# Differential Equation Axiomatization

## The Impressive Power of Differential Ghosts

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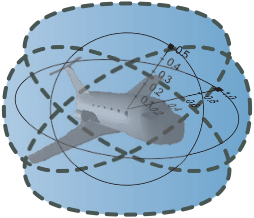


# Outline

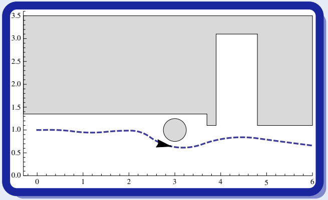
1. Differential Dynamic Logic
2. Proofs for Differential Equations
   - Differential Invariants / Cuts / Ghosts
3. Completeness for Differential Equation Invariants
   - Darboux are Differential Ghosts
   - Derived Semialgebraic Invariants
   - Real Induction
   - Derived Local Progress
   - Completeness for Invariants
4. Summary

# Hybrid Systems: Example Robot Control

## Challenge (Hybrid Systems)

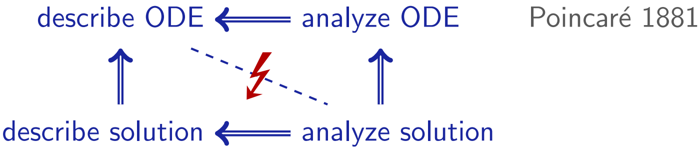
Fixed law describing state evolution with both

- Discrete dynamics (control decisions)
- Continuous dynamics (differential equations)



# Contributions: Differential Equation Axiomatization

- Classical approach: ① Given ODE ② Solve ODE ③ Analyze solution
- Descriptive power of ODEs: ODE much easier than its solution
- ⚡ Analyzing ODEs via their solutions undoes their descriptive power!




1. Now: Logical foundations of differential equation invariants
2. Identify axioms for differential equations
3. Completeness for differential equation invariants
4. Uniformly substitutable axioms, not infinite axiom schemata
5. Decide invariance by proof

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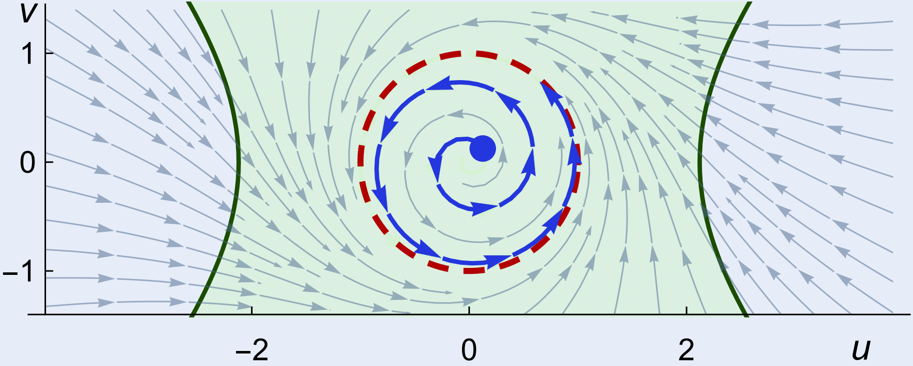
# Hybrid Systems = Differential Equations + Discrete

## Concept (Differential Dynamic Logic) (JAR'08,LICS'12)

$$u^2 \le v^2 + \frac{9}{2} \rightarrow [u' = -v + \frac{u}{4}(1-u^2-v^2), v' = u + \frac{v}{4}(1-u^2-v^2)]\, u^2 \le v^2 + \frac{9}{2}$$

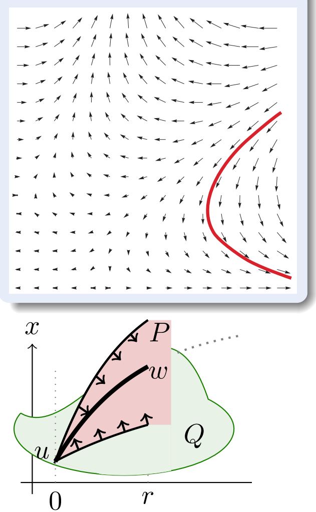
$$u^2 + v^2 = 1 \rightarrow [u' = -v + \frac{u}{4}(1-u^2-v^2), v' = u + \frac{v}{4}(1-u^2-v^2)]\, u^2 + v^2 = 1$$



# Outline

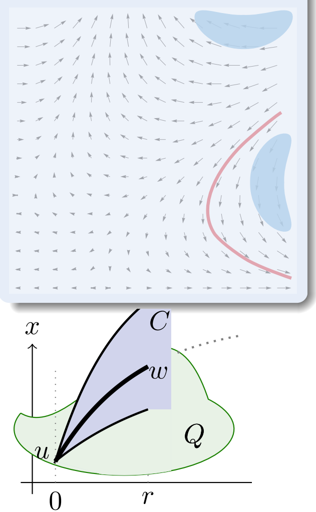
# Differential Invariants for Differential Equations

## Differential Invariant




$x' = f(x) \,\&\, Q$

## Differential Cut

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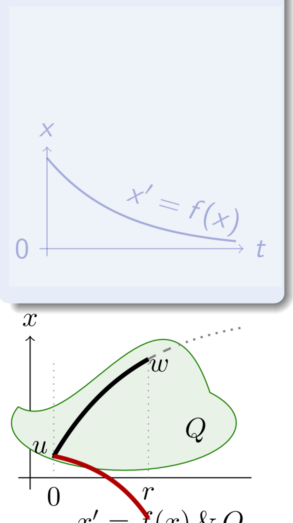


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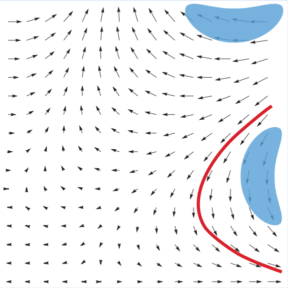
# Differential Invariants for Differential Equations

## Differential Invariant



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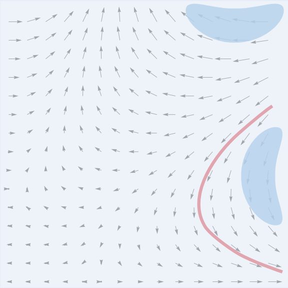

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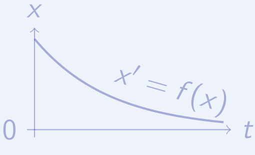


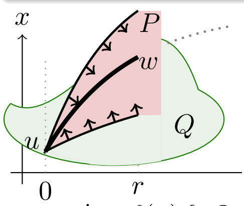


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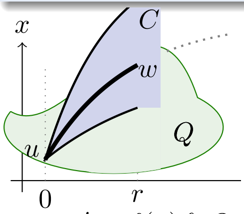


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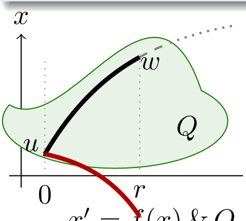


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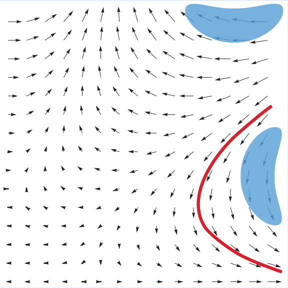
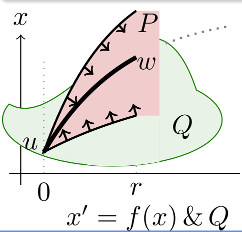
# Differential Invariants for Differential Equations

## Differential Invariant



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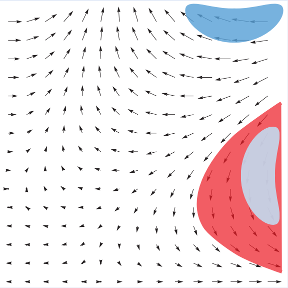

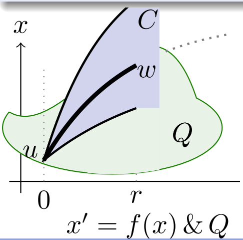


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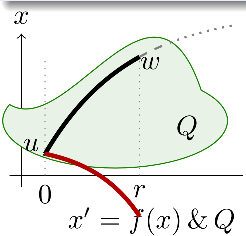

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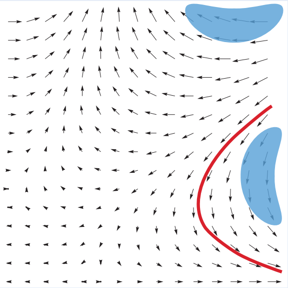

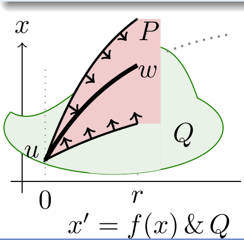


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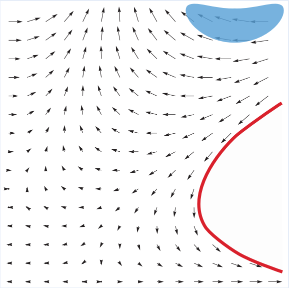

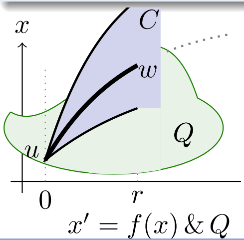


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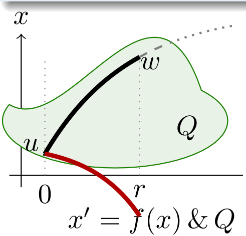

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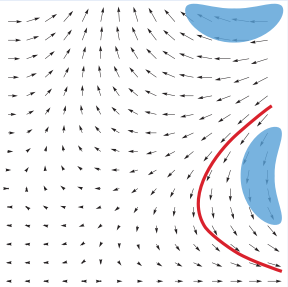

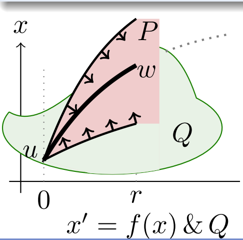


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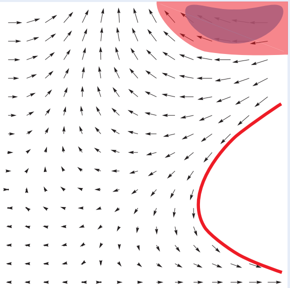

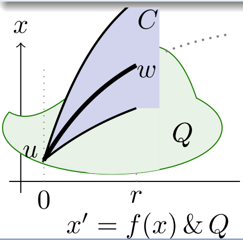


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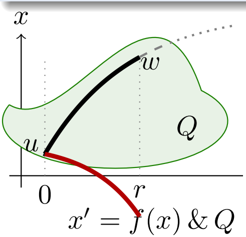

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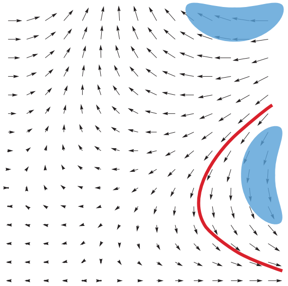

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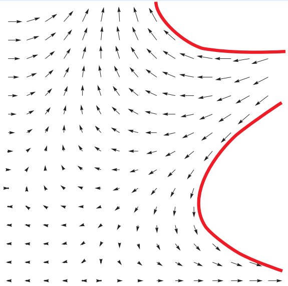

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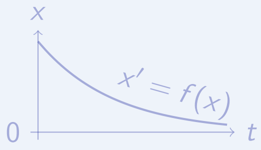
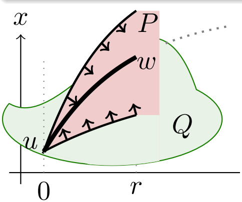
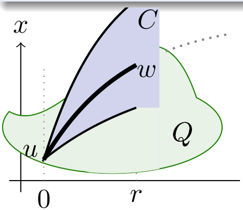



$x' = f(x) \,\&\, Q$

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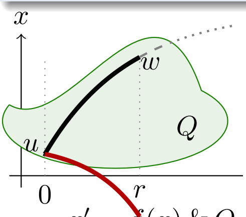


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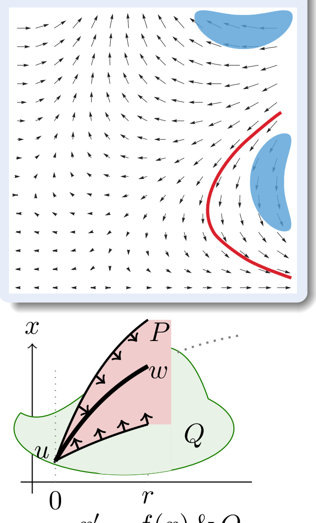


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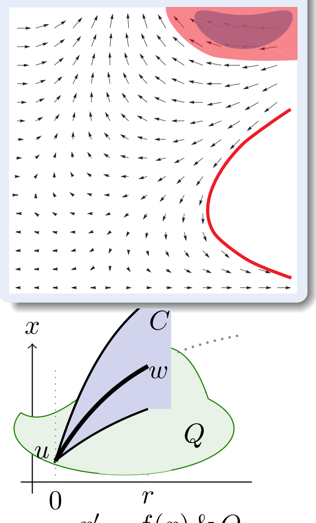


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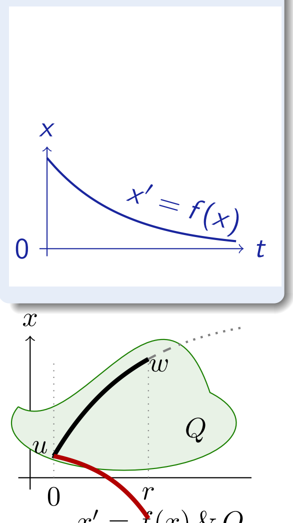


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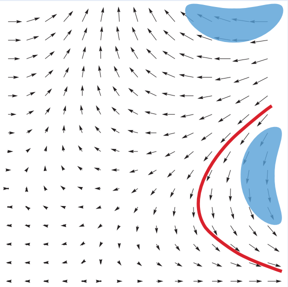

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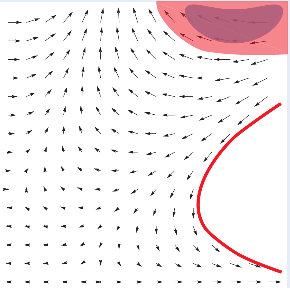

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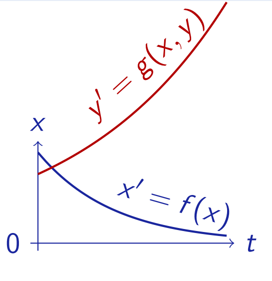


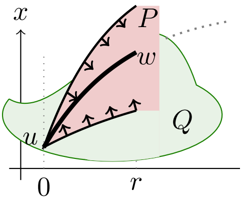


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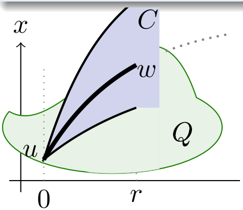


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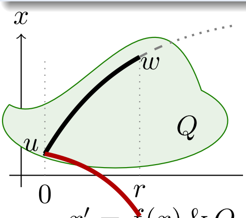


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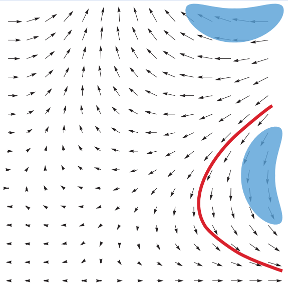
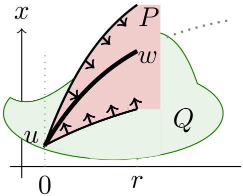
# Differential Invariants for Differential Equations

## Differential Invariant



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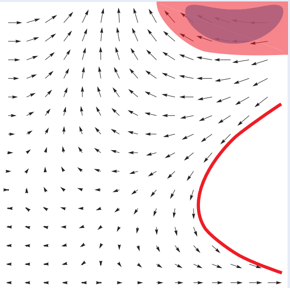

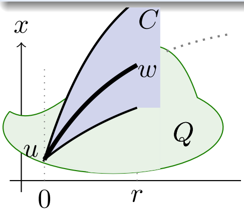


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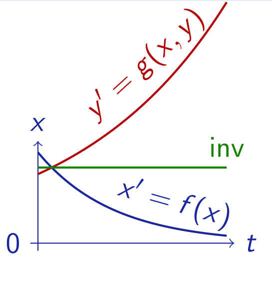
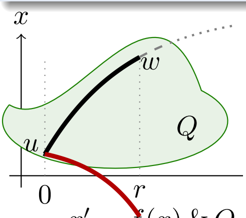


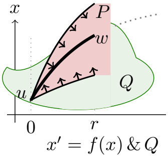




$x' = f(x) \,\&\, Q$

# Differential Invariants for Differential Equations

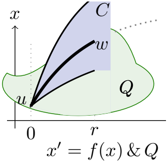
**Differential Invariant**

$$\frac{Q \vdash [x' := f(x)](P)'}{P \vdash [x' = f(x) \,\&\, Q]P}$$



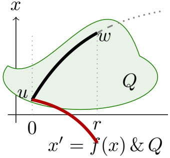

**Differential Cut**

$$\frac{P \vdash [x' = f(x) \,\&\, Q]C \quad P \vdash [x' = f(x) \,\&\, Q \land C]P}{P \vdash [x' = f(x) \,\&\, Q]P}$$




**Differential Ghost**

$$\frac{P \leftrightarrow \exists y\, G \quad G \vdash [x' = f(x), y' = g(x,y) \,\&\, Q]G}{P \vdash [x' = f(x) \,\&\, Q]P}$$




deductive power adds DI $\prec$ DC $\prec$ DG

JLogComput'10,LMCS'12, LICS'12,JAR'17,LICS'18

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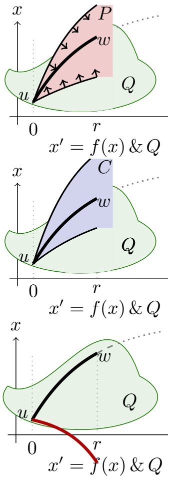
**Differential Ghost**

$$\frac{P \leftrightarrow \exists y\, G \quad G \vdash [x' = f(x), y' = g(x,y) \,\&\, Q]G}{P \vdash [x' = f(x) \,\&\, Q]P}$$

if new $y' = g(x, y)$ has long enough solution

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# Differential Equation Axiomatization

**Theorem (Algebraic Completeness) (LICS'18)**

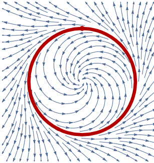
dL *calculus is a sound & complete axiomatization of algebraic invariants of polynomial differential equations. They are decidable by DI,DC,DG*

**Theorem (Semialgebraic Completeness) (LICS'18)**

dL *calculus with RI is a sound & complete axiomatization of semialgebraic invariants of differential equations. They are decidable in* dL

# ODE Axiomatization: Derived Darboux Rules

Gaston Darboux 1878



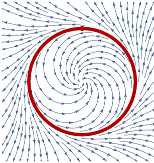
**Darboux equalities are DG**

$$\frac{Q \vdash p' = gp}{p = 0 \vdash [x' = f(x) \,\&\, Q]p = 0} \quad (g \in \mathbb{R}[x])$$

Definable $p'$ for Lie-derivative w.r.t. ODE

# ODE Axiomatization: Derived Darboux Rules

Gaston Darboux 1878



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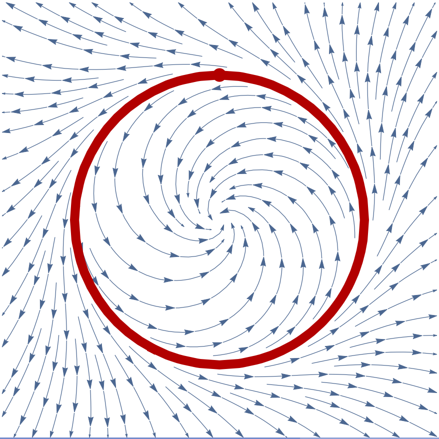
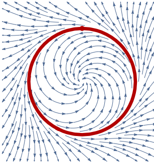


$$\frac{\vdash 2uu' + 2vv' = 2(u^2+v^2)(u^2+v^2-1)}{.. \vdash \begin{bmatrix} u' = -v-u+u^3+uv^2 \\ v' = u-v+u^2v+v^3 \end{bmatrix} u^2+v^2-1=0}$$

# ODE Axiomatization: Derived Darboux Rules

**Darboux equalities are DG**

$$\frac{Q \vdash p' = gp}{p = 0 \vdash [x' = f(x) \,\&\, Q]p = 0} \quad (g \in \mathbb{R}[x])$$



**Proof Idea.**

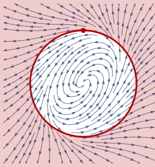
1. DG counterweight $y' = -gy$ to reduce $p = 0$ to $py = 0 \wedge y \neq 0$.
2. DG counter-counterweight $z' = gz$ to reduce $y \neq 0$ to $yz = 1$.
3. $py = 0$ and $yz = 1$ are now differential invariants by construction. □

# ODE Axiomatization: Derived Darboux Rules

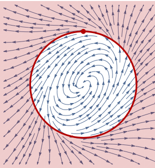
Thomas Grönwall 1919

**Darboux inequalities are DG**

$$\frac{Q \vdash p' \geq gp}{p \succcurlyeq 0 \vdash [x' = f(x) \,\&\, Q] p \succcurlyeq 0} \quad (g \in \mathbb{R}[x])$$



# ODE Axiomatization: Derived Darboux Rules



**Darboux inequalities are DG**

$$\frac{Q \vdash p' \geq gp}{p \succcurlyeq 0 \vdash [x' = f(x) \,\&\, Q] p \succcurlyeq 0} \quad (g \in \mathbb{R}[x])$$

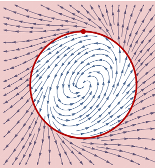
**Proof Idea.**

1. DG counterweight $y' = -gy$ to reduce $p \succcurlyeq 0$ to $py \succcurlyeq 0 \wedge y > 0$.
2. DG counter-counterweight $z' = \frac{g}{2} z$ to reduce $y > 0$ to $yz^2 = 1$.
3. $yz^2 = 1$ and (after DC with $y > 0$) $py \succcurlyeq 0$ are differential invariants by construction as $(py)' = p'y - gyp \geq 0$ from premise since $y > 0$. □

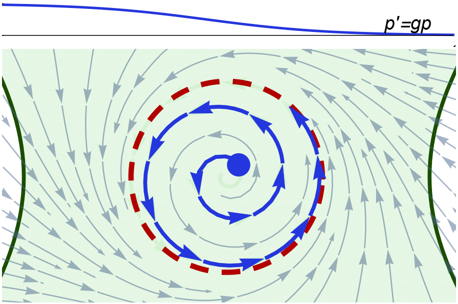
# ODE Axiomatization: Derived Darboux Rules

**Darboux inequalities are DG**

$$\frac{Q \vdash p' \geq gp}{p \succcurlyeq 0 \vdash [x' = f(x) \,\&\, Q] p \succcurlyeq 0} \quad (g \in \mathbb{R}[x])$$



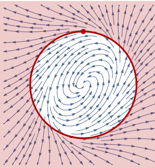
$$\frac{(1-u^2-v^2)' \geq -\frac{1}{2}(u^2+v^2)(1-u^2-v^2)}{\ldots \vdash \left[\begin{array}{l} u' = -v + \frac{u}{4}(1-u^2-v^2) \\ v' = \;\; u + \frac{v}{4}(1-u^2-v^2) \end{array}\right] 1-u^2-v^2 > 0}$$

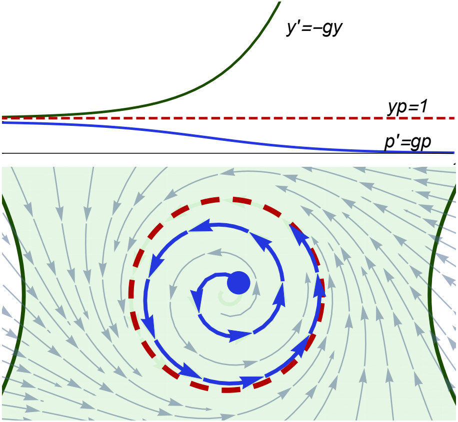


# ODE Axiomatization: Derived Darboux Rules

**Darboux inequalities are DG**

$$\frac{Q \vdash p' \geq gp}{p \succcurlyeq 0 \vdash [x' = f(x) \,\&\, Q]p \succcurlyeq 0} \quad (g \in \mathbb{R}[x])$$





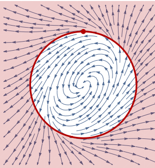

$$\frac{(1-u^2-v^2)' \geq -\frac{1}{2}(u^2+v^2)(1-u^2-v^2)}{\ldots \vdash \left[\begin{array}{l} u' = -v + \frac{u}{4}(1-u^2-v^2) \\ v' = \quad u + \frac{v}{4}(1-u^2-v^2) \\ y' = \frac{1}{2}(u^2+v^2)y \end{array}\right] 1-u^2-v^2 > 0}$$

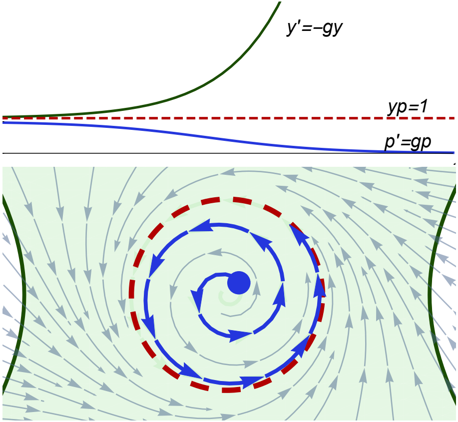
$$(1-u^2-v^2)y > 0$$

# ODE Axiomatization: Derived Darboux Rules

**Darboux inequalities are DG**

$$\frac{Q \vdash p' \geq gp}{p \succcurlyeq 0 \vdash [x' = f(x) \,\&\, Q]p \succcurlyeq 0} \quad (g \in \mathbb{R}[x])$$






$$\frac{(1-u^2-v^2)' \geq -\frac{1}{2}(u^2+v^2)(1-u^2-v^2)}{\ldots \vdash \left[\begin{aligned} u' &= -v + \tfrac{u}{4}(1-u^2-v^2) \\ v' &= u + \tfrac{v}{4}(1-u^2-v^2) \\ y' &= \tfrac{1}{2}(u^2+v^2)y \\ z' &= -\tfrac{1}{4}(u^2+v^2)z \end{aligned}\right] 1-u^2-v^2 > 0}$$

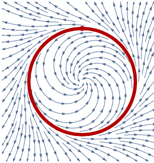
$$(1-u^2-v^2)y > 0$$

$$yz^2 = 1$$

# ODE Axiomatization: Derived Darboux Rules

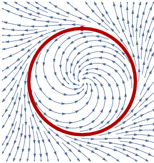
**Vectorial Darboux are VDG**

$$\frac{Q \vdash \mathbf{p}' = G\mathbf{p}}{\mathbf{p} = 0 \vdash [x' = f(x) \,\&\, Q]\mathbf{p} = 0} \quad (G \in \mathbb{R}[x]^{n \times n})$$



Definable $\mathbf{p}'$ for component-wise Lie-derivative w.r.t. ODE

# ODE Axiomatization: Derived Darboux Rules



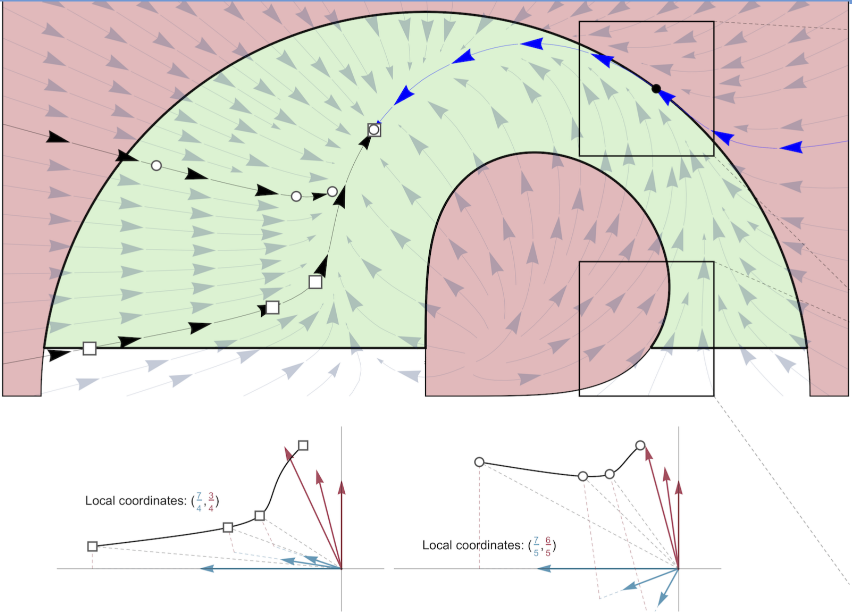
**Vectorial Darboux are VDG**

$$\frac{Q \vdash \mathbf{p}' = G\mathbf{p}}{\mathbf{p} = 0 \vdash [x' = f(x)\,\&\,Q]\mathbf{p} = 0} \quad (G \in \mathbb{R}[x]^{n\times n})$$

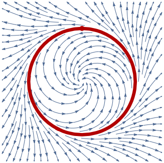
**Proof Idea.**

1. DG counterweight $\mathbf{y}' = -G^T\mathbf{y}$ to change $\mathbf{p} = 0$ to $\mathbf{p} \cdot \mathbf{y} = 0$.
2. But: $\mathbf{p} \cdot \mathbf{y} = 0 \not\Rightarrow \mathbf{p} = 0$ even if $\mathbf{y} \neq 0$.
3. Redo: time-varying independent DG matrix $Y' = -YG$ with $Y\mathbf{p} = 0$.
4. $Y\mathbf{p} = 0 \Rightarrow \mathbf{p} = 0$ if $\det Y \neq 0$.
5. DC $\det Y \neq 0$ which proves by dbx using Liouville's identity:
$$\det(Y)' = -\operatorname{tr}(G)\det(Y)$$
6. Continuous change of basis $Y^{-1}$ balancing out motion of $\mathbf{p}$: constant!
7. Continuous change to new evolving variables is sound by DG. □

# Time is defined so that motion looks simple ≈Poincaré



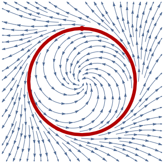
# ODE Axiomatization: Derived Invariant Rules



**Vectorial Darboux are VDG**

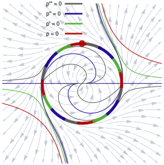
$$\frac{Q \vdash \mathbf{p}' = G\mathbf{p}}{\mathbf{p} = 0 \vdash [x' = f(x) \,\&\, Q]\mathbf{p} = 0}$$

# ODE Axiomatization: Derived Invariant Rules



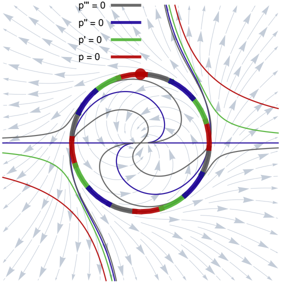
## Vectorial Darboux are VDG

$$\frac{Q \vdash \mathbf{p}' = G\mathbf{p}}{\mathbf{p} = 0 \vdash [x' = f(x) \,\&\, Q]\mathbf{p} = 0}$$




## Differential Radical Invariants are vdbx

$$\frac{\Gamma, Q \vdash \bigwedge_{i=0}^{N-1} p^{(i)} = 0 \quad Q \vdash p^{(N)} = \sum_{i=0}^{N-1} g_i p^{(i)}}{\Gamma \vdash [x' = f(x) \,\&\, Q]p = 0}$$



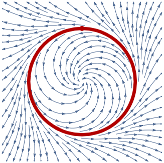
# ODE Axiomatization: Derived Invariant Rules

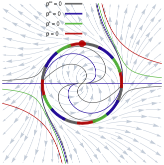
## Vectorial Darboux are VDG

$$\frac{Q \vdash \mathbf{p}' = G\mathbf{p}}{\mathbf{p} = 0 \vdash [x' = f(x)\,\&\,Q]\mathbf{p} = 0}$$

## Differential Radical Invariants are vdbx

$$\frac{\Gamma, Q \vdash \bigwedge_{i=0}^{N-1} p^{(i)} = 0 \quad Q \vdash p^{(N)} = \sum_{i=0}^{N-1} g_i p^{(i)}}{\Gamma \vdash [x' = f(x)\,\&\,Q]p = 0}$$






## Proof Idea.

by vdbx with $G = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \dots & 0 & 1 \\ g_0 & g_1 & \dots & g_{N-2} & g_{N-1} \end{pmatrix}$, $\mathbf{p} = \begin{pmatrix} p \\ p^{(1)} \\ p^{(2)} \\ \vdots \\ p^{(N-1)} \end{pmatrix}$

**Vectorial Darboux are VDG**

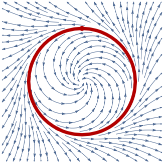
$$\frac{Q \vdash \mathbf{p}' = G\mathbf{p}}{\mathbf{p} = 0 \vdash [x' = f(x) \,\&\, Q]\mathbf{p} = 0}$$

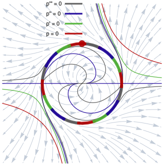
$p'^* = 0$

**Differential Radical Invariants are vdbx**

$$\frac{\Gamma, Q \vdash \bigwedge_{i=0}^{N-1} p^{(i)} = 0 \quad Q \vdash p^{(N)} = \sum_{i=0}^{N-1} g_i p^{(i)}}{\Gamma \vdash [x' = f(x) \,\&\, Q]p = 0}$$

$N$ exists





# ODE Axiomatization: Derived Invariant Rules

**Vectorial Darboux are VDG**

$$\frac{Q \vdash \mathbf{p}' = G\mathbf{p}}{\mathbf{p} = 0 \vdash [x' = f(x) \,\&\, Q]\mathbf{p} = 0}$$

$p'^* = 0$

**Differential Radical Invariants are vdbx**

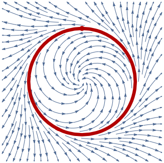
$$\frac{\Gamma, Q \vdash \bigwedge_{i=0}^{N-1} p^{(i)} = 0 \quad Q \vdash p^{(N)} = \sum_{i=0}^{N-1} g_i p^{(i)}}{\Gamma \vdash [x' = f(x) \,\&\, Q]p = 0}$$

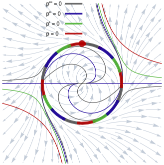
$N$ exists

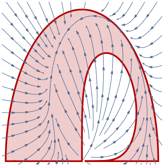
**Semialgebraic Invariants are derived**

$$\frac{p{=}0 \vdash p'{\geq}0 \quad .. \quad p{=}0 \wedge .. \wedge p^{(N-2)}{=}0 \vdash p^{(N-1)}{\geq}0}{p \geq 0 \vdash [x' = f(x)]p \geq 0}$$

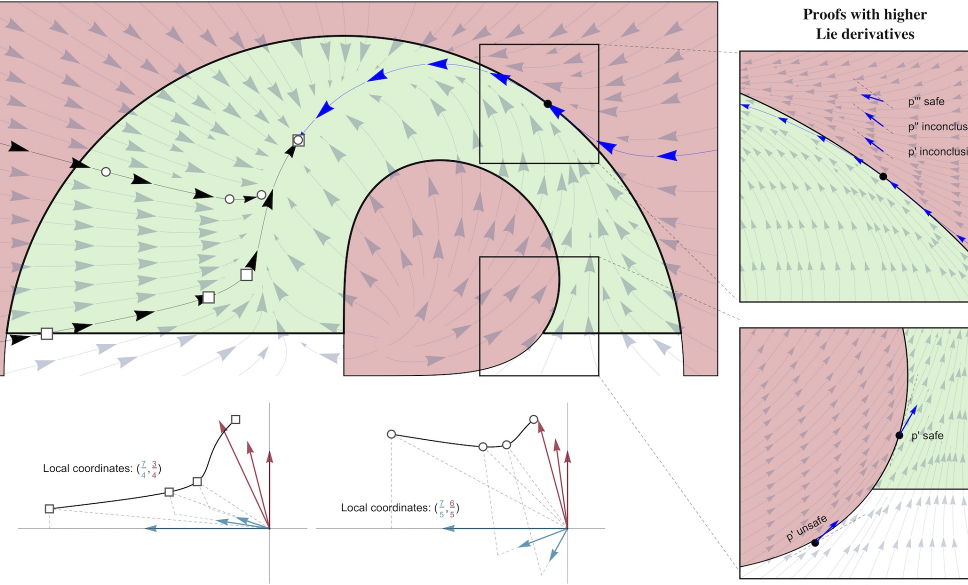
$p'^* \geq 0$





# ODE Axiomatization from Higher Derivatives and Ghosts




**Proofs with higher Lie derivatives**

**Sound and complete ODE invariance proofs**

**Proofs use continuously changing basis to keep invariants at constant local coordinates**

# ODE Axiomatization: Derived Semialgebraic Rules

**Semialgebraic invariants are derived**

$$\frac{P \vdash \bigvee_{i=0}^{M} \big( \bigwedge_{j=0}^{m(i)} {p_{ij}}'^{*}{\geq}0 \wedge \bigwedge_{j=0}^{n(i)} {q_{ij}}'^{*}{>}0 \big) \quad \neg P \vdash \bigvee_{i=0}^{N} \big( \bigwedge_{j=0}^{a(i)} {r_{ij}}'^{*-}{\geq}0 \wedge \bigwedge_{j=0}^{b(i)} {s_{ij}}'^{*-}{>}0 \big)}{P \vdash [x' = f(x)]P}$$

$$P \equiv \bigvee_{i=0}^{M} \big( \bigwedge_{j=0}^{m(i)} p_{ij} \geq 0 \wedge \bigwedge_{j=0}^{n(i)} q_{ij} > 0 \big) \quad \neg P \equiv \bigvee_{i=0}^{N} \big( \bigwedge_{j=0}^{a(i)} r_{ij} \geq 0 \wedge \bigwedge_{j=0}^{b(i)} s_{ij} > 0 \big)$$

$$p'^{*}{=}0 \equiv \bigwedge_{i=0}^{N-1} p^{(i)} = 0 \qquad p'^{*}{\geq}0 \equiv p'^{*}{>}0 \vee p'^{*}{=}0 \qquad p^{(N)} = \sum_{i=0}^{N-1} g_i p^{(i)}$$

$$q'^{*}{>}0 \equiv q \geq 0 \wedge \big(q = 0 \rightarrow q' \geq 0\big) \wedge \big(q = 0 \wedge q' = 0 \rightarrow q^{(2)} \geq 0\big) \wedge \ldots$$
$$\wedge \big(q = 0 \wedge q' = 0 \wedge \cdots \wedge q^{(N-2)} = 0 \rightarrow q^{(N-1)} > 0\big)$$

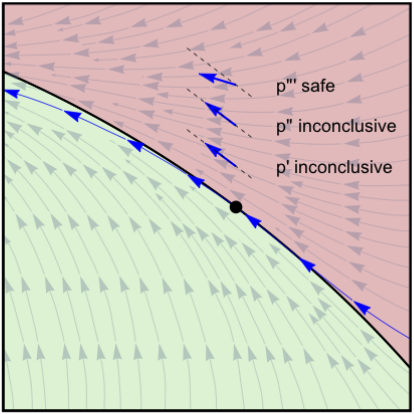
Definable $p'^{*-}$ for *all/most significant* Lie derivatives w.r.t. backwards ODE

# ODE Axiomatization: Derived Semialgebraic Rules

Semialgebraic inva…

$$P \vdash \bigvee_{i=0}^{M} \Big( \bigwedge_{j=0}^{m(i)} p_{ij}{'^{*}} \; \ldots \; {}^{-}{\geq}0 \land \bigwedge_{j=0}^{b(i)} s_{ij}{'^{*-}}{>}0 \Big)$$

### Seriously?




Fortunately, it's just a derived rule!

$$P \equiv \bigvee_{i=0}^{M} \Big( \bigwedge_{j=0}^{m(i)} p \; \ldots \; r_{ij} \geq 0 \land \bigwedge_{j=0}^{b(i)} s_{ij} > 0 \Big)$$

$$p'^{*}{=}0 \equiv \bigwedge_{i=0}^{N-1} p^{(i)} = \ldots \; p^{(N)} = \sum_{i=0}^{N-1} g_i p^{(i)}$$

$$q'^{*}{>}0 \equiv q \geq 0 \land \big( \ldots \rightarrow q^{(2)} \geq 0 \big) \land \ldots$$

$$\land \big( q = 0 \land q \ldots > 0 \big)$$

Definable $p'^{*-}$ for *all/most significant* Lie derivatives w.r.t. backwards ODE

# Extended Axiomatization for Semialgebraics

## Real Induction

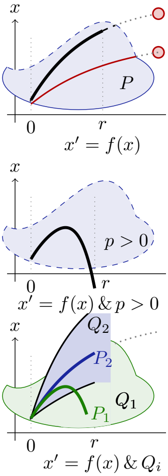
$$[x' = f(x)]P \leftrightarrow \forall y\, [x' = f(x)\,\&\, P \lor x{=}y] \big(x{=}y \rightarrow P \land \langle x' = f(x)\,\&\, P\rangle x{\neq}y\big)$$

## Continuous Existence

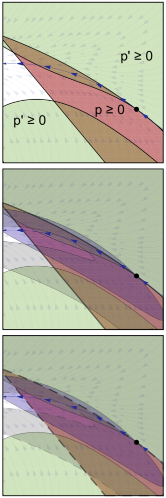
$$p > 0 \rightarrow \langle x' = f(x)\,\&\, p > 0\rangle \circ$$

## Unique Solutions

$$\langle x' = f(x)\,\&\, Q_1\rangle P_1 \land \langle x' = f(x)\,\&\, Q_2\rangle P_2 \rightarrow \langle x' = f(x)\,\&\, Q_1 \land Q_2\rangle (P_1 \lor P_2)$$



# ODE Axiomatization: Derived Local Progress Rules




## Local Progress Step

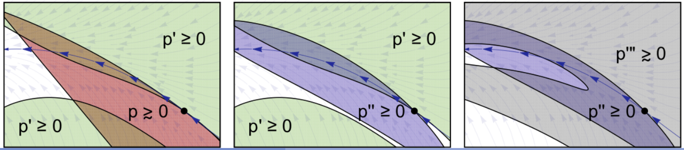
$$p > 0 \;\lor\; p = 0 \land \langle x' = f(x) \,\&\, p' \geq 0 \rangle \circ$$
$$\rightarrow \langle x' = f(x) \,\&\, p \geq 0 \rangle \circ$$

## Local Progress $\geq$

$$p'^{*} \geq 0 \rightarrow \langle x' = f(x) \,\&\, p \geq 0 \rangle \circ$$

## Local Progress $>$

$$p'^{*} > 0 \rightarrow \langle x' = f(x) \,\&\, p > 0 \rangle \bigcirc$$



# Differential Equation Axiomatization

**Theorem (Algebraic Completeness)** (LICS'18)

dL *calculus is a sound & complete axiomatization of algebraic invariants of polynomial differential equations. They are decidable with a derived axiom (on open Q for completeness):*

$$\text{(DRI)}\quad [x' = f(x)\,\&\,Q]p = 0 \leftrightarrow \big(Q \to p'^{*} = 0\big)$$

**Theorem (Semialgebraic Completeness)** (LICS'18)

dL *calculus with RI is a sound & complete axiomatization of semialgebraic invariants of differential equations. They are decidable with derived axiom*

$$\text{(SAI)}\quad \forall x\,(P \to [x' = f(x)]P) \leftrightarrow \forall x\,\big(P \to P'^{*}\big) \land \forall x\,\big(\neg P \to (\neg P)'^{*-}\big)$$

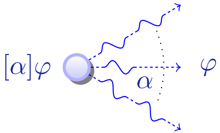
Definable $p'^{*}$ is short for *all/most significant* Lie derivatives w.r.t. ODE
Definable $p'^{*-}$ is w.r.t. backwards ODE. Also for DNF $P$.

# Outline

1. Differential Dynamic Logic
2. Proofs for Differential Equations
   - Differential Invariants / Cuts / Ghosts
3. Completeness for Differential Equation Invariants
   - Darboux are Differential Ghosts
   - Derived Semialgebraic Invariants
   - Real Induction
   - Derived Local Progress
   - Completeness for Invariants
4. Summary

**differential dynamic logic**

dL = DL + HP




1. Poincaré: qualitative ODE
2. Complete axiomatization
3. Algebraic ODE invariants
4. Semialgebraic ODE invariants
5. Algebraic hybrid systems
6. Local ODE progress
7. Decide by dL proof/disproof
8. Uniform substitution axioms

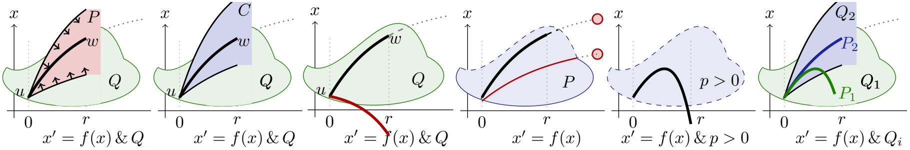
Properties

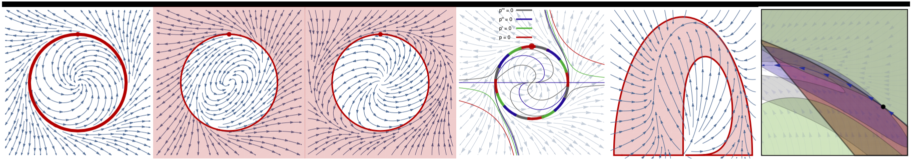
1. MVT
2. Prefix
3. Picard-Lind
4. $\mathbb{R}$-complete
5. Existence
6. Uniqueness

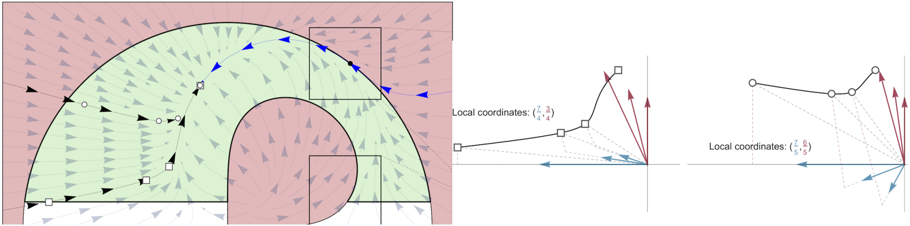
1. Differential invariants
2. Differential cuts
3. Differential ghosts
4. Real induction
5. Continuous existence
6. Unique solutions

Impressive power of differential ghosts

# Differential Equation Axiomatization vs. Derived Rules

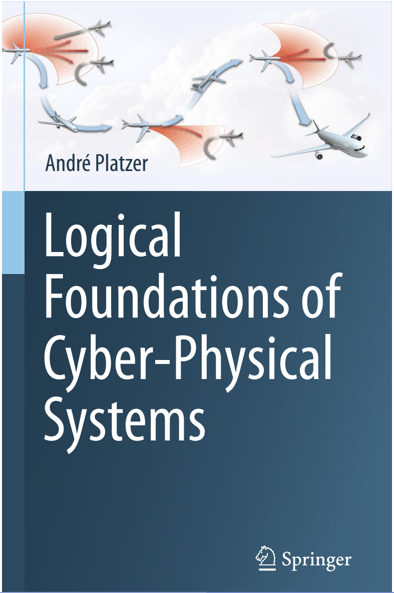




A. Platzer. *Logical Foundations of Cyber-Physical Systems*. Springer 2018

- **I Part: Elementary Cyber-Physical Systems**
- 1. Differential Equations & Domains
- 2. Choice & Control
- 3. Safety & Contracts
- 4. Dynamical Systems & Dynamic Axioms
- 5. Truth & Proof
- 6. Control Loops & Invariants
- 7. Events & Responses
- 8. Reactions & Delays
- **II Part: Differential Equations Analysis**
- 9. Differential Equations & Differential Invariants
- 10. Differential Equations & Proofs
- 11. Ghosts & Differential Ghosts
- 12. Differential Invariants & Proof Theory
- **III Part: Adversarial Cyber-Physical Systems**
- 13-16. Hybrid Systems & Hybrid Games
- **IV Part: Comprehensive CPS Correctness**



André Platzer and Yong Kiam Tan.
Differential equation axiomatization: The impressive power of differential ghosts.
In Anuj Dawar and Erich Grädel, editors, *LICS*, pages 819–828, New York, 2018. ACM.
doi:10.1145/3209108.3209147.

André Platzer.
Differential-algebraic dynamic logic for differential-algebraic programs.
*J. Log. Comput.*, 20(1):309–352, 2010.
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