0 \to [x' = (x-3)^4 + y^5, y' = y^2] x^3 \geq -1}$$

$$\dfrac{\dfrac{\dfrac{*}{5y^4 y^2 \geq 0}}{5y^4 y' \geq 0}}{y^5 \geq 0 \to [x' = (x-3)^4 + y^5, y' = y^2] y^5 \geq 0}$$

Ex: Differential Cuts

$$\cfrac{\cfrac{\cfrac{\cfrac{*}{y^5 \geq 0 \rightarrow 2x^2((x-3)^4 + y^5) \geq 0}}{y^5 \geq 0 \rightarrow 2x^2 x' \geq 0}}{x^3 \geq -1 \rightarrow [x' = (x-3)^4 + y^5, y' = y^2 \,\&\, y^5 \geq 0]x^3 \geq -1 \quad \triangleright}}{x^3 \geq -1 \wedge y^5 \geq 0 \rightarrow [x' = (x-3)^4 + y^5, y' = y^2]x^3 \geq -1}$$

$$\cfrac{\cfrac{\cfrac{*}{5y^4 y^2 \geq 0}}{5y^4 y' \geq 0}}{y^5 \geq 0 \rightarrow [x' = (x-3)^4 + y^5, y' = y^2]y^5 \geq 0}$$

Differential Cuts

$$\frac{\phi \rightarrow [x' = \theta \,\&\, H]C \qquad \phi \rightarrow [x' = \theta \,\&\, (H \land C)]\phi}{\phi \rightarrow [x' = \theta \,\&\, H]\phi}$$

Differential Cuts

$$\frac{\phi \to [x' = \theta \,\&\, H]C \qquad \phi \to [x' = \theta \,\&\, (H \land C)]\phi}{\phi \to [x' = \theta \,\&\, H]\phi}$$

Theorem (Gentzen's Cut Elimination)

$$\frac{A \to B \lor C \quad A \land C \to B}{A \to B}$$ *cut can be eliminated*

Differential Cuts

$$\frac{\phi \to [x' = \theta \,\&\, H]C \qquad \phi \to [x' = \theta \,\&\, (H \land C)]\phi}{\phi \to [x' = \theta \,\&\, H]\phi}$$

Theorem (Gentzen's Cut Elimination)

$$\frac{A \to B \lor C \qquad A \land C \to B}{A \to B}$$ *cut can be eliminated*

Theorem (No Differential Cut Elimination) (LMCS 2012)

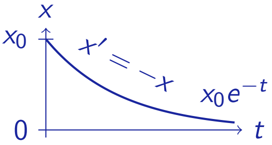
Deductive power with differential cut exceeds deductive power without.

$$\mathcal{DCI} > \mathcal{DI}$$

Ex: Exponentials

Counterexample ()

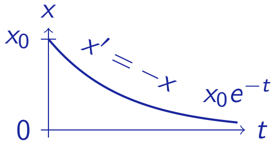
$$\overline{x > 0 \rightarrow [x' = -x]x > 0}$$



Ex: Exponentials

Counterexample ()

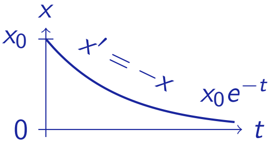
$$\frac{\overline{x' > 0}}{x > 0 \rightarrow [x' = -x]x > 0}$$



Ex: Exponentials

Counterexample ()

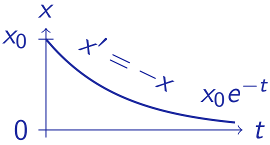
$$\cfrac{\cfrac{\overline{\;-x > 0\;}}{x' > 0}}{x > 0 \rightarrow [x' = -x]x > 0}$$



Ex: Exponentials

Counterexample (Cannot prove)

$$\cfrac{\cfrac{\cfrac{\text{not valid}}{-x > 0}}{x' > 0}}{x > 0 \rightarrow [x' = -x]x > 0}$$



Differential Auxiliaries

Example (Successful proof)

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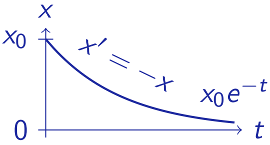
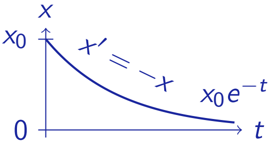


Differential Auxiliaries

Example (Successful proof)

$$\dfrac{\dfrac{}{x > 0 \leftrightarrow \exists y\, xy^2 = 1} \qquad \dfrac{}{xy^2 = 1 \rightarrow [x' = -x, y' = \frac{y}{2}]xy^2 = 1}}{x > 0 \rightarrow [x' = -x]x > 0}$$



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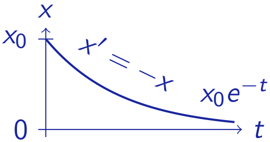
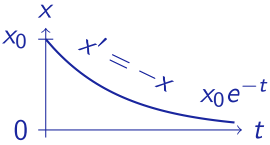


Differential Auxiliaries

Example (Successful proof)

$$
\dfrac{\dfrac{*}{x>0 \leftrightarrow \exists y\, xy^2=1} \qquad \dfrac{\dfrac{}{x'y^2+x2yy'=0}}{xy^2=1 \rightarrow [x'=-x, y'=\frac{y}{2}]xy^2=1}}{x>0 \rightarrow [x'=-x]x>0}
$$



Differential Auxiliaries

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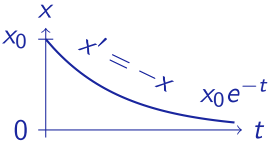

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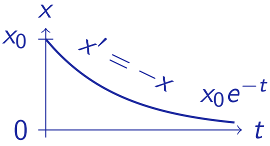

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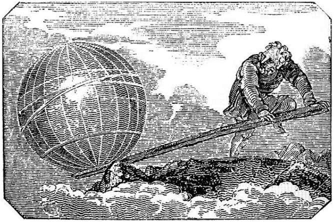

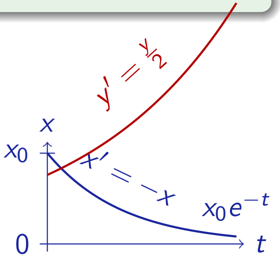

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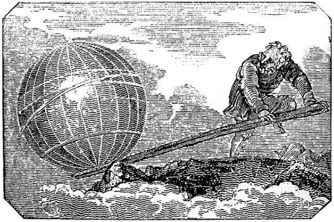
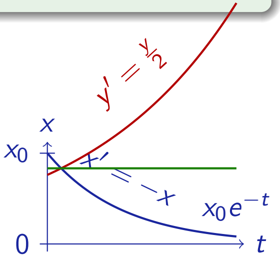




Differential Auxiliaries

$$\frac{\phi \leftrightarrow \exists y\, \psi \quad \psi \rightarrow [x' = \theta, y' = \vartheta \,\&\, H]\psi}{\phi \rightarrow [x' = \theta \,\&\, H]\phi}$$

if $y' = \vartheta$ has solution $y : [0, \infty) \to \mathbb{R}^n$

Theorem (Auxiliary Differential Variables) (LMCS 2012)

Deductive power with differential auxiliaries exceeds deductive power without.

$$\mathcal{DCI} + DA > \mathcal{DCI}$$

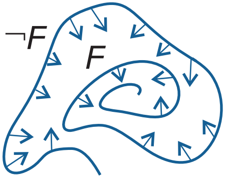
Outline

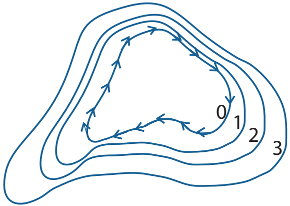
Equational Differential Invariants

Theorem (Lie)

$$\frac{H \rightarrow p' = 0}{\forall c\, (p = c \rightarrow [x' = f(x)\, \&\, H]p = c)}$$

equivalence if H open





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Decidable whether polynomial p invariant function of $x' = f(x)$ on open H

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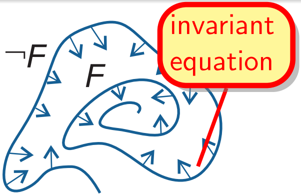

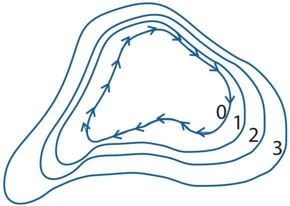

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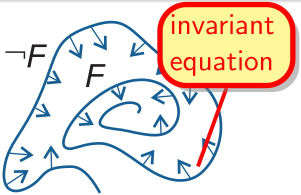

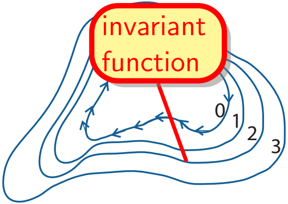

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Structure of Invariant Functions

Lemma (Structure of invariant functions)

Invariant functions of $x' = \theta \,\&\, H$ *form an* $\mathbb{R}$*-algebra.*

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Corollary

Only need generating system of algebra.

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Corollary

Only need generating system of algebra.

$$p \text{ invariant}, F \text{ function} \quad \Rightarrow \quad F(p) \text{ invariant}$$

Structure of Invariant Equations

$$\mathcal{I}_=(\Gamma) := \{p \in \mathbb{R}[\vec{x}] \quad : \quad \vDash \Gamma \to [x' = \theta \,\&\, H]p = 0\}$$
$$\mathcal{DCI}_=(\Gamma) := \{p \in \mathbb{R}[\vec{x}] \quad : \quad \vdash_{DI_= + DC} \Gamma \to [x' = \theta \,\&\, H]p = 0\}$$

Lemma (Structure of invariant equations)

$\mathcal{DCI}_=(\Gamma) \subseteq \mathcal{I}_=(\Gamma)$ chain of differential ideals ($(\theta \cdot \nabla)p \in \mathcal{DCI}_=(\Gamma)$ for all $p \in \mathcal{DCI}_=(\Gamma)$). The varieties are generated by a single polynomial.

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Proof.

4. $p \in \mathcal{DCI}_=(\Gamma)$ and $r \in \mathbb{R}[\vec{x}]$ implies $rp \in \mathcal{DCI}_=(\Gamma)$, because

$$(\theta \cdot \nabla)(rp) = p(\theta \cdot \nabla)r + r\underbrace{(\theta \cdot \nabla)p}_{0} = \underbrace{p}_{0}(\theta \cdot \nabla)r = 0$$

   and $\Gamma \to p = 0$ implies $\Gamma \to rp = 0$

5. $p = 0 \land q = 0$ iff $p^2 + q^2 = 0$, differential, Hilbert basis theorem . . .

Outline

Full Rank Assumptions

Theorem (Lie — necessary)

$$(\overleftarrow{DI}_p)\quad \frac{\bigwedge_{i=1}^{n} p_i = 0 \rightarrow [x' = f(x)\,\&\,H]\bigwedge_{i=1}^{n} p_i = 0}{H \land \bigwedge_{i=1}^{n} p_i = 0 \rightarrow \bigwedge_{i=1}^{n} (\theta \cdot \nabla) p_i = 0}$$

Premises, conclusions equivalent if rank $\frac{\partial p_i}{\partial x_j} = n$ *on* $H \land \bigwedge_{i=1}^{n} p_i = 0$.

Full Rank Assumptions

Theorem (… — sufficient)

$$(\overrightarrow{DI}_p) \quad \frac{H \rightarrow \bigwedge_{i=1}^{n} (\theta \cdot \nabla) p_i = \sum_j Q_{i,j} p_j}{\bigwedge_{i=1}^{n} p_i = 0 \rightarrow [x' = f(x) \,\&\, H] \bigwedge_{i=1}^{n} p_i = 0}$$

Theorem (Lie — necessary)

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Premises, conclusions equivalent if rank $\frac{\partial p_i}{\partial x_j} = n$ *on* $H \wedge \bigwedge_{i=1}^{n} p_i = 0$.

$$\dfrac{\dfrac{\dfrac{\dfrac{*}{e = x \rightarrow -2yx + 2xy = 0}}{e = x \rightarrow (-y)2x + e2y = 0}}{e = x \rightarrow -y\frac{\partial(x^2+y^2)}{\partial x} + e\frac{\partial(x^2+y^2)}{\partial y} = 0}}{\ldots \rightarrow [x' = -y, y' = e, e' = -y \,\&\, e = x](x^2 + y^2 = 1 \land e = x)}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{*}{-y = -y}}{-y\frac{\partial e}{\partial e} = -y\frac{\partial x}{\partial x}}}{e = x \rightarrow [x' = -y, y' = e, e' = -y]e = x \quad \triangleright}}{x^2 + y^2 = 1 \land e = x \rightarrow [x' = -y, y' = e, e' = -y](x^2 + y^2 = 1 \land e = x)}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{*}{e = x \rightarrow -2yx + 2xy = 0}}{e = x \rightarrow (-y)2x + e2y = 0}}{e = x \rightarrow -y\frac{\partial(x^2+y^2)}{\partial x} + e\frac{\partial(x^2+y^2)}{\partial y} = 0}}{\ldots \rightarrow \quad y^2 = 1 \land e = x)}$$

Successful Proof

Lie & differential cuts separate aircraft

$$\dfrac{\dfrac{\dfrac{*}{-y = -y}}{-y\frac{\partial e}{\partial e} = -y\frac{\partial x}{\partial x}}}{\dfrac{e = x \rightarrow [x' = -y, y' = e, e' = -y]e = x \qquad \triangleright}{x^2 + y^2 = 1 \land e = x \rightarrow [x' = -y, y' = e, e' = -y](x^2 + y^2 = 1 \land e = x)}}$$

$$\begin{pmatrix} \frac{\partial(x^2+y^2-1)}{\partial x} & \frac{\partial(x^2+y^2-1)}{\partial y} & \frac{\partial(x^2+y^2-1)}{\partial e} \\ \frac{\partial(e-x)}{\partial x} & \frac{\partial(e-x)}{\partial y} & \frac{\partial(e-x)}{\partial e} \end{pmatrix} = \begin{pmatrix} 2x & 2y & 0 \\ -1 & 0 & 1 \end{pmatrix}$$

Full rank 2 at invariant $x^2 + y^2 = 1$

Outline

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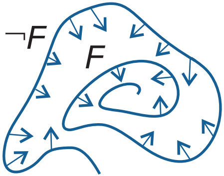

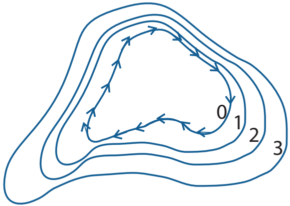

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Inverse Characteristic Method

Theorem (Inverse characteristic method)

(Sufficiently smooth) f *is invariant function of* $x' = f(x)$ *on* H *iff* f *solves*

$$(\theta \cdot \nabla) f = 0 \quad \text{on } H$$

Proof.

$\Leftarrow$ Lie ☐

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- If ODE too complicated, consider PDE instead???

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Proof.

$\Leftarrow$ Lie □

- If ODE too complicated, consider PDE instead???
- Yes, but inverse characteristic PDE is simple (first-order, linear, homogeneous)
- Makes rich PDE theory available for differential invariants
- Oracle PDE solver sufficient

Ex: Deconstructed Aircraft (IV)

Example (Generate Differential Invariants)

$$x^2 + y^2 = 1 \land e = x \rightarrow [x' = -y, y' = e, e' = -y](\underbrace{x^2 + y^2 = 1}_{(3)} \land \underbrace{e = x}_{(4)})$$

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Example (Inverse Characteristic PDE)

$$\leadsto \quad -y\frac{\partial f}{\partial x} + e\frac{\partial f}{\partial y} - y\frac{\partial f}{\partial e} = 0$$

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$\rightsquigarrow$ Differential invariants: $-e + x = 0, \quad -y^2 - 2ex + x^2 = -1$

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$$F \land \omega \neq 0 \rightarrow [x_1' = d_1, x_2' = d_2, d_1' = -\omega d_2, d_2' = \omega d_1]F$$

$$F \equiv d_1^2 + d_2^2 = \omega^2 p^2 \qquad \land\, d_1 = -\omega x_2 \qquad \land\, d_2 = \omega x_1$$

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Ex: Aircraft

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$d_2 - \omega x_1$

$d_1 + \omega x_2$

$d_1^2 + 2\omega x_1 d_2 - \omega^2 x_1^2$

Example (Inverse Characteristic PDE)

$$\leadsto \quad d_1 \frac{\partial f}{\partial x_1} + d_2 \frac{\partial f}{\partial x_2} - \omega d_2 \frac{\partial f}{\partial d_1} + \omega d_1 \frac{\partial f}{\partial d_2} = 0$$

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Example (Generate Differential Invariants)

$$F \wedge \omega \neq 0 \rightarrow [x_1' = d_1, x_2' = d_2, d_1' = -\omega d_2, d_2' = \omega d_1]F$$

$$F \equiv d_1^2 + d_2^2 = \omega^2 p^2 \ (4) \ \wedge d_1 = -\omega x_2 \ (5) \ \wedge d_2 = \omega x_1 \ (6)$$

$$d_2 - \omega x_1 \overset{(5)}{\rightsquigarrow} 0$$

$$d_1 + \omega x_2 \overset{(6)}{\rightsquigarrow} 0$$

$$d_1^2 + 2\omega x_1 d_2 - \omega^2 x_1^2 \overset{(6)}{\rightsquigarrow} d_1^2 + 2d_2^2 - \omega^2 x_1^2 \overset{(5)}{\rightsquigarrow} d_1^2 + 2d_2^2 - d_2^2 \overset{(4)}{\rightsquigarrow} \omega^2 p^2$$

Example (Inverse Characteristic PDE)

$$\rightsquigarrow \quad d_1 \frac{\partial f}{\partial x_1} + d_2 \frac{\partial f}{\partial x_2} - \omega d_2 \frac{\partial f}{\partial d_1} + \omega d_1 \frac{\partial f}{\partial d_2} = 0$$

$$\rightsquigarrow \quad f(x_1, x_2, d_1, d_2) = g\Big(\underbrace{d_2 - \omega x_1}_{(1)}, \underbrace{\frac{d_1 + \omega x_2}{\omega}}_{(2)}, \frac{1}{2}\underbrace{(d_1^2 + 2\omega d_2 x_1 - \omega^2 x_1^2)}_{(3)}\Big)$$

Ex: Aircraft

Example (Generate Differential Invariants)

$$F \land \omega \neq 0 \rightarrow [x_1' = d_1, x_2' = d_2, d_1' = -\omega d_2, d_2' = \omega d_1]F$$

$$F \equiv d_1^2 + d_2^2 = \omega^2 p^2 \ (4) \ \land d_1 = -\omega x_2 \ (5) \ \land d_2 = \omega x_1 \ (6)$$

$$d_2 - \omega x_1 \overset{(5)}{\leadsto} 0$$

$$d_1 + \omega x_2 \overset{(6)}{\leadsto} 0$$

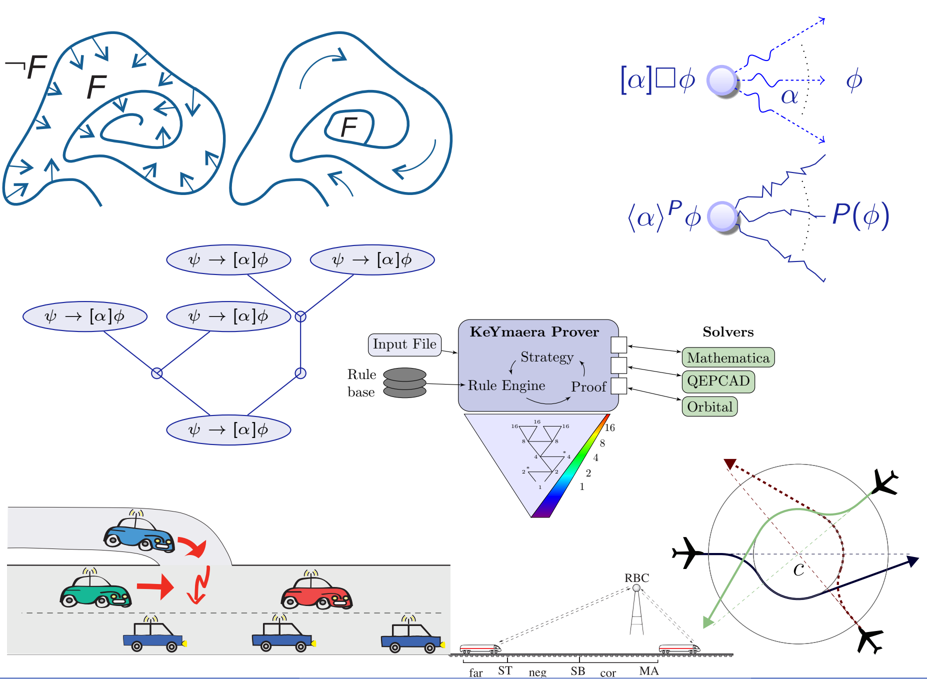
$$d_1^2 + 2\omega x_1 d_2 - \omega^2 x_1^2 \overset{(6)}{\leadsto} d_1^2 + 2d_2^2 - \omega^2 x_1^2 \overset{(5)}{\leadsto} d_1^2 + 2d_2^2 - d_2^2 \overset{(4)}{\leadsto} \omega^2 p^2$$

Example (Inverse Characteristic PDE)

$$\leadsto \quad d_1 \frac{\partial f}{\partial x_1} + d_2 \frac{\partial f}{\partial x_2} - \omega d_2 \frac{\partial f}{\partial d_1} + \omega d_1 \frac{\partial f}{\partial d_2} = 0$$

$$\leadsto \quad f(x_1, x_2, d_1, d_2) = g\Big(\underbrace{d_2 - \omega x_1}_{(1)}, \underbrace{\frac{d_1 + \omega x_2}{\omega}}_{(2)}, \frac{1}{2}\underbrace{(d_1^2 + 2\omega d_2 x_1 - \omega^2 x_1^2)}_{(3)}\Big)$$

Outline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Rule base
Rule Engine
Proof
Solvers
Mathematica
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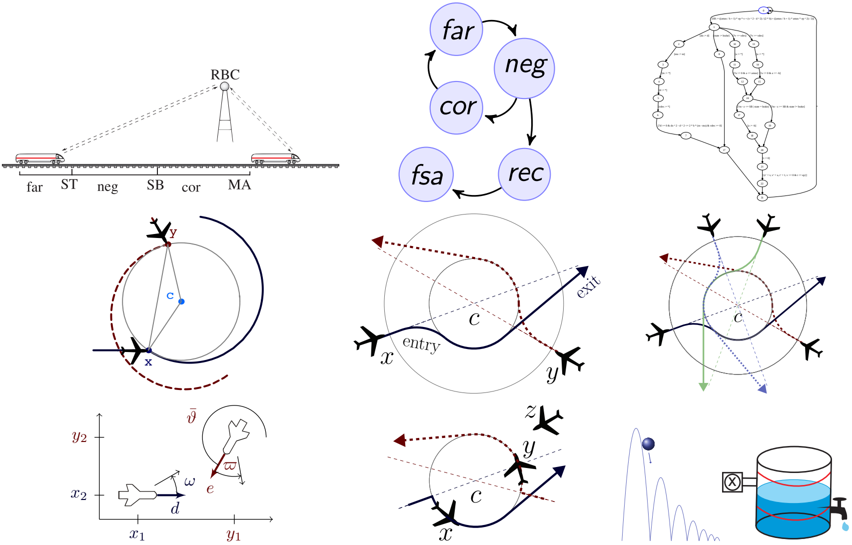
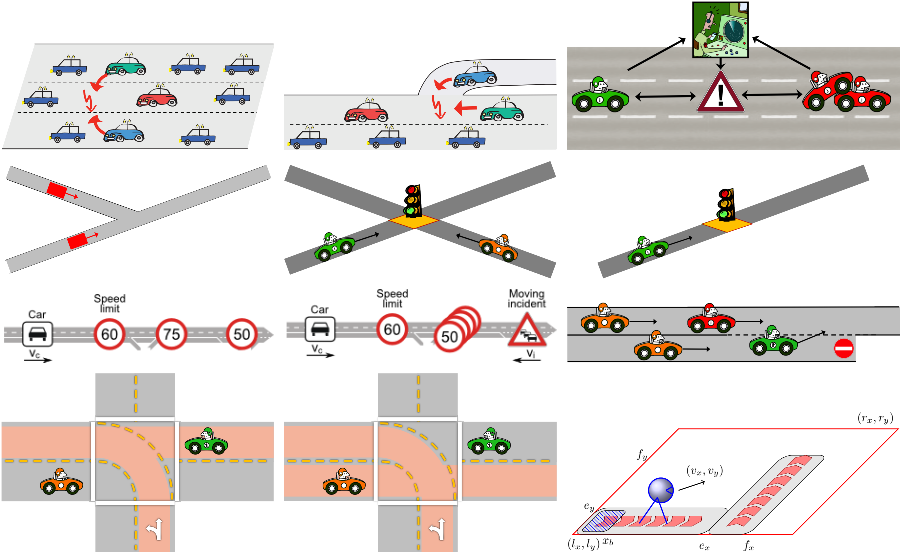
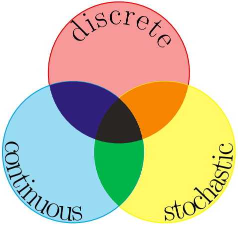
Successful Hybrid Systems Proofs



Successful Hybrid Systems Proofs

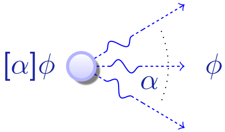


Outline



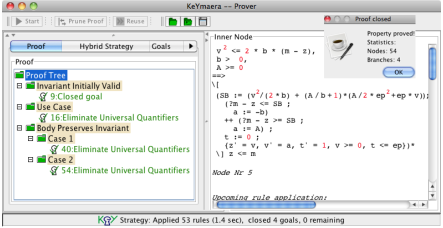
differential dynamic logic

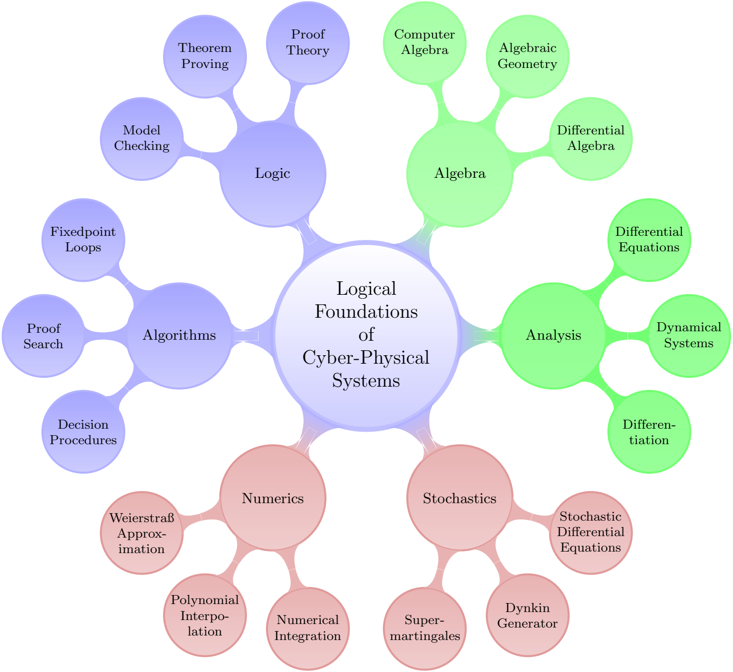
$$d\mathcal{L} = \text{DL} + \text{HP}$$



- Logic for hybrid systems++
- Sound & complete / ODE
- Differential invariants
- No differential cut elimination
- Differential auxiliaries
- Algebra / differential ideal
- Inverse characteristic PDE

KeYmaera

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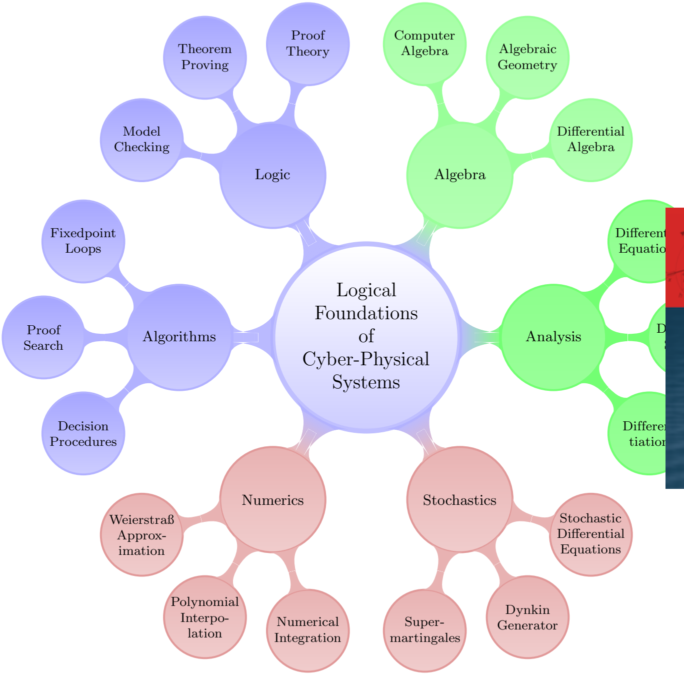
Model Checking
Theorem Proving
Proof Theory
Algebra
Computer Algebra
Algebraic Geometry
Differential Algebra
Analysis
Differential Equations
Dynamical Systems
Differen-tiation
Algorithms
Fixedpoint Loops
Proof Search
Decision Procedures
Numerics
Weierstraß Approx-imation
Polynomial Interpo-lation
Numerical Integration
Stochastics
Stochastic Differential Equations
Dynkin Generator
Super-martingales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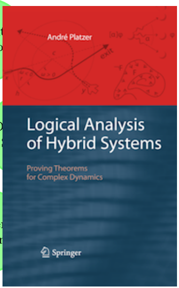
André Platzer
Logical Analysis of Hybrid Systems
Proving Theorems for Complex Dynamics
Springer