# Differential Refinement Logic

Sarah M. Loos and André Platzer
Computer Science Department
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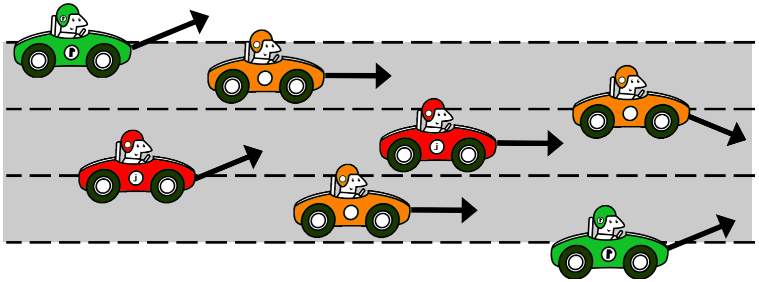
# Verified Cyber-Physical Systems





[FM11]

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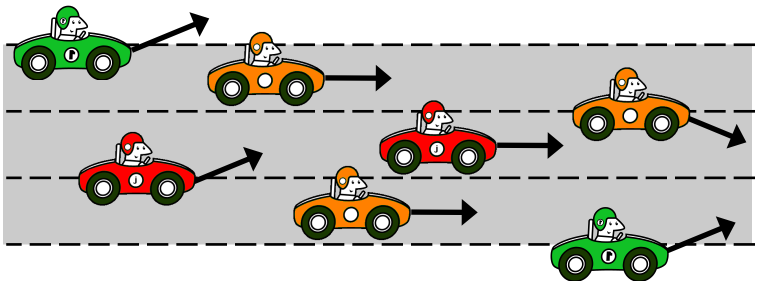

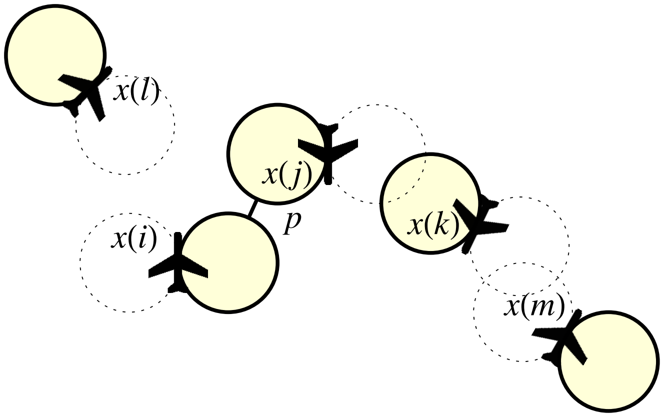




[FM11, HSCC13]

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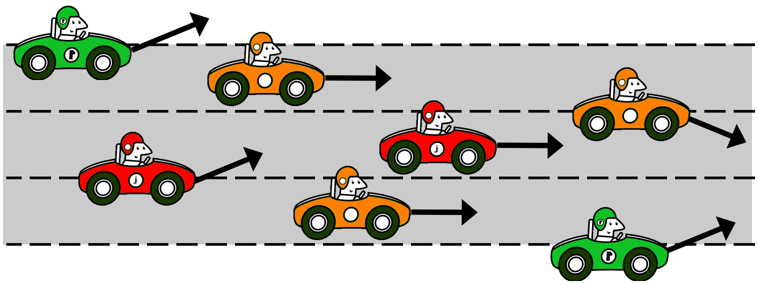
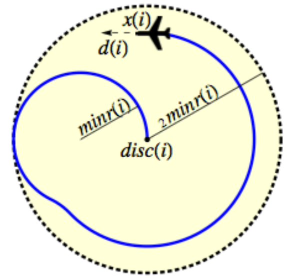
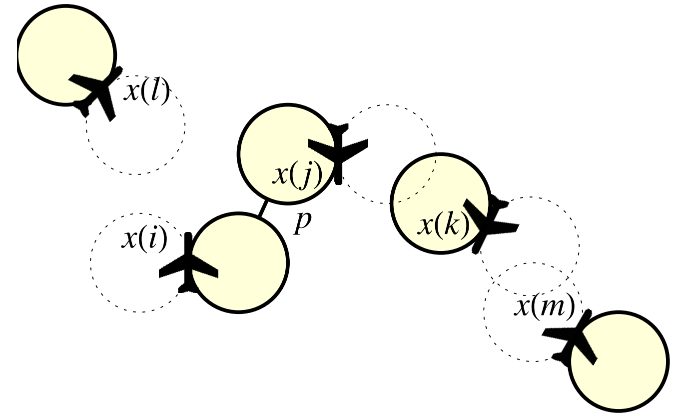
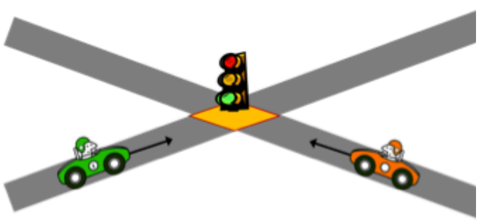
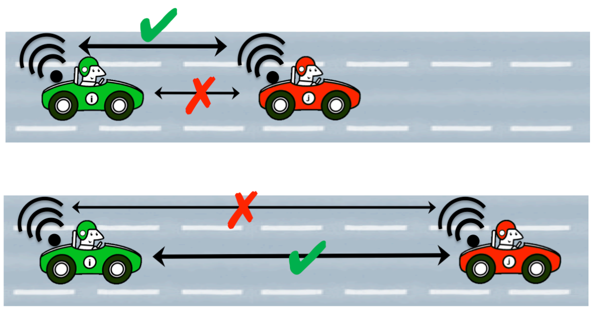
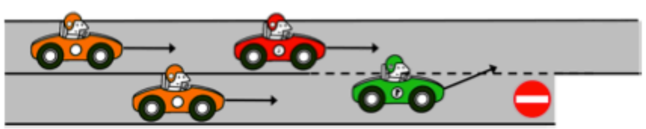
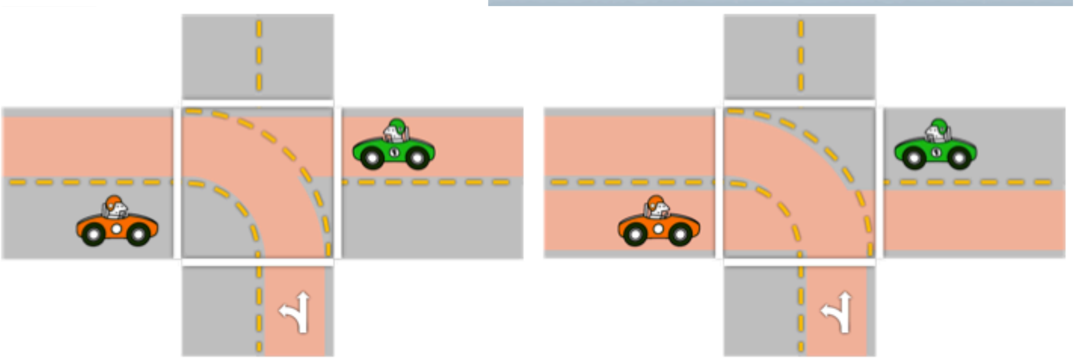
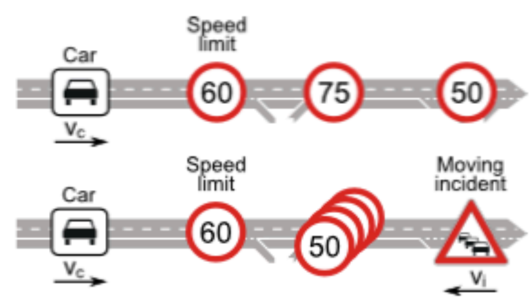








[FM11, ITSC11, ICCPS12, HSCC13, ITSC13]

# Roadmap

## Differential Refinement Logic (dRL)

$$\boldsymbol{\alpha} \leq \boldsymbol{\beta}$$

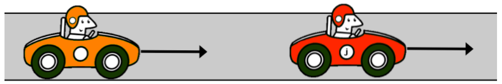
### Proof Calculus

$$\frac{\Gamma \vdash [\beta]\phi, \Delta \qquad \Gamma \vdash \alpha \leq \beta, \Delta}{\Gamma \vdash [\alpha]\phi, \Delta}([\leq])$$

### Time-triggered vs. Event-triggered

$$\mathtt{time}^* \leq \mathtt{event}^*$$

### Verified Car Control



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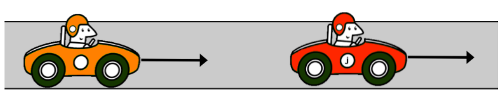

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$$\Big((?\phi; a := \theta \cup a := -B); x'' = a \,\&\, \psi\Big)^*$$

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$$\big((?\phi; \boldsymbol{a} := * \cup \boldsymbol{a} := -\boldsymbol{B}); \boldsymbol{x}'' = \boldsymbol{a}\big)^*$$

# So, what does dRL look like exactly?

Syntax of a dRL formula:

$$\boldsymbol{\phi, \psi ::= \ \theta_1 \leq \theta_2 \mid \neg\phi \mid \phi \wedge \psi \mid \forall x \phi}$$

$\boxed{\mathrm{FOL}_{\mathbb{R}}}$

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dL

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$\boxed{\text{refinement}}$

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Syntax of a hybrid program:

$$\begin{aligned}\alpha, \beta ::= \ & x := \theta \mid x' = \theta \ \& \ \psi \mid ?\psi \\ & \mid \alpha \cup \beta \mid \alpha ; \beta \mid \alpha^*\end{aligned}$$

d$\mathcal{L}$

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$$\mid [\alpha]\phi \mid \langle\alpha\rangle\phi$$
$$\mid \boxed{\alpha \leq \beta}$$

dRL extends d$\mathcal{L}$ by adding refinement directly into the grammar of formulas

Syntax of a hybrid program:

$$\alpha, \beta ::= x := \theta \mid x' = \theta \ \& \ \psi \mid ?\psi$$
$$\mid \alpha \cup \beta \mid \alpha; \beta \mid \alpha^*$$

# Semantics of hybrid programs

***Hybrid Programs*** model cyber-physical systems

$$v \xrightarrow{\alpha} w$$

$\rho(\alpha) = \{(v, w) :$ when starting in state $v$ and then following transitions of $\alpha$, state $w$ can be reached. $\}$



[Platzer08]

# Semantics of hybrid programs

$$v \xrightarrow{x := \theta} w$$

iff $v = w$ except for the value of $x$



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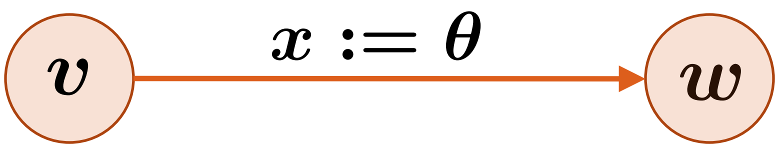


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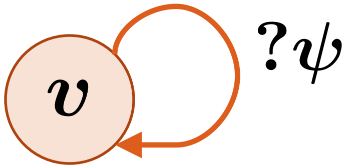


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[Platzer08]

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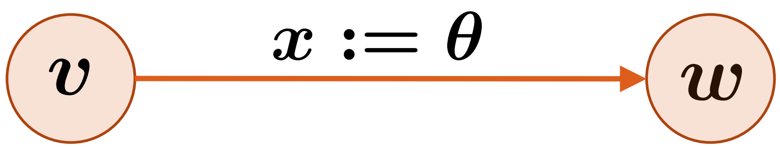


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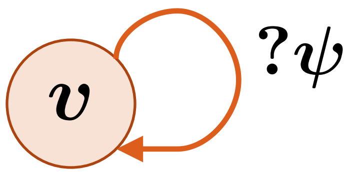


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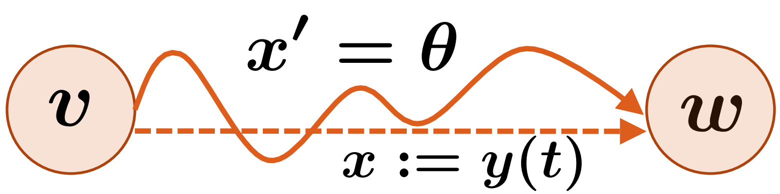


If $y(t)$ solves $x' = \theta$



[Platzer08]

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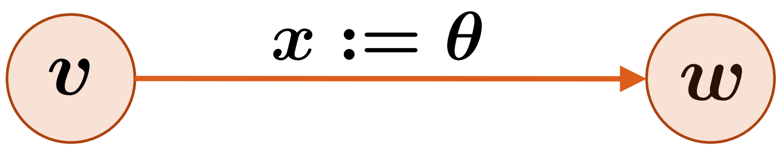


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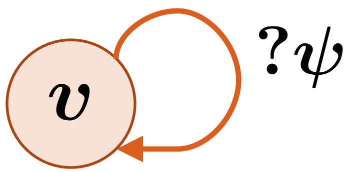


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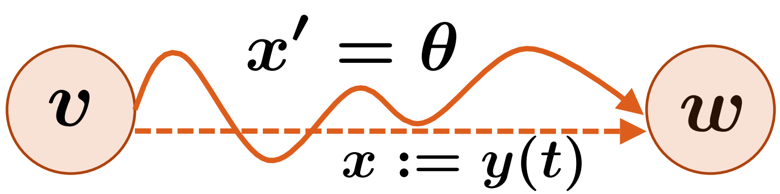


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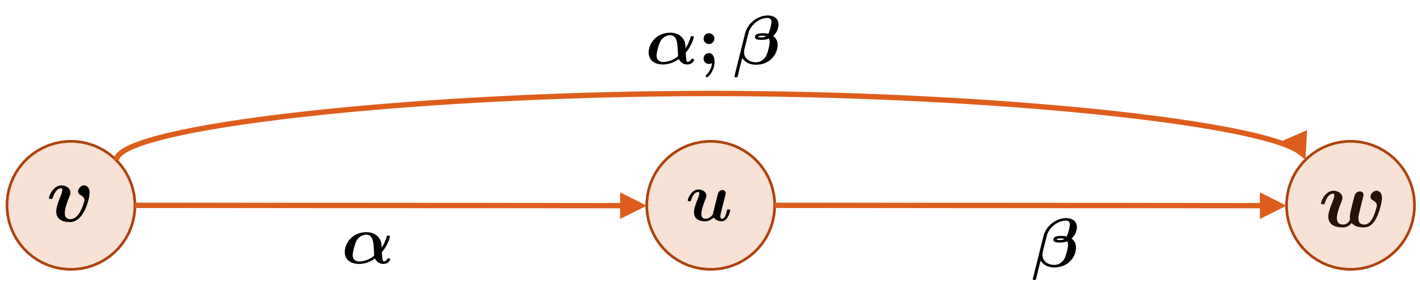




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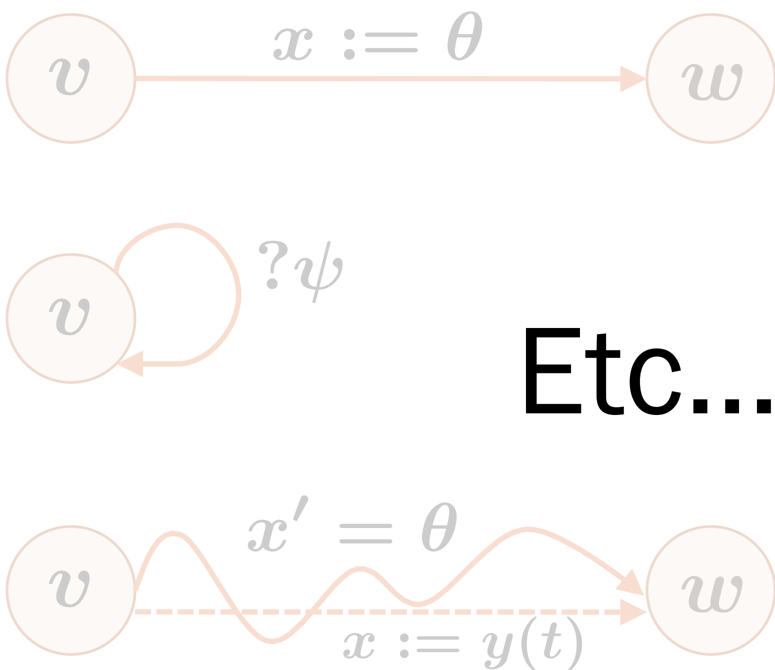


iff $v = w$ except for the value of $x$

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Etc...

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[Platzer08]

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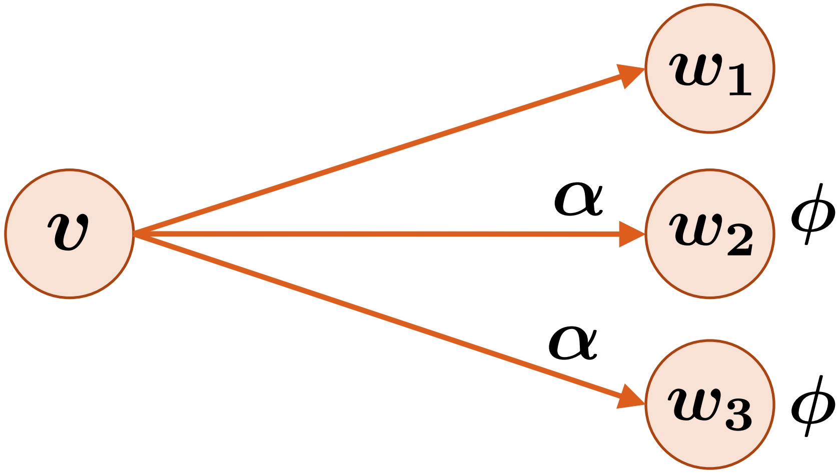

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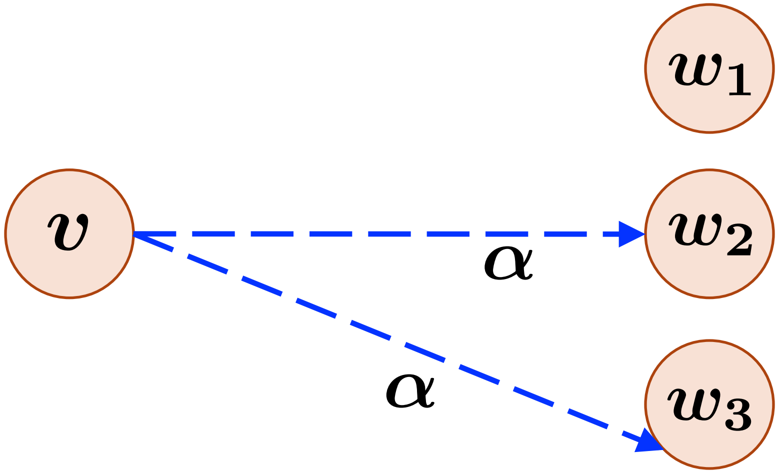

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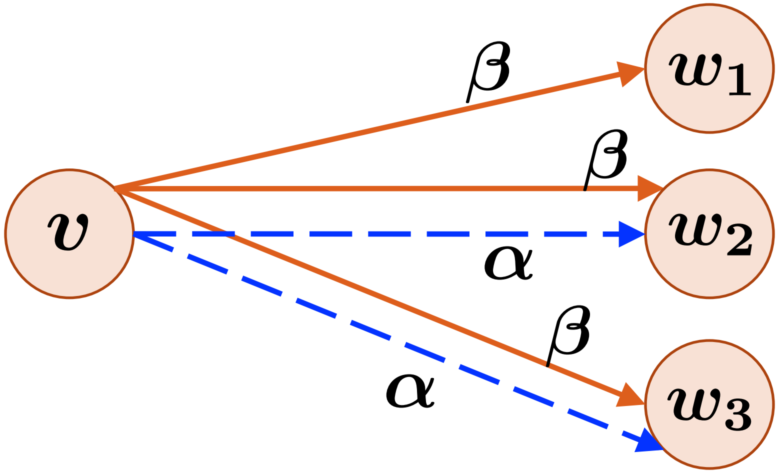

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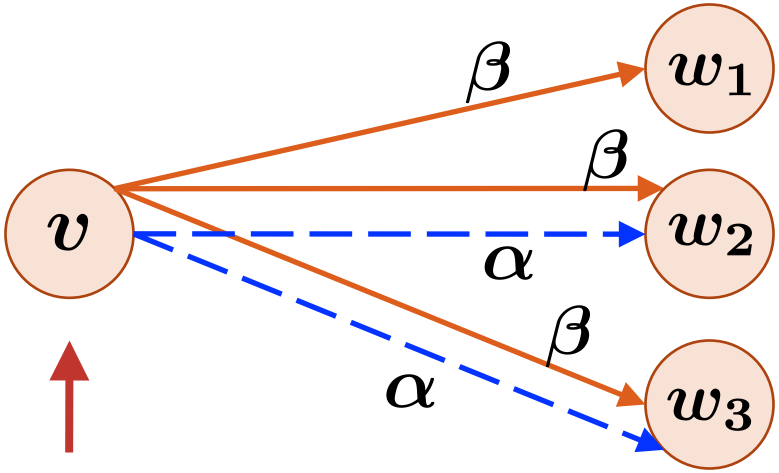

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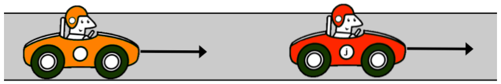

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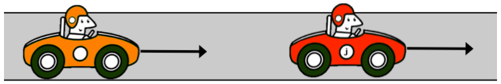

# Combining refinement and box modality

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for all $\boldsymbol{G \in \Gamma, D \in \Delta}$

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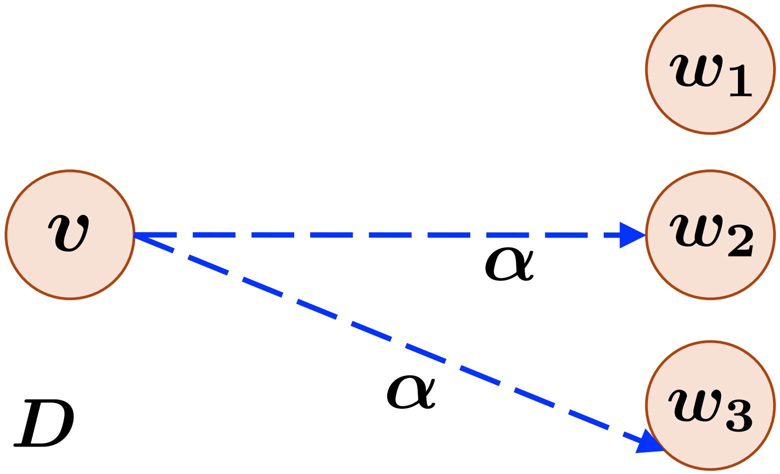


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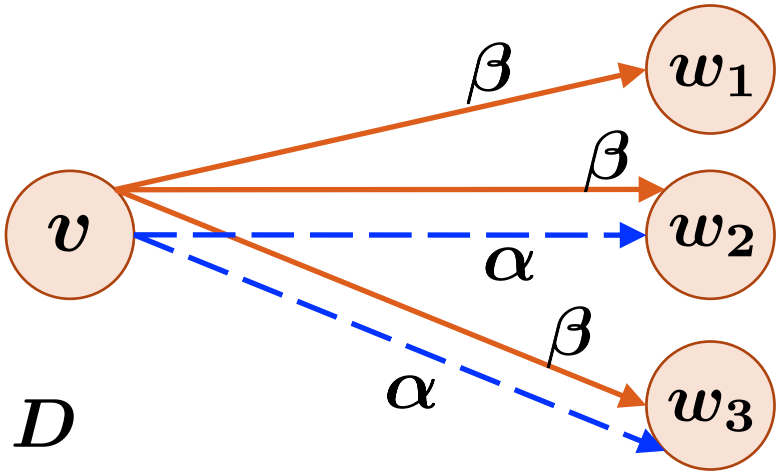


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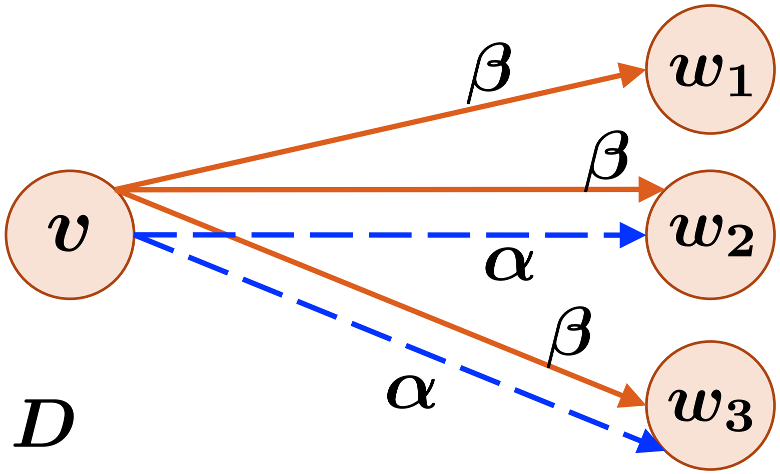


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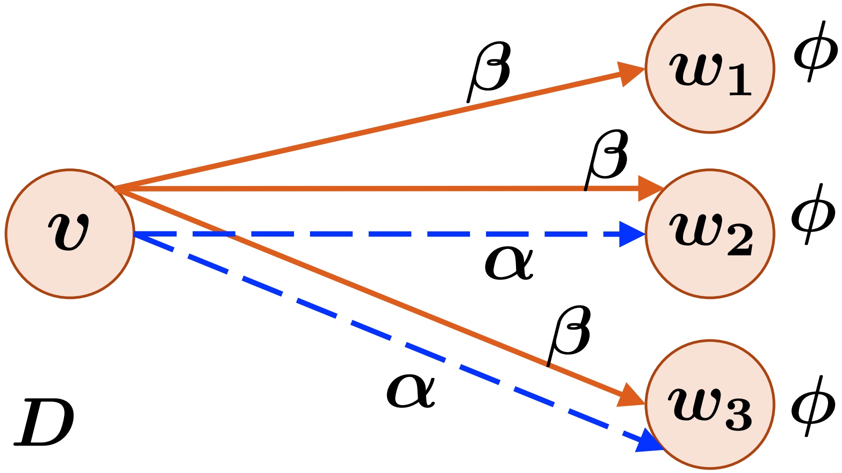


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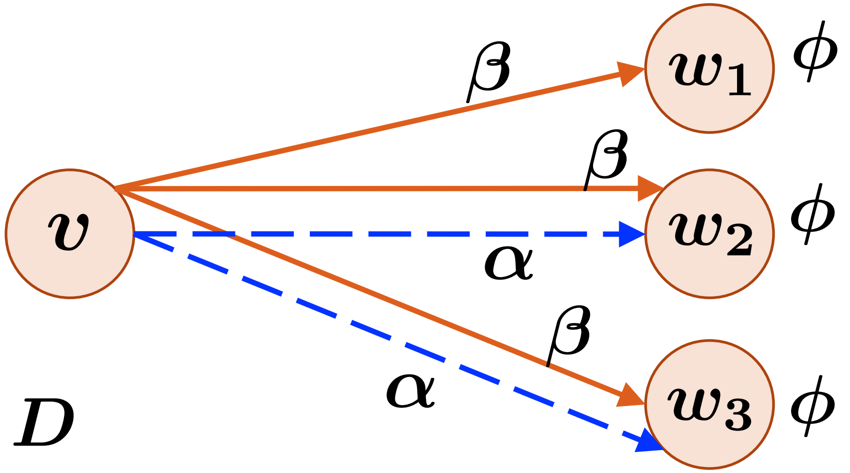


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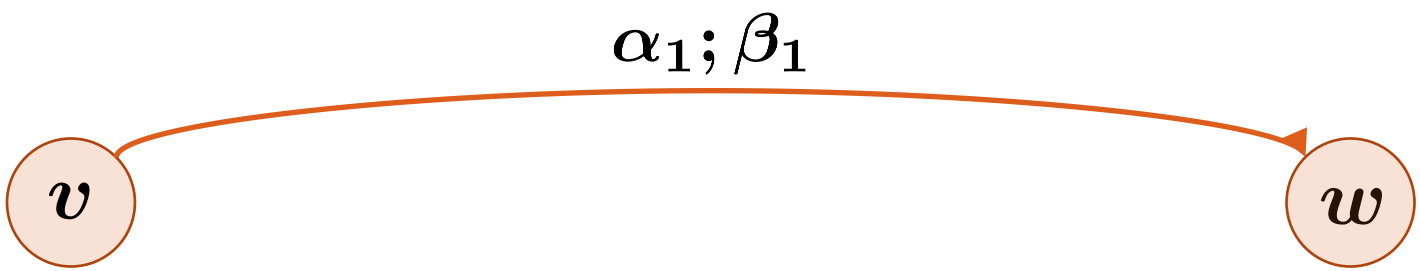
# Sequential Composition

$$\frac{\Gamma \vdash \alpha_1 \leq \alpha_2, \Delta \qquad\qquad \Gamma \vdash (\beta_1 \leq \beta_2), \Delta}{\Gamma \vdash (\alpha_1;\beta_1) \leq (\alpha_2;\beta_2), \Delta}(\,;\,)$$



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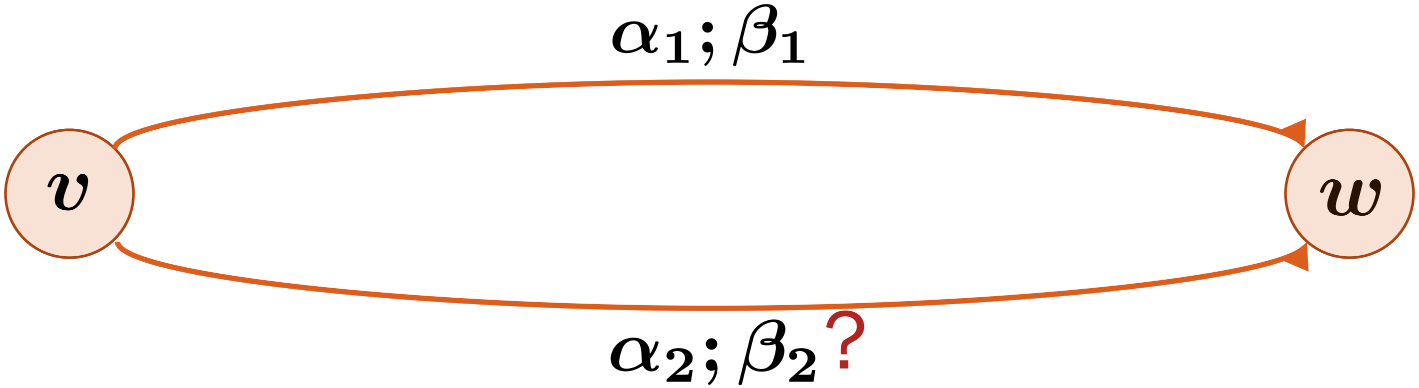

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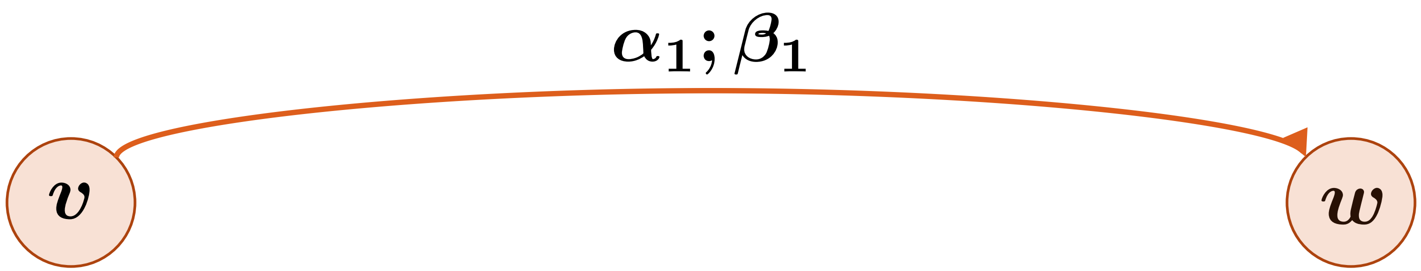

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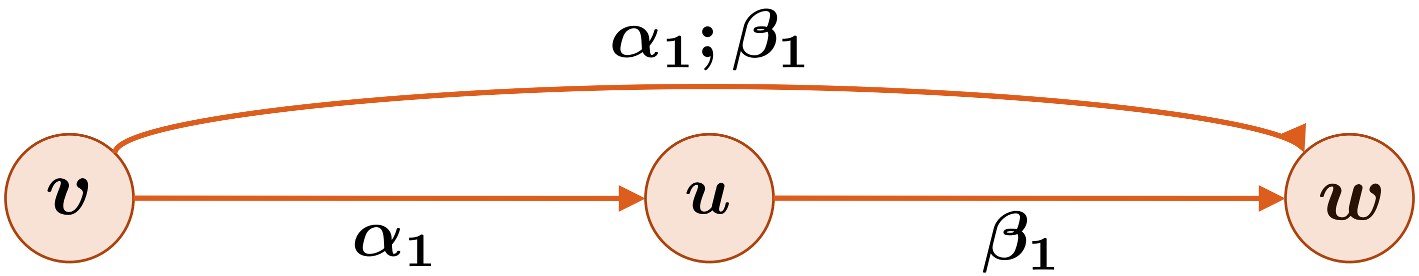

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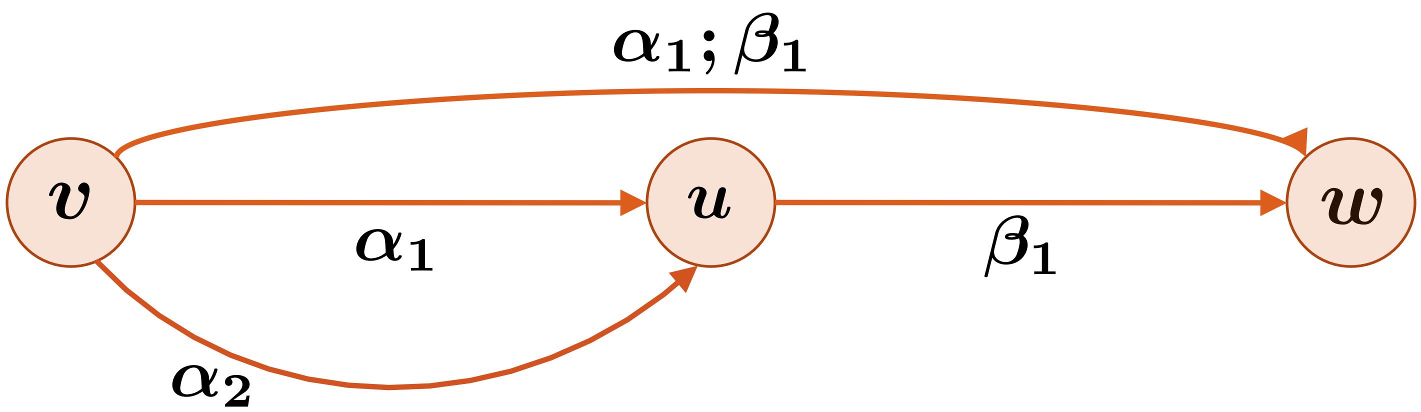

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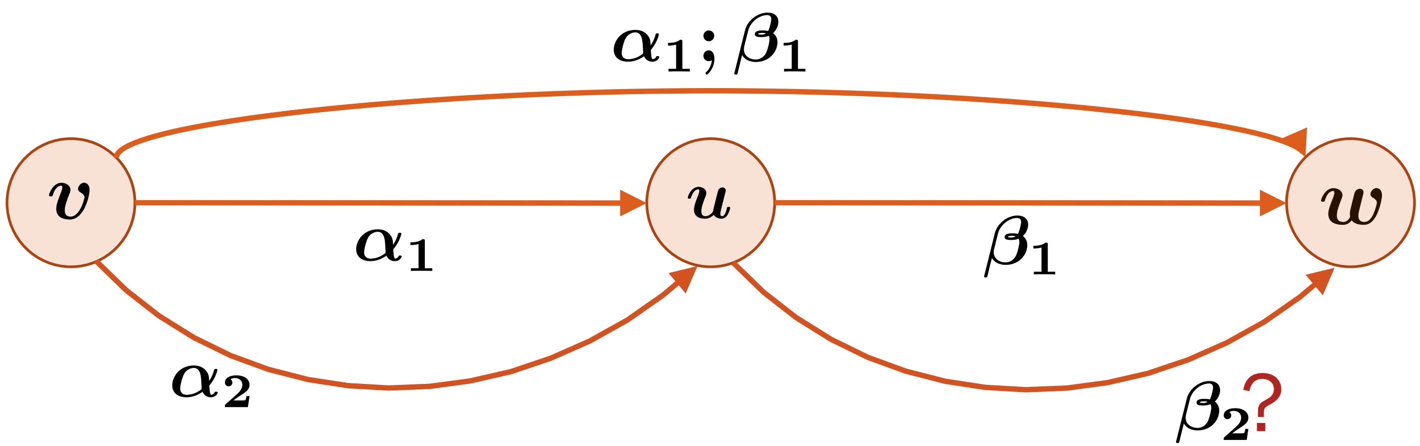

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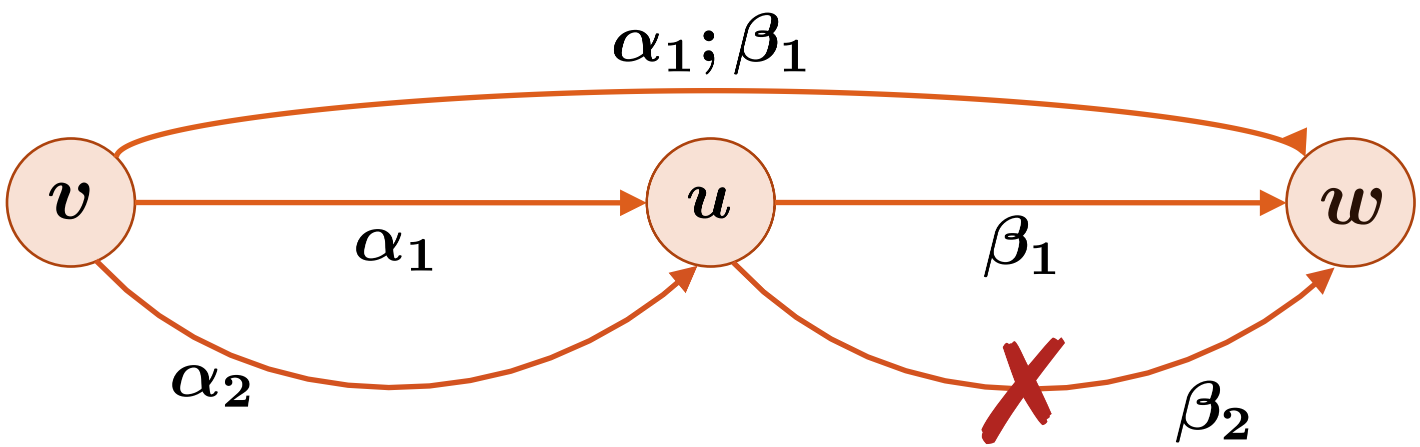

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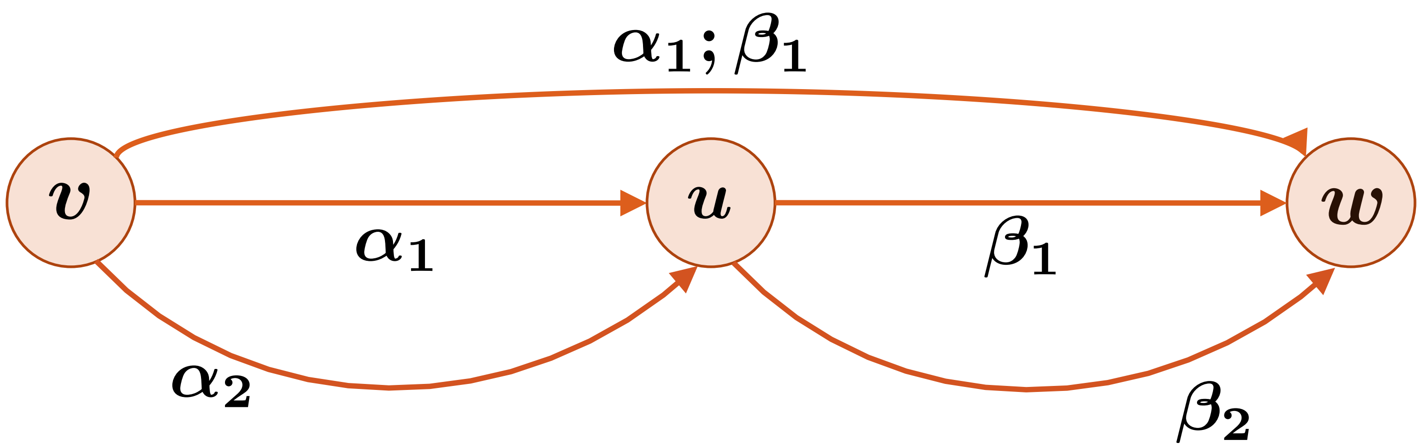

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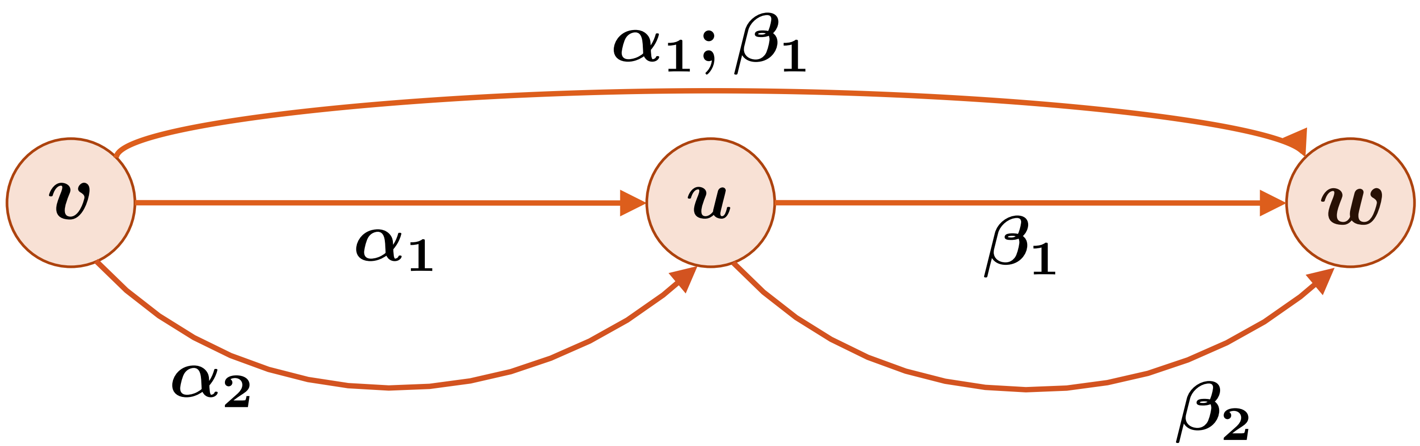

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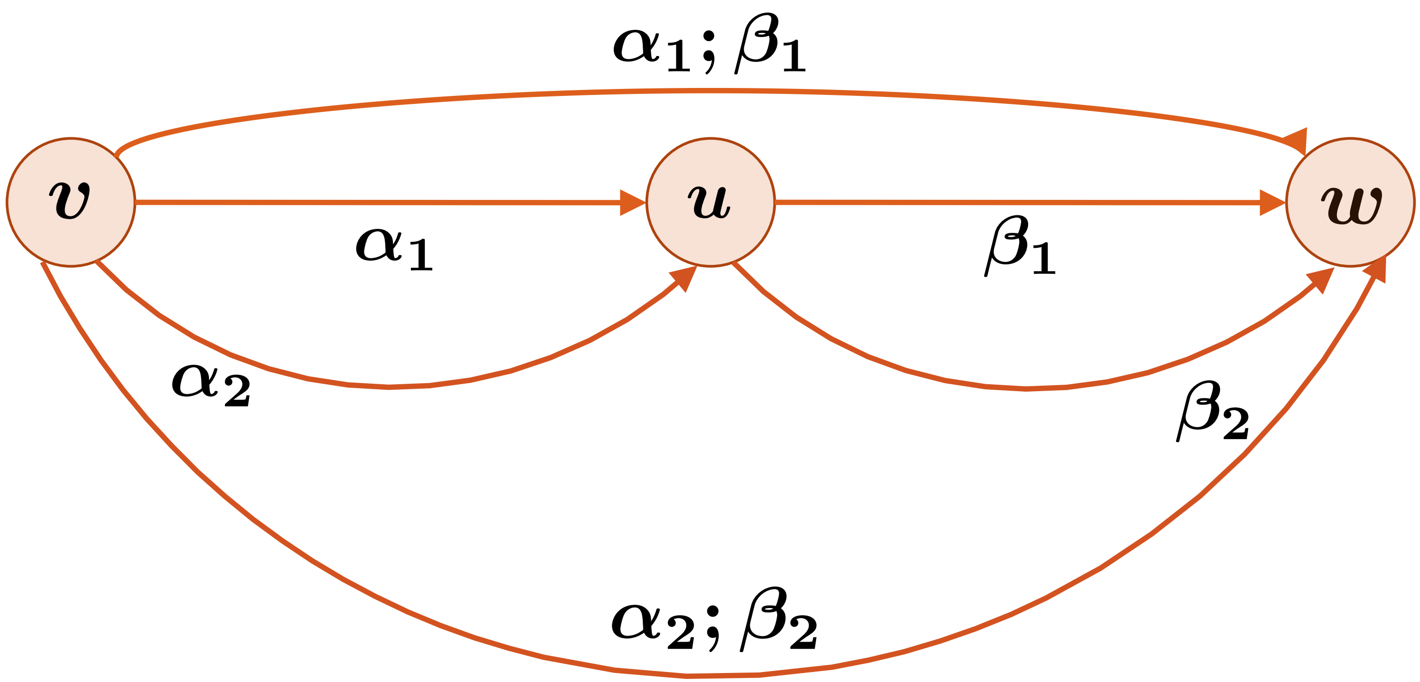

# Differential Equations

$$(\boldsymbol{x'} = \mathbf{1}) \stackrel{?}{\leq} (\boldsymbol{x'} = \mathbf{9})$$

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$$(\boldsymbol{x}' = \mathbf{1}) \overset{?}{\leq} (\boldsymbol{x}' = \mathbf{9})$$

$$\boldsymbol{x} \in [\boldsymbol{x}_{\mathbf{0}}, \boldsymbol{\infty})$$

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$$(x' = 1) \overset{?}{\leq} (x' = 9)$$

$$x \in [x_0, \infty) \qquad\qquad x \in [x_0, \infty)$$

# Differential Equations

$$(\boldsymbol{x' = 1}) = (\boldsymbol{x' = 9})$$

$$\boldsymbol{x \in [x_0, \infty)} \qquad\qquad \boldsymbol{x \in [x_0, \infty)}$$

# Differential Equations

$$\frac{\Gamma \vdash \forall x \left( \frac{\theta_1}{||\theta_1||} = \frac{\theta_2}{||\theta_2||} \wedge (||\theta_1|| = 0 \leftrightarrow ||\theta_2|| = 0) \right), \Delta}{\Gamma \vdash (x' = \theta_1) = (x' = \theta_2), \Delta} \; (mdf)$$

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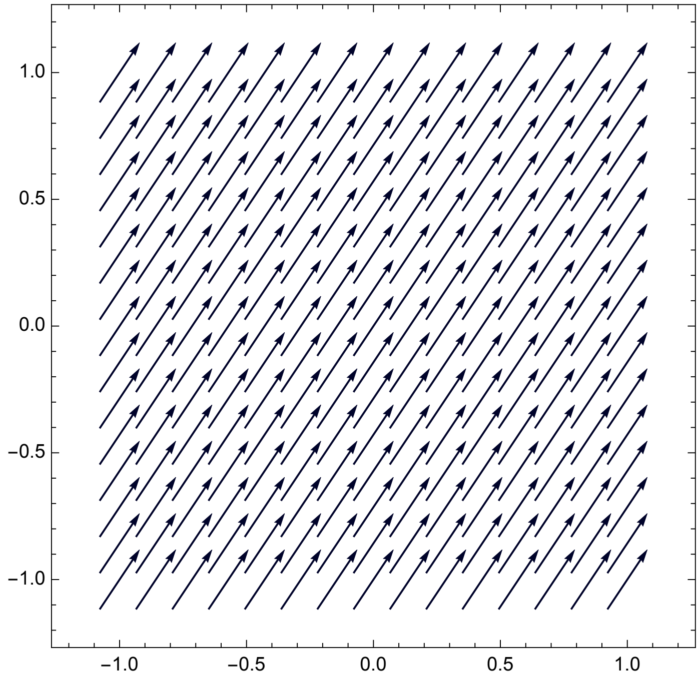


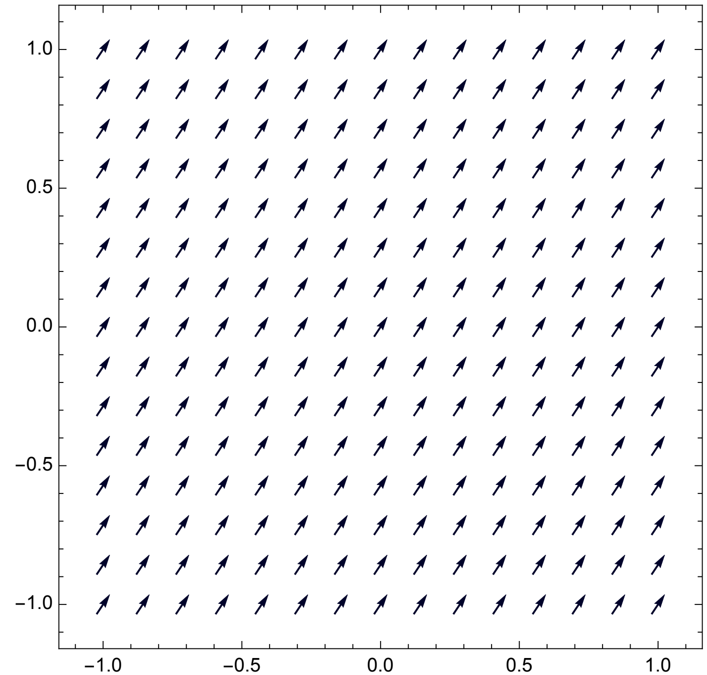


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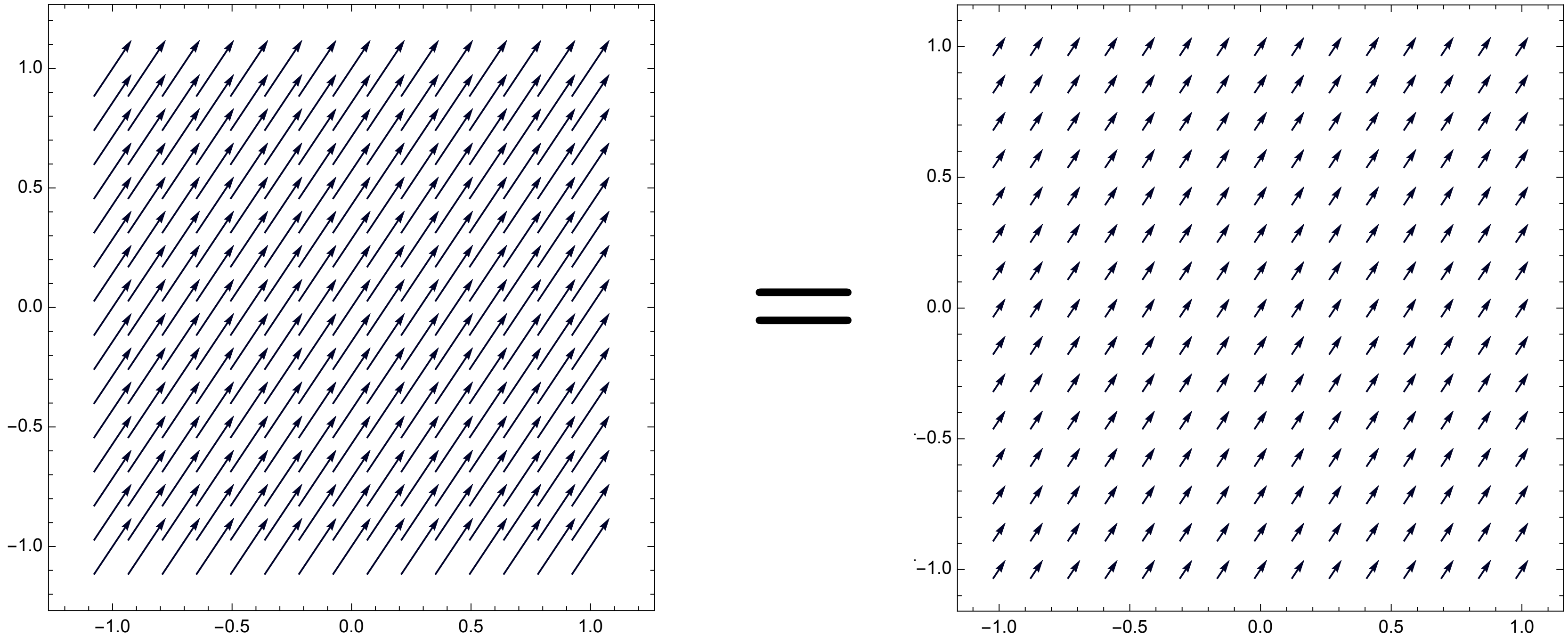


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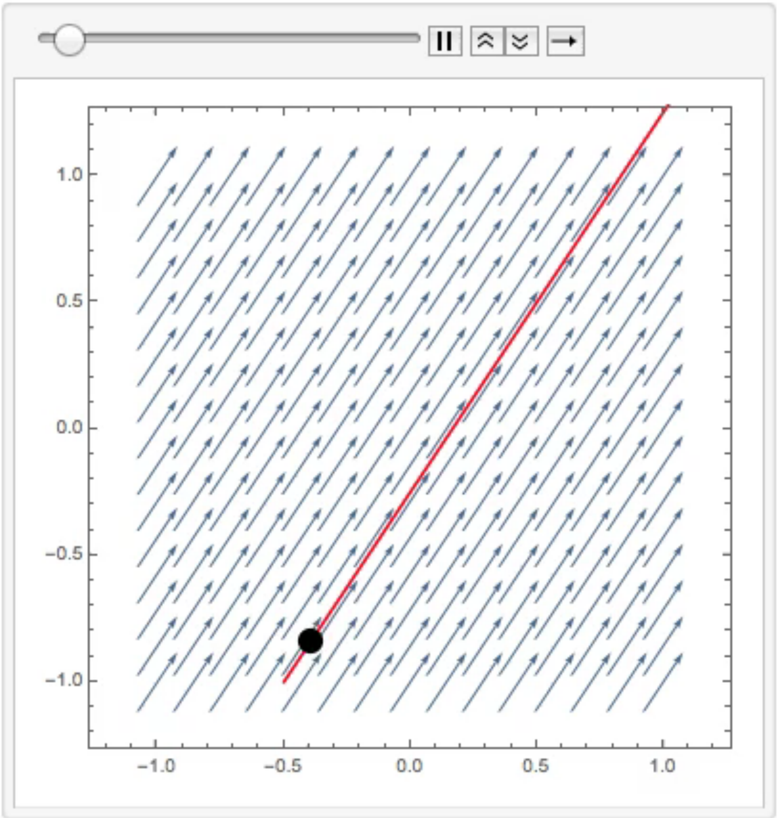


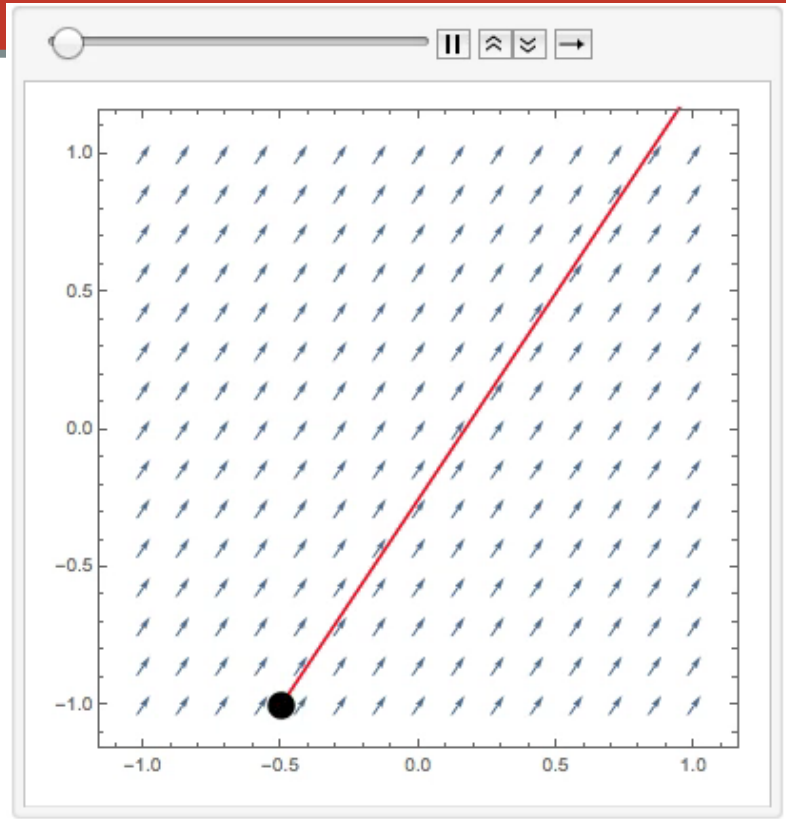


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# Roadmap

## Differential Refinement Logic (dRL)

$$\alpha \leq \beta$$

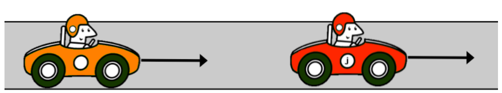
### Proof Calculus

$$\frac{\Gamma \vdash [\beta]\phi, \Delta \qquad \Gamma \vdash \alpha \leq \beta, \Delta}{\Gamma \vdash [\alpha]\phi, \Delta}([\leq])$$

### Time-triggered vs. Event-triggered

$$\mathtt{time}^* \leq \mathtt{event}^*$$

### Verified Car Control



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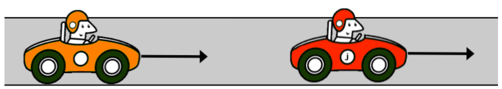

# Two Modeling Paradigms

## Time-triggered

- Discrete sensing

## Event-triggered

- Continuous sensing

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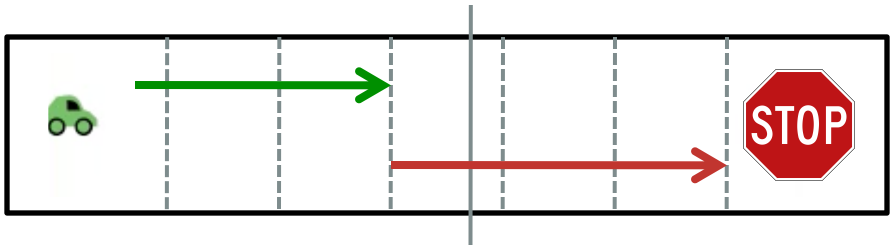
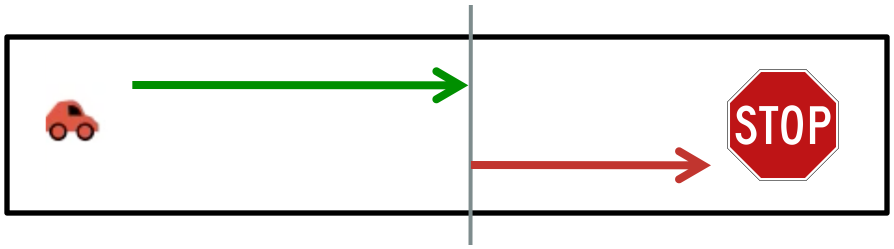



## Event-triggered

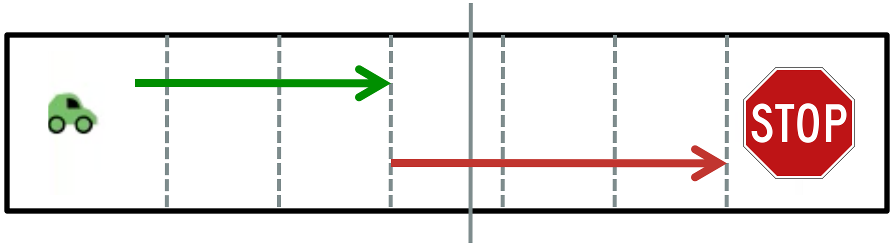
- Continuous sensing



# Two Modeling Paradigms

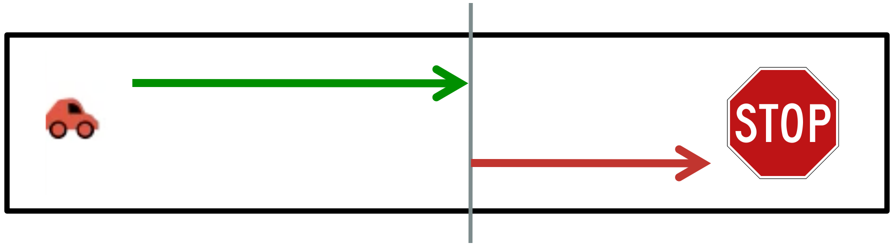
## Time-triggered

- Discrete sensing
- Realistic, easy to implement
- Difficult to design controllers
- Challenging to verify




## Event-triggered

- Continuous sensing
- Unrealistic, hard to implement
- Easier to design controllers
- Easier to verify



# Two Modeling Paradigms

Time-triggered $\leq$ Event-triggered

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- Difficult to design controllers
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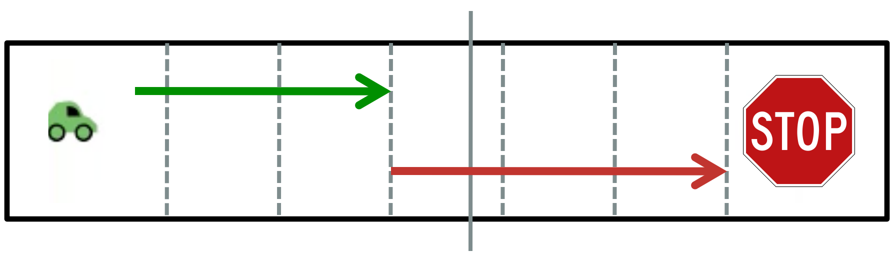


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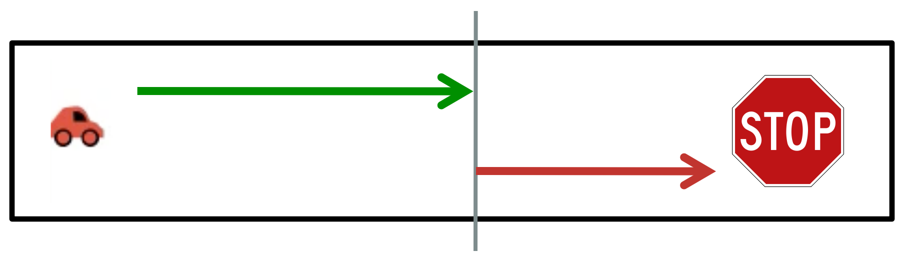

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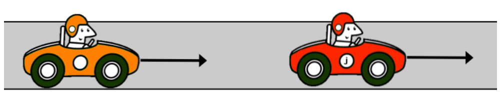

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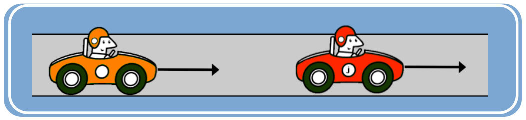
### Time-triggered vs. Event-triggered

$$\texttt{time}^* \leq \texttt{event}^*$$

### Verified Car Control

# Local Lane Control using Refinement

| Proof statistics for local lane controller, with and without refinement | | | |
|---|---|---|---|
|  | Interactive Steps | Computation Time (seconds) | Proof Nodes |
| Time-triggered [FM11] | **656** | **329.8** | **924** |
| Event-triggered | 4 | 73.3 | 140 |
| Controllers satisfy refinement | 0 | 0.6 | 16 |
| “Brake” for epsilon time | 0 | 2.7 | 30 |
| “Accelerate” for epsilon time | 79 | 8.4 | 126 |
| Time-triggered (dRL) | **83** | **85.0** | **312** |

# Contributions

## Differential Refinement Logic

- Maintains a modular and hierarchical proof structure
- Abstracts implementation-specific designs
- Leverages iterative system design
- Prove time-triggered model refines event-triggered
- Encouraging evidence of reduced user interaction and computation time

# Appendix

# Comparing dRL and dL

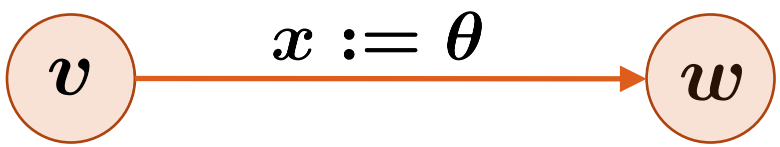
$$\vDash_{\mathsf{dR}\mathcal{L}} \alpha \leq \beta \iff \vDash_{\mathsf{d}\mathcal{L}} \forall \bar{x}\, (\langle \alpha \rangle (x = \bar{x}) \rightarrow \langle \beta \rangle (x = \bar{x}))$$

We have proved that the refinement relation can be embedded in dL. As a result, dL and dRL are equivalent in terms of *expressibility* and *provability*.

However, we can analyze dRL on familiar (challenging) case studies. We can consider:

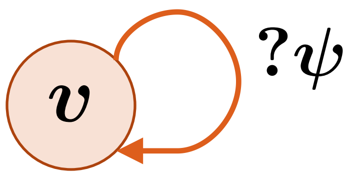
- Number of proof steps
- Computation time
- Qualitative difficulty to complete proof
- Proof structure

# Semantics of hybrid programs



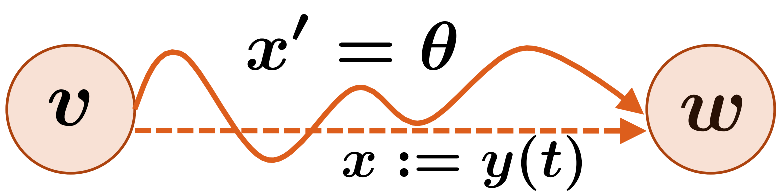

iff $v = w$ except for the value of $x$

$$\rho(x := \theta) = \{(v, w) : w = v \text{ except } [[x]]_w = [[\theta]]_v\}$$




Iff $\psi$ holds in state $v$

$$\rho(?\psi) = \{(v, v) : v \models \psi\}$$



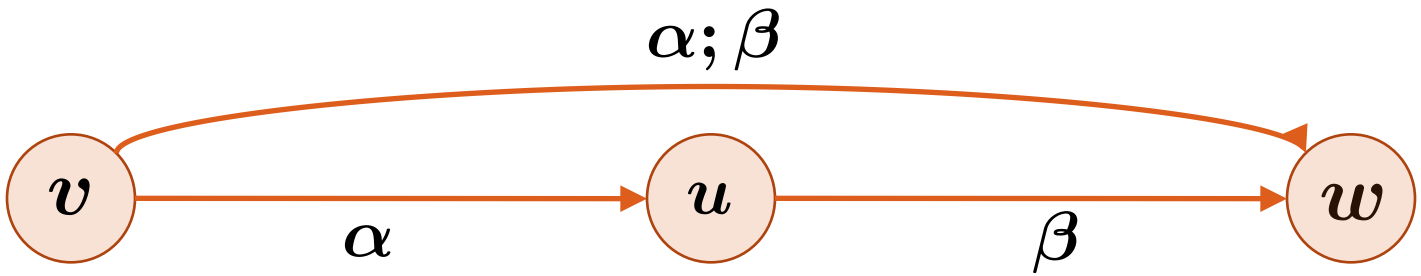

If $y(t)$ solves $x' = \theta$

$$\rho(x' = \theta) = \{(\varphi(0), \varphi(t)) : \varphi(s) \models x' = \theta \text{ for all } 0 \leq s \leq t\}$$



[Platzer08]

# Semantics of hybrid programs




$$\rho(\alpha;\beta) = \{(v,w) : (v,u) \in \rho(\alpha), (u,w) \in \rho(\beta)\}$$



[Platzer08]

# Combining refinement and diamond modality

$$\frac{\Gamma \vdash \boxed{[\beta]\phi}, \Delta \qquad \Gamma \vdash \alpha \leq \beta, \Delta}{\Gamma \vdash \boxed{[\alpha]\phi}, \Delta}([\leq])$$

$$\frac{\Gamma \vdash \boxed{\langle\alpha\rangle\phi}, \Delta \qquad \Gamma \vdash \alpha \leq \beta, \Delta}{\Gamma \vdash \boxed{\langle\beta\rangle\phi}, \Delta}(\langle\leq\rangle)$$

# Nondeterministic Assignment

$$\frac{}{\Gamma \vdash (x := \theta) \leq (x := *), \Delta}(:= *)$$

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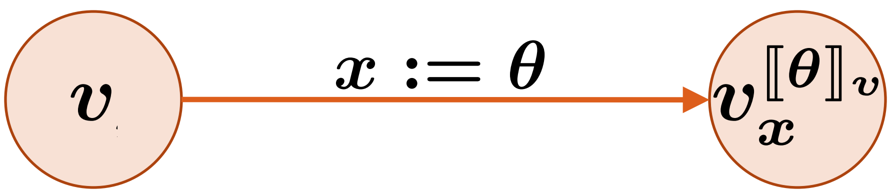

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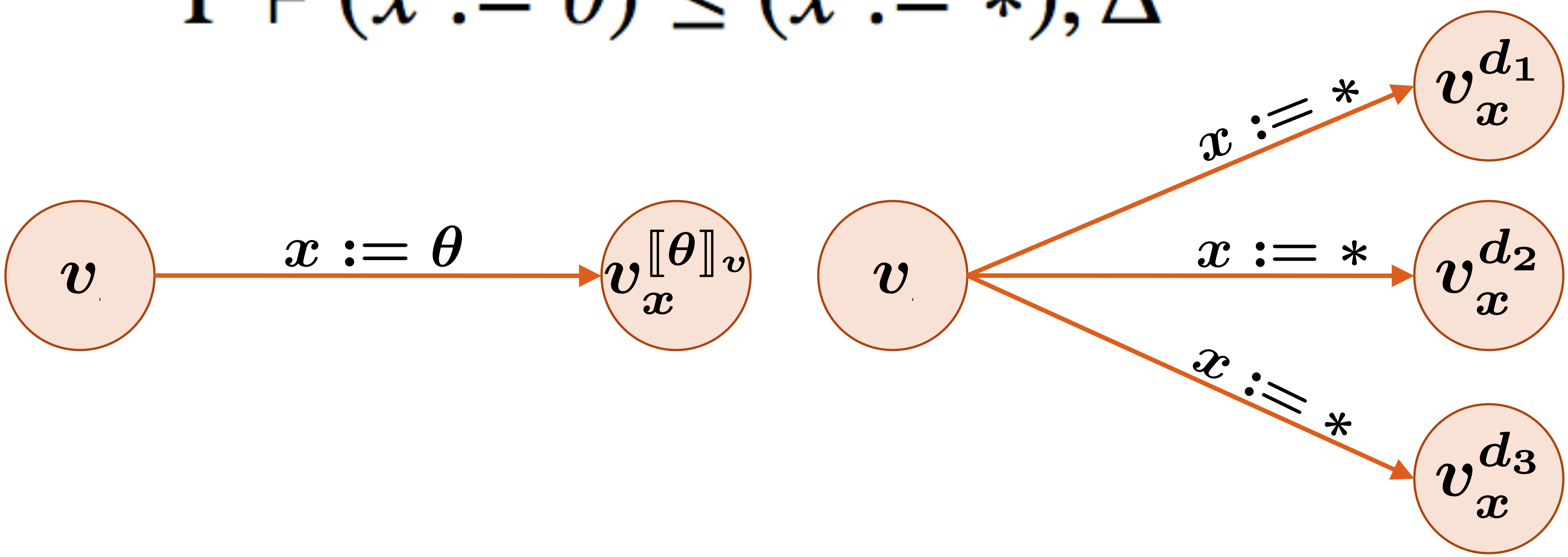

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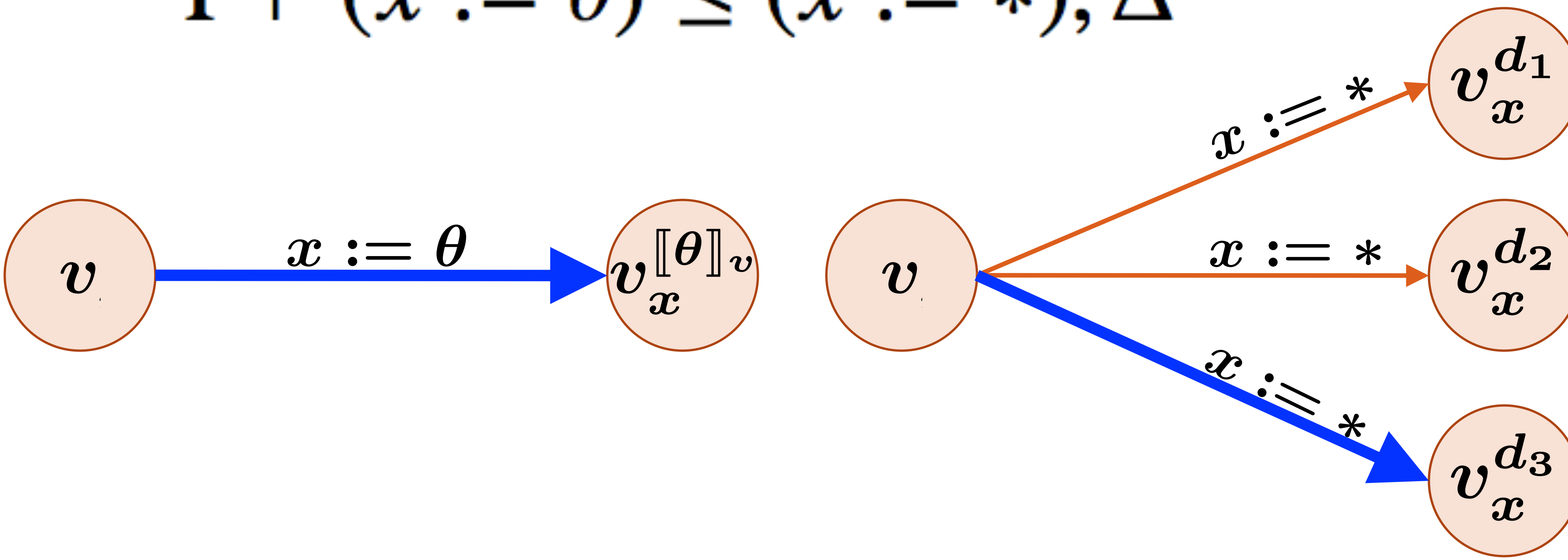
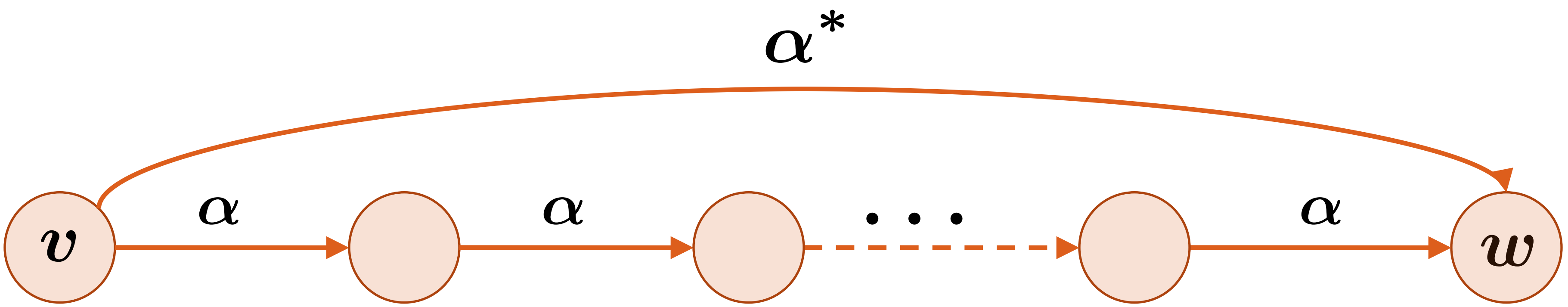


# Nondeterministic Repetition

$$\frac{\Gamma \vdash (\alpha \leq \beta), \Delta}{\Gamma \vdash \alpha^* \leq \beta^*, \Delta}(unloop)$$



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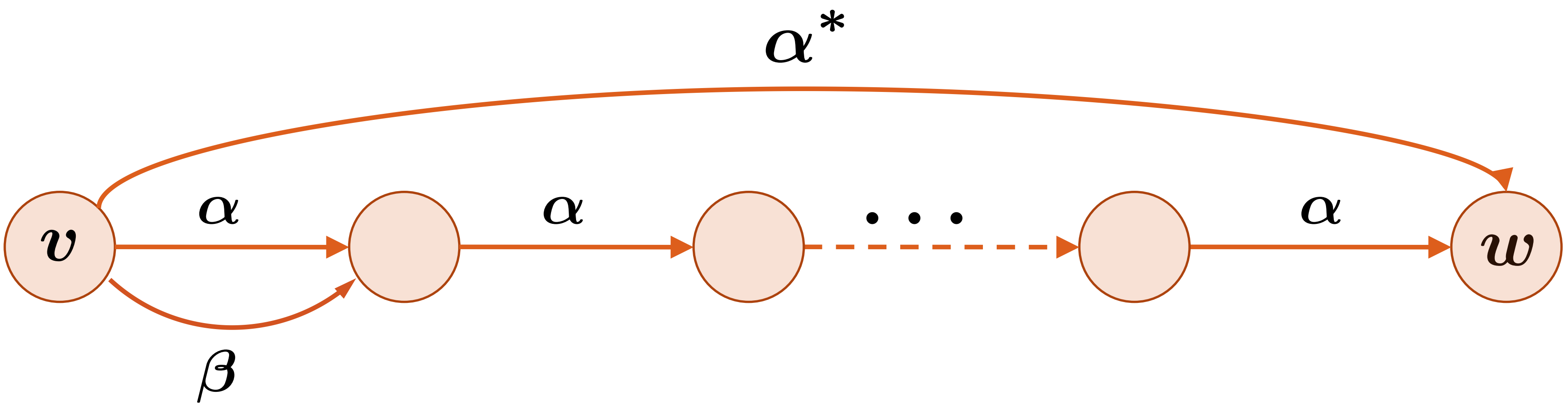

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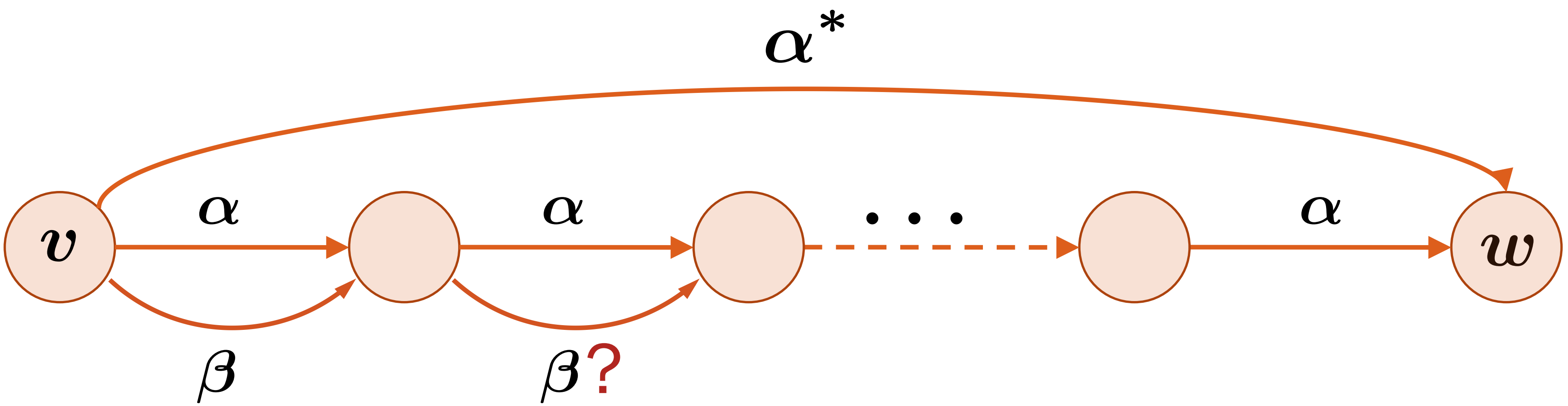

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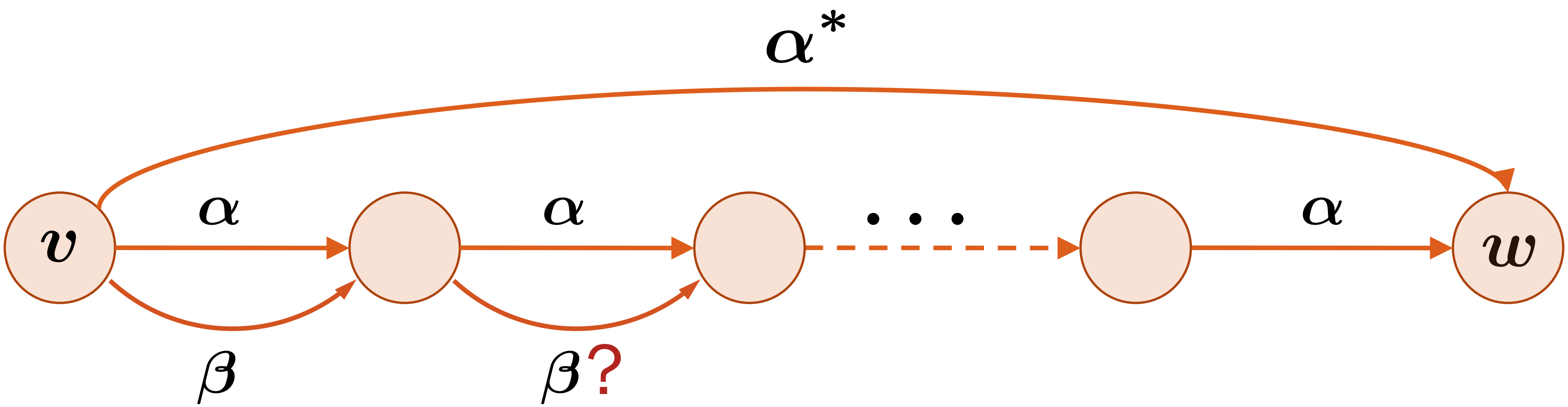

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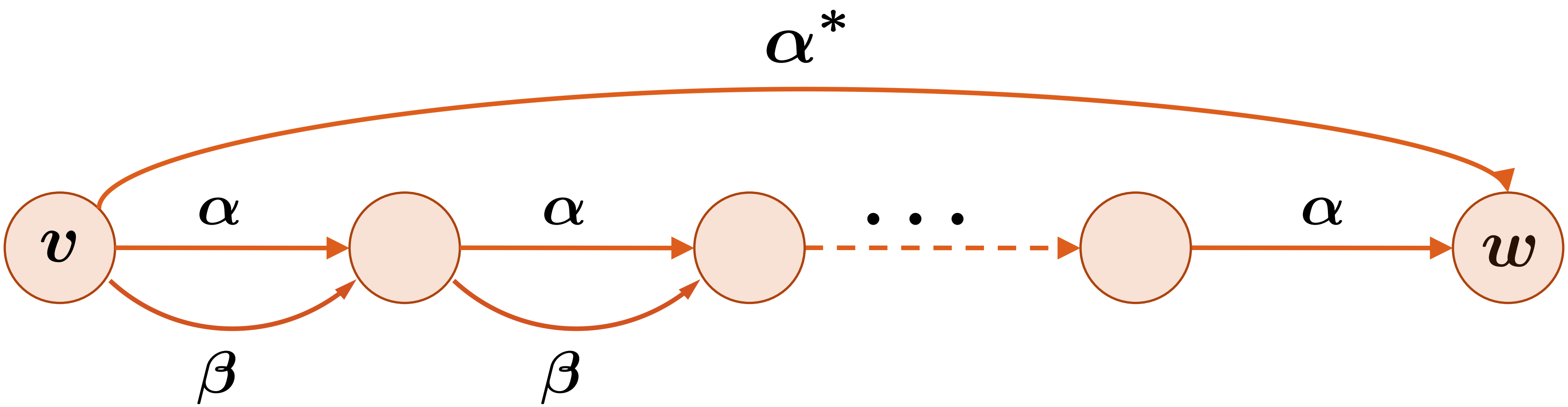

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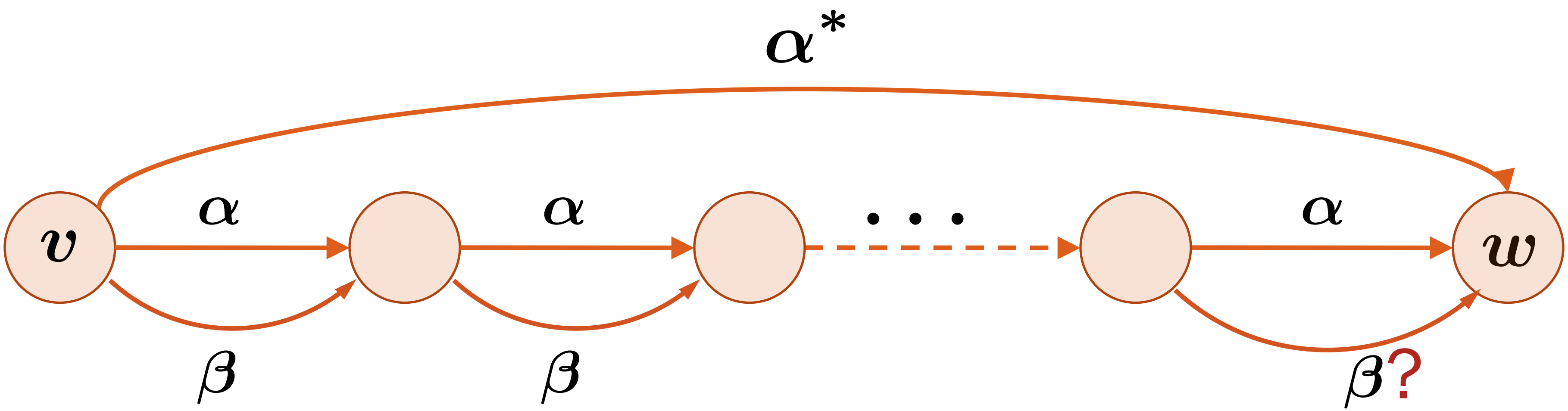

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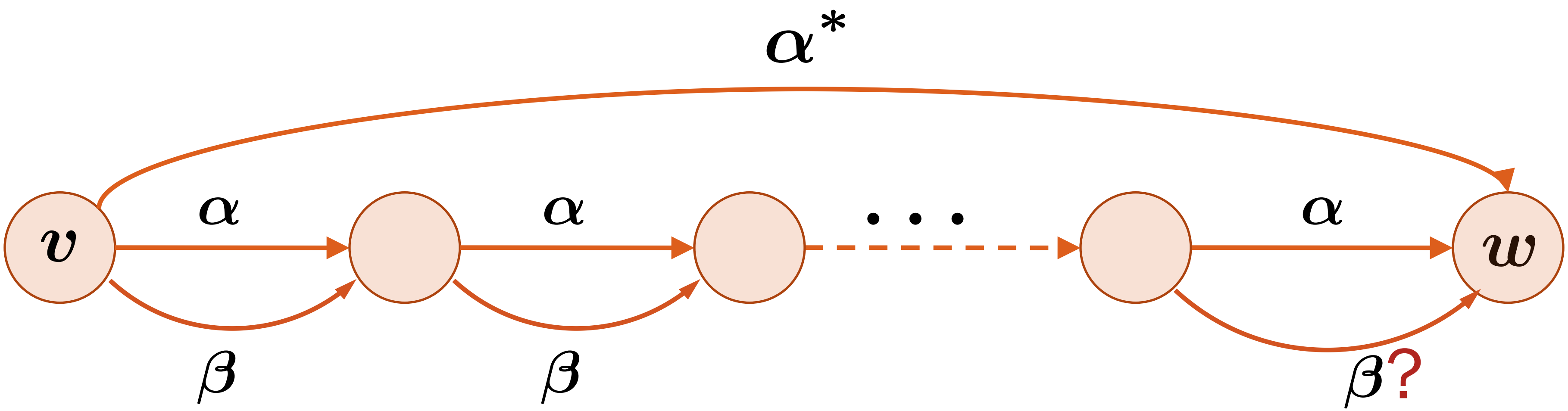

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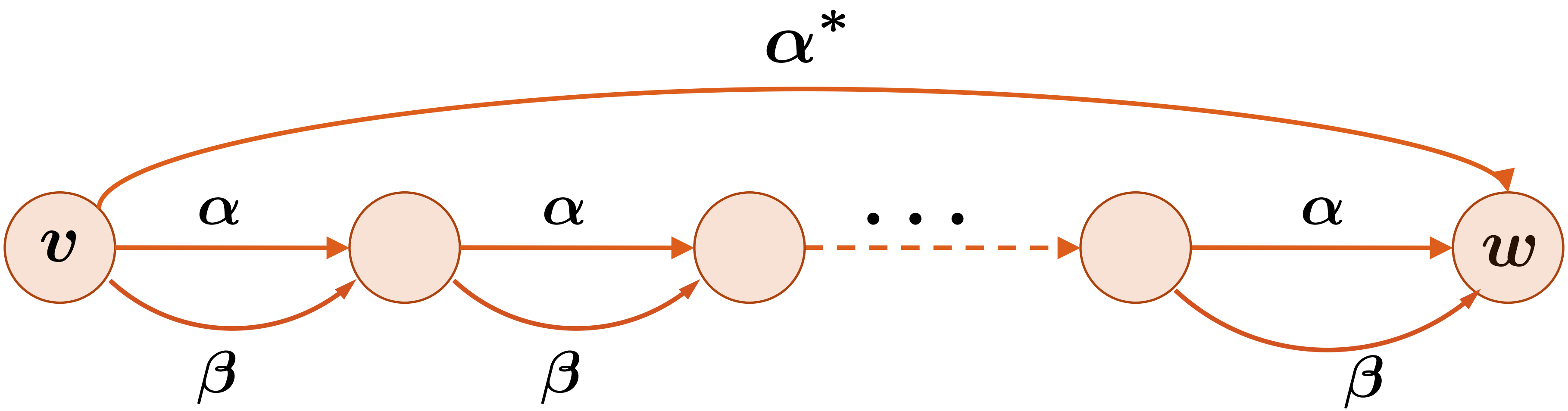

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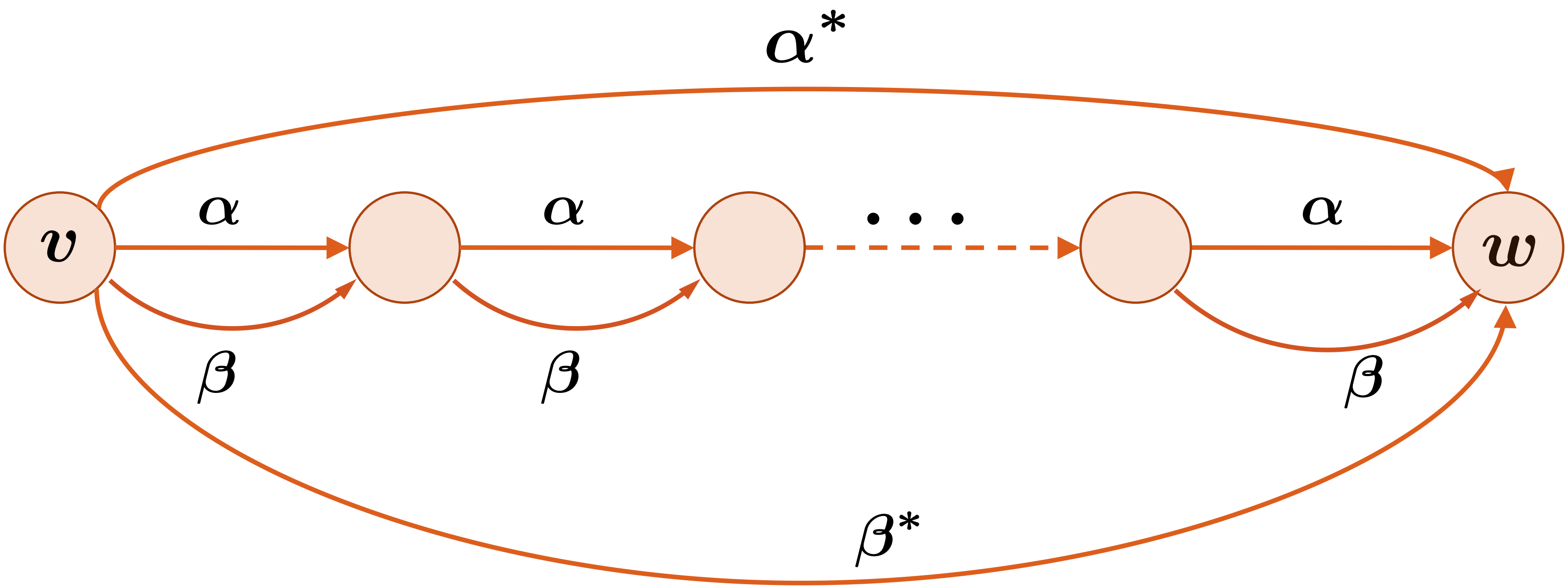

# Nondeterministic Repetition (KAT style)

$$\frac{\Gamma \vdash (\alpha;\gamma) \leq \gamma, \Delta \qquad \Gamma \vdash \beta \leq \gamma, \Delta}{\Gamma \vdash \alpha^*;\beta \leq \gamma, \Delta}(loop_l)$$

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$$v \xrightarrow{\alpha} w_1 \xrightarrow{\cdots} w_2 \xrightarrow{\alpha} w_3 \xrightarrow{\beta} w_4$$

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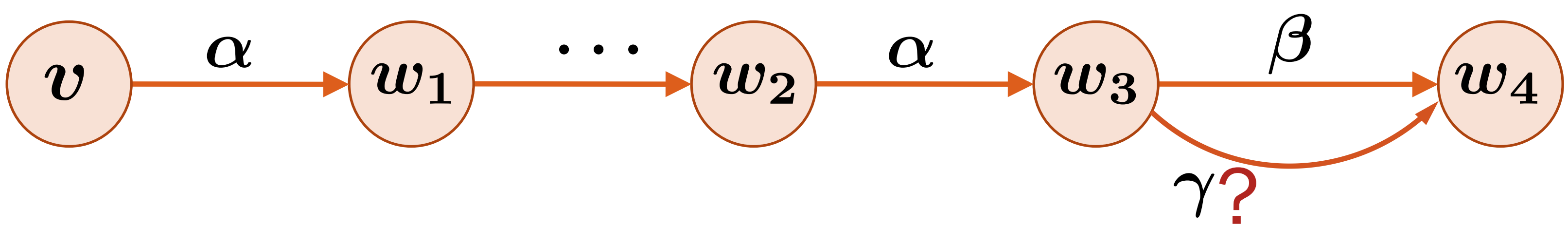

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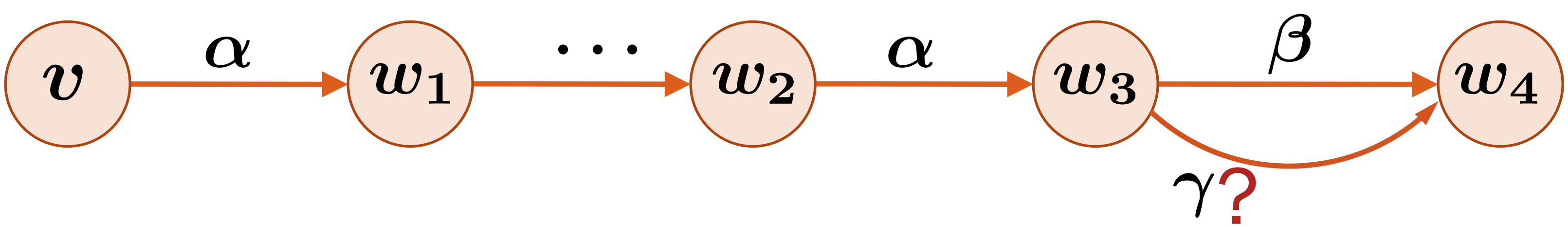

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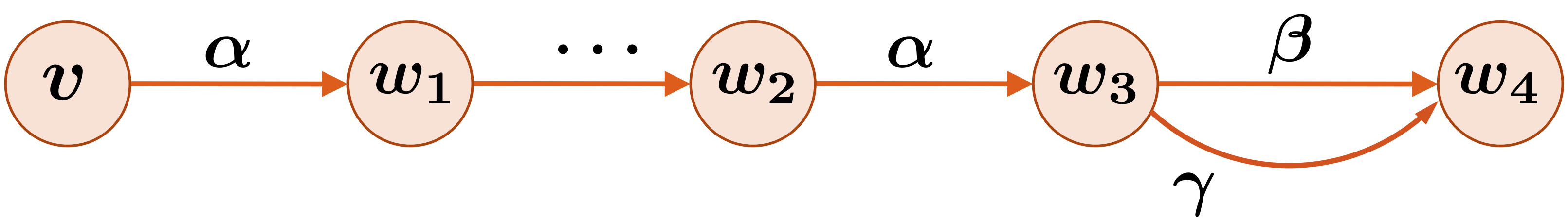

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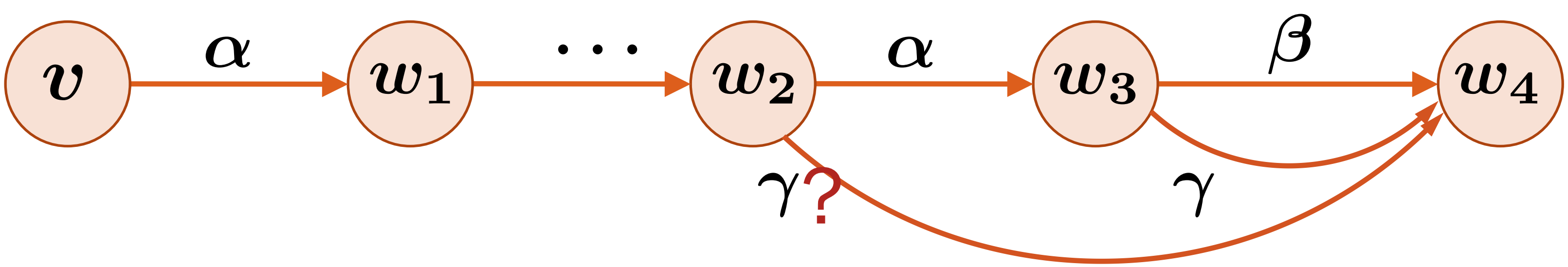

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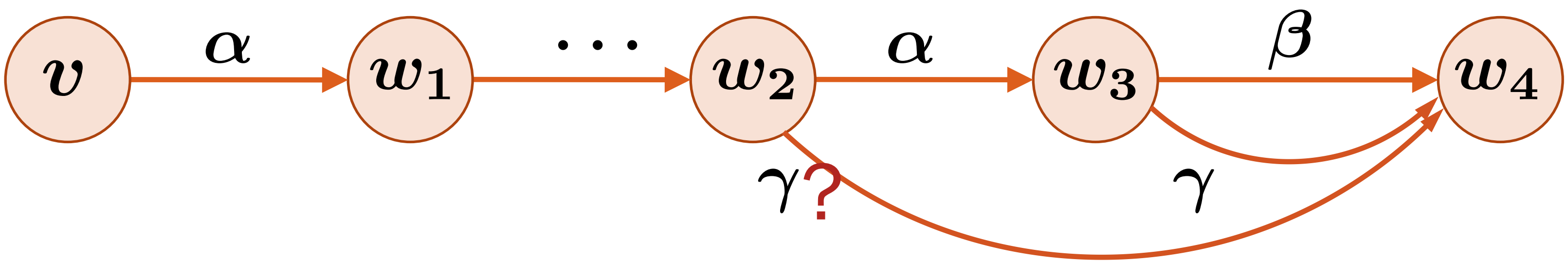

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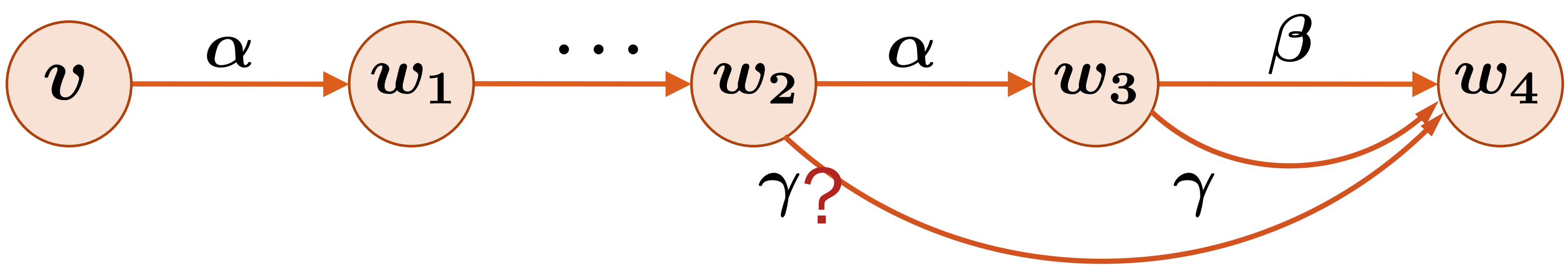

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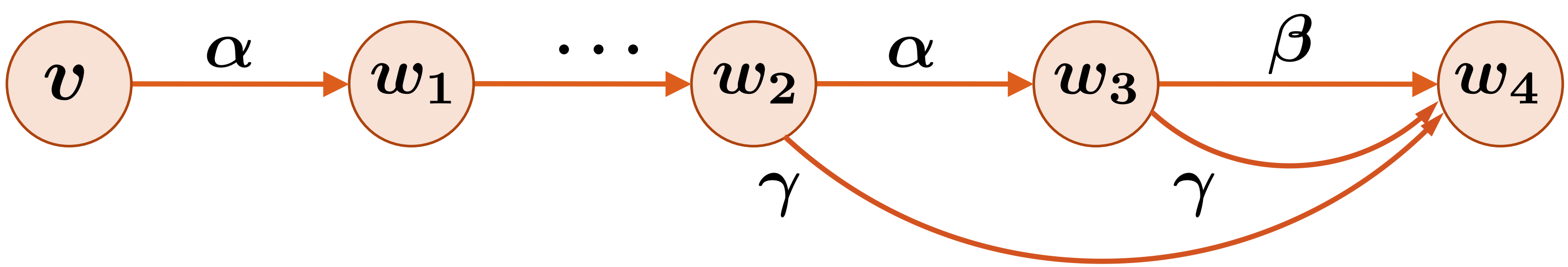
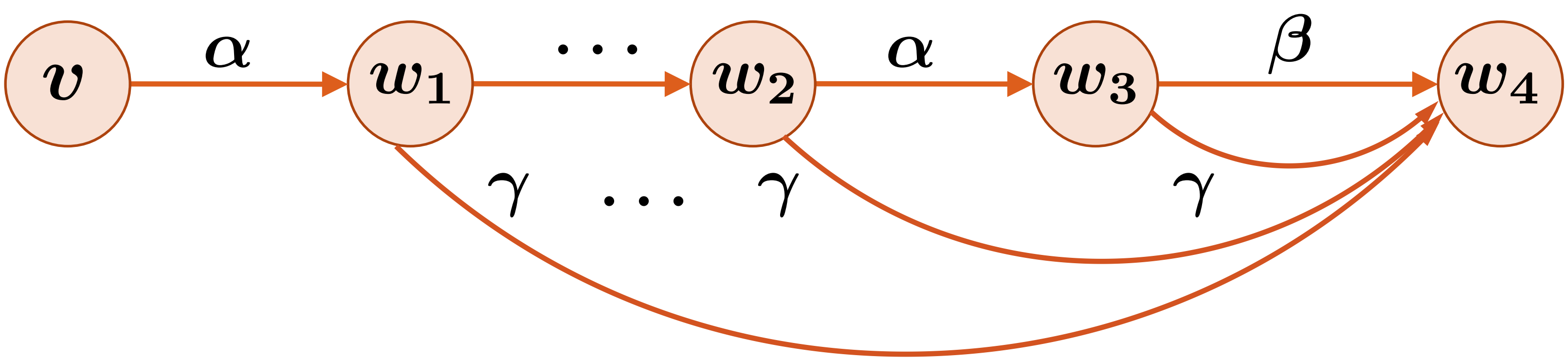


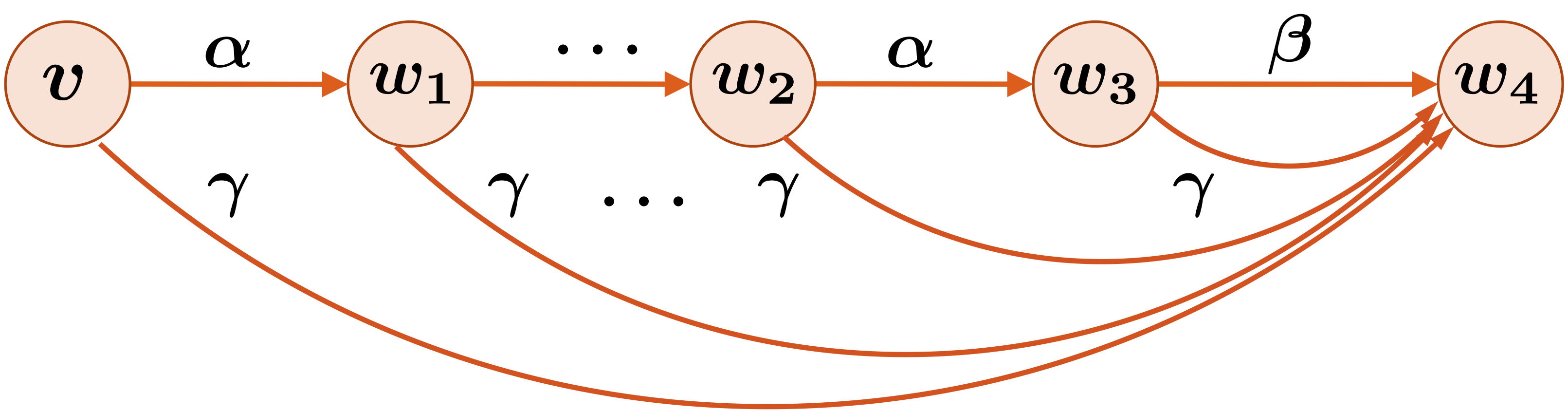
# Nondeterministic Repetition (KAT style)

$$\frac{\boxed{\Gamma \vdash [\alpha^*](\alpha;\gamma) \leq \gamma, \Delta} \qquad \Gamma \vdash [\alpha^*]\beta \leq \gamma, \Delta}{\Gamma \vdash \alpha^*;\beta \leq \gamma, \Delta} (loop_l)$$



# Nondeterministic Repetition (KAT style)

$$\frac{\boxed{\Gamma \vdash [\alpha^*](\alpha;\gamma) \leq \gamma, \Delta} \qquad \Gamma \vdash [\alpha^*]\beta \leq \gamma, \Delta}{\Gamma \vdash \alpha^*;\beta \leq \gamma, \Delta}(loop_l)$$



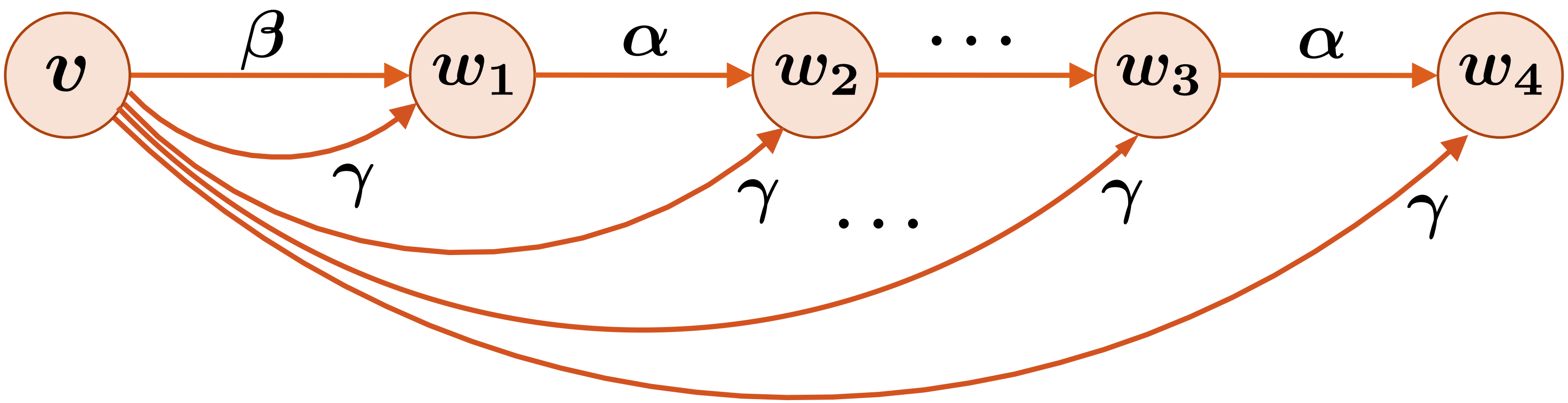
# Nondeterministic Repetition (KAT style)

$$\frac{\Gamma \vdash \beta \leq \gamma, \Delta \qquad \Gamma \vdash (\gamma; \alpha) \leq \gamma, \Delta}{\Gamma \vdash \beta; \alpha^* \leq \gamma, \Delta}(loop_r)$$

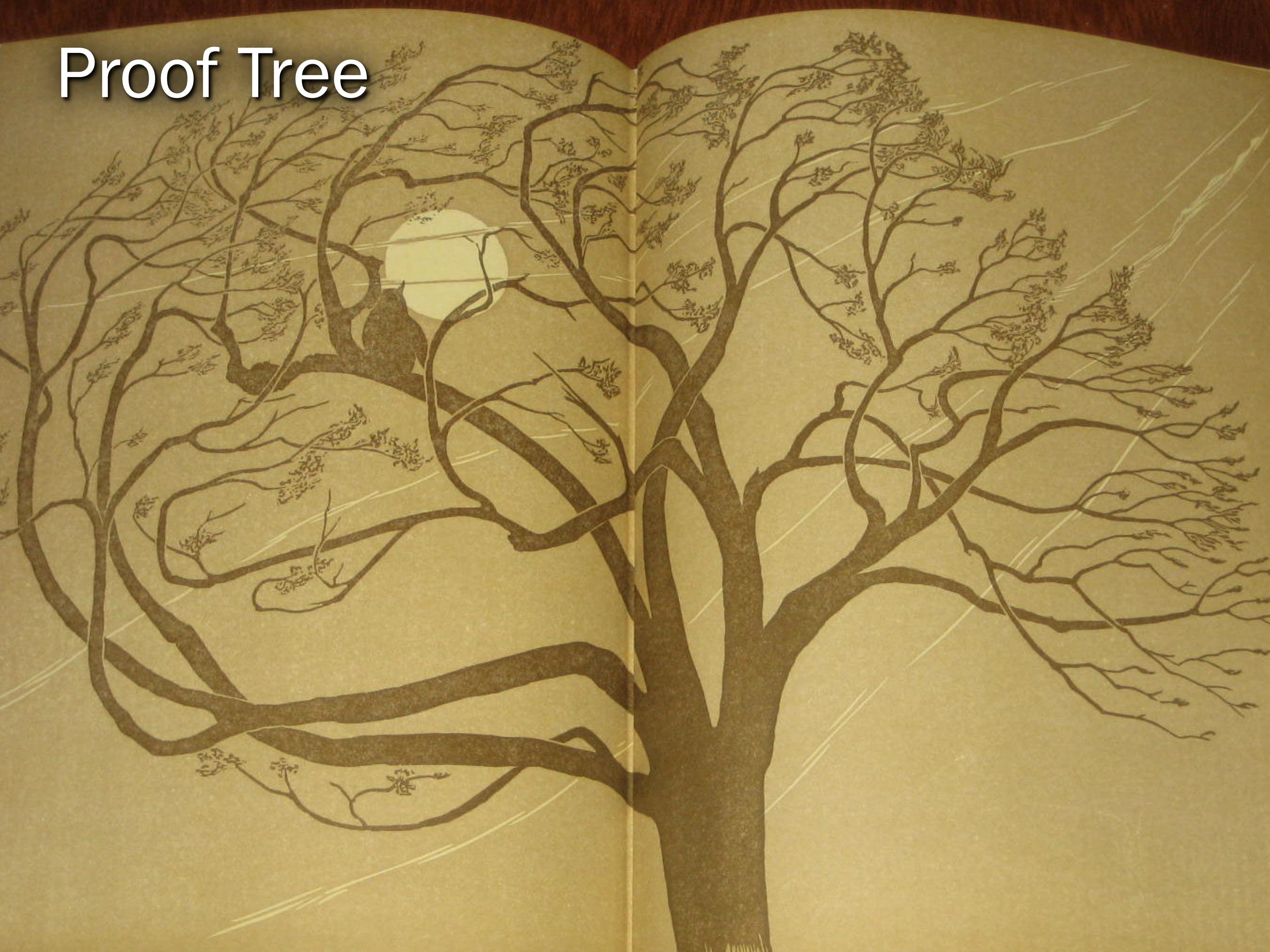
$$\boldsymbol{v} \xrightarrow{\boldsymbol{\beta}} \boldsymbol{w_1} \xrightarrow{\boldsymbol{\alpha}} \boldsymbol{w_2} \xrightarrow{\cdots} \boldsymbol{w_3} \xrightarrow{\boldsymbol{\alpha}} \boldsymbol{w_4}$$

# Nondeterministic Repetition (KAT style)

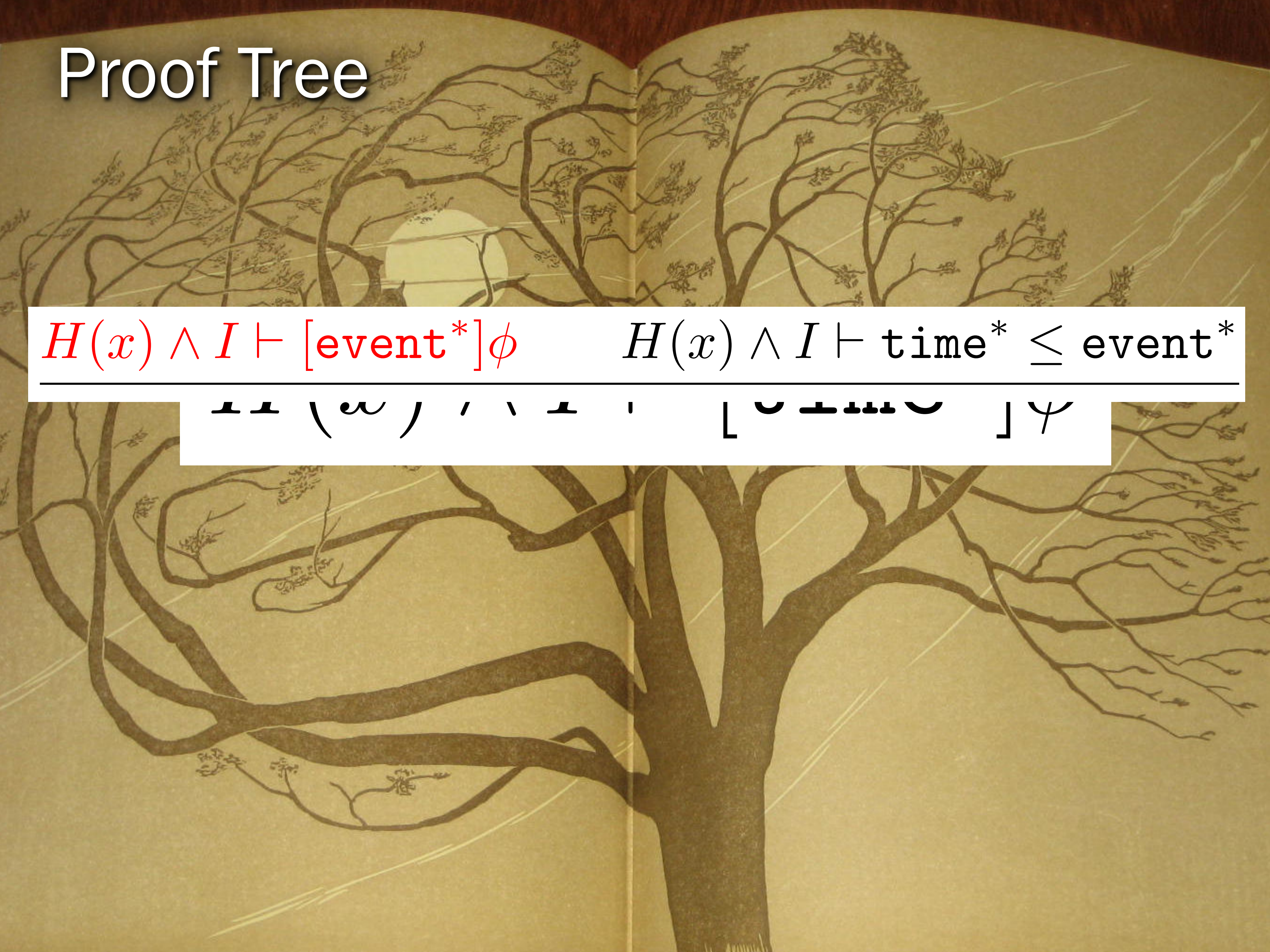
$$\frac{\Gamma \vdash \beta \leq \gamma, \Delta \qquad \Gamma \vdash (\gamma; \alpha) \leq \gamma, \Delta}{\Gamma \vdash \beta; \alpha^* \leq \gamma, \Delta}(loop_r)$$



# Proof Tree

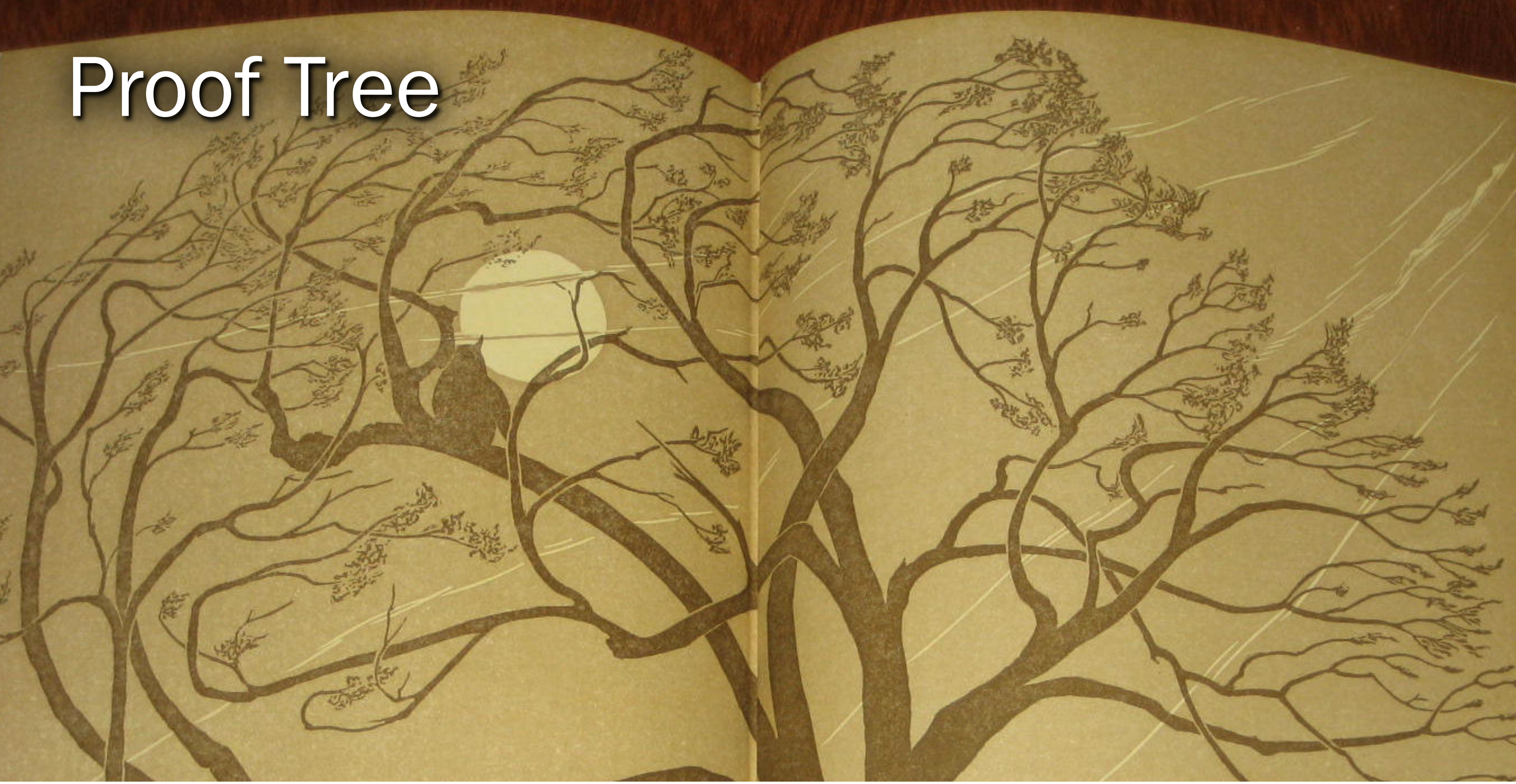


# Proof Tree



$$\frac{\color{red}{H(x) \wedge I \vdash [\mathtt{event}^*]\phi} \qquad H(x) \wedge I \vdash \mathtt{time}^* \leq \mathtt{event}^*}{H(x) \wedge I \vdash [\mathtt{time}^*]\phi}$$

# Proof Tree



$$\frac{H(x) \land I \vdash [a := c](dyn_t \le dyn_{Ev}) \qquad H(x) \land I \vdash [a := *; ?\mathrm{Safe}_\varepsilon(x,a)](dyn_t \le dyn_{Ev})}{H(x) \land I \vdash [a := c \;\cup\; (a := *; ?\mathrm{Safe}_\varepsilon(x,a))](dyn_t \le dyn_{Ev})}\;[\cup]$$

$$\frac{H(x) \land I \vdash [a := c \;\cup\; (a := *; ?\mathrm{Safe}_\varepsilon(x,a))](dyn_t \le dyn_{Ev})}{H(x) \land I \vdash [ctrl_t](dyn_t \le dyn_{Ev})}\;\mathit{subst}$$

$$\frac{H(x) \land I \vdash ctrl_t \le ctrl_{Ev} \qquad H(x) \land I \vdash [ctrl_t](dyn_t \le dyn_{Ev})}{H(x) \land I \vdash ctrl_t; dyn_t \le ctrl_{Ev}; dyn_{Ev}}\;;_A$$

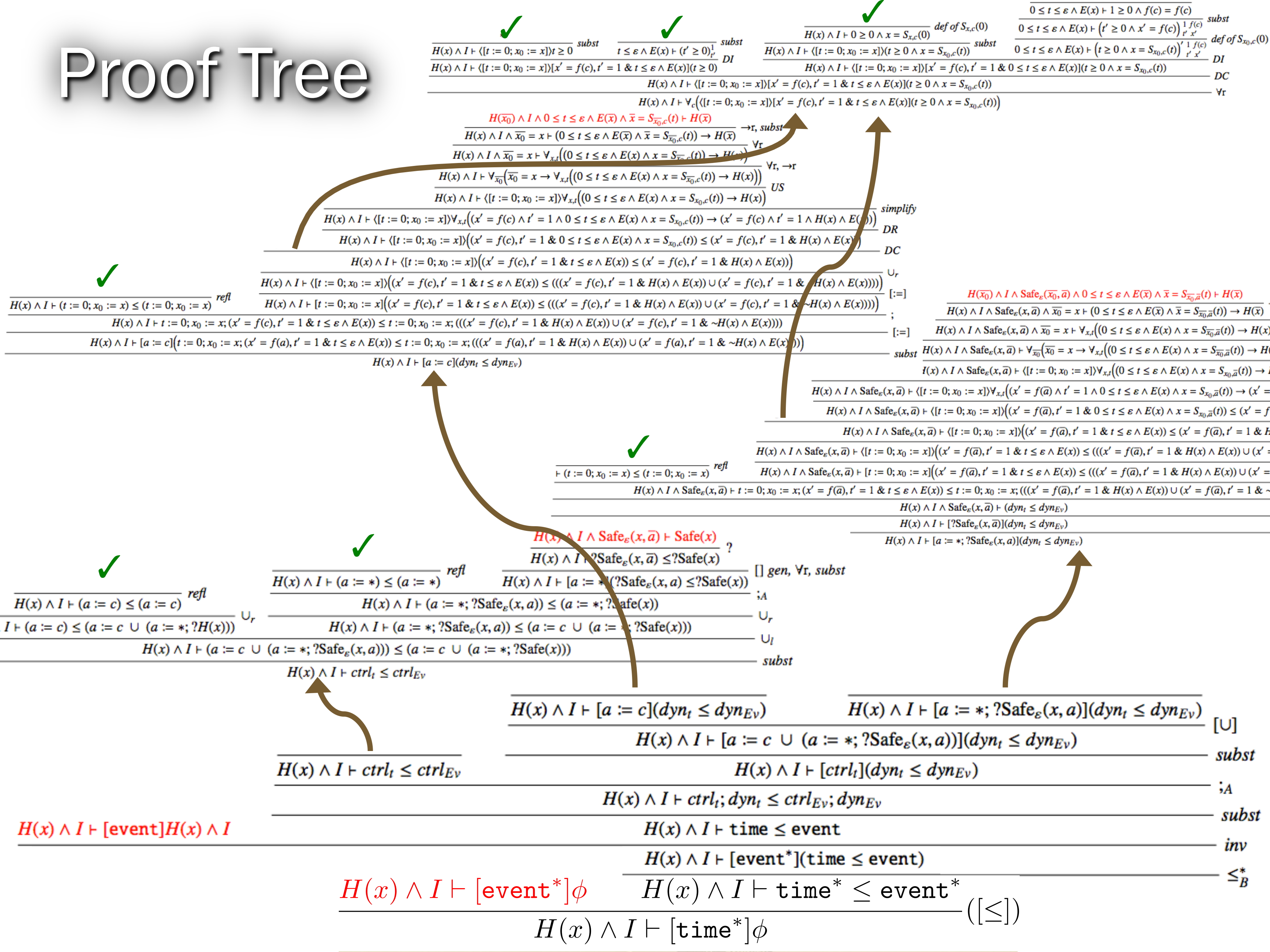
$$\frac{H(x) \land I \vdash ctrl_t; dyn_t \le ctrl_{Ev}; dyn_{Ev}}{H(x) \land I \vdash \mathtt{time} \le \mathtt{event}}\;\mathit{subst}$$

$$\frac{\color{red}{H(x) \land I \vdash [\mathtt{event}]H(x) \land I} \qquad H(x) \land I \vdash \mathtt{time} \le \mathtt{event}}{H(x) \land I \vdash [\mathtt{event}^*](\mathtt{time} \le \mathtt{event})}\;\mathit{inv}$$

$$\frac{H(x) \land I \vdash [\mathtt{event}^*](\mathtt{time} \le \mathtt{event})}{H(x) \land I \vdash \mathtt{time}^* \le \mathtt{event}^*}\;\le_B^*$$

$$\frac{\color{red}{H(x) \land I \vdash [\mathtt{event}^*]\phi} \qquad H(x) \land I \vdash \mathtt{time}^* \le \mathtt{event}^*}{H(x) \land I \vdash [\mathtt{time}^*]\phi}\;([\le])$$

# Proof Tree



H(x) ∧ I ⊢ [a := c](dyn_t ≤ dyn_Ev)
H(x) ∧ I ⊢ [a := ∗; ?Safe_ε(x, a)](dyn_t ≤ dyn_Ev)
[∪]
H(x) ∧ I ⊢ [a := c ∪ (a := ∗; ?Safe_ε(x, a))](dyn_t ≤ dyn_Ev)
subst
H(x) ∧ I ⊢ ctrl_t ≤ ctrl_Ev
H(x) ∧ I ⊢ [ctrl_t](dyn_t ≤ dyn_Ev)
;A
H(x) ∧ I ⊢ ctrl_t; dyn_t ≤ ctrl_Ev; dyn_Ev
subst
H(x) ∧ I ⊢ [event]H(x) ∧ I
H(x) ∧ I ⊢ time ≤ event
inv
H(x) ∧ I ⊢ [event*](time ≤ event)
≤*_B
H(x) ∧ I ⊢ [event*]ϕ
H(x) ∧ I ⊢ time* ≤ event*
([≤])
H(x) ∧ I ⊢ [time*]ϕ

# Proof Tree

# Open goals

$$\dfrac{\dfrac{\checkmark}{H(x)\wedge I \vdash \langle [t:=0;x_0:=x]\rangle t\geq 0}\,\text{subst} \quad \dfrac{\dfrac{\checkmark}{t\leq\varepsilon\wedge E(x)\vdash (t'\geq 0)'^{1}_{t'}}\,\text{subst}}{}\,\text{DI}}{H(x)\wedge I\vdash \langle[t:=0;x_0:=x]\rangle[x'=f(c),t'=1\,\&\,t\leq\varepsilon\wedge E(x)](t\geq 0)}$$

$$\dfrac{\dfrac{\dfrac{\checkmark}{H(x)\wedge I\vdash 0\geq 0\wedge x=S_{x,c}(0)}\,\text{def of }S_{x,c}(0)}{H(x)\wedge I\vdash\langle[t:=0;x_0:=x]\rangle(t\geq 0\wedge x=S_{x_0,c}(t))}\,\text{subst} \quad \dfrac{\dfrac{\dfrac{0\leq t\leq\varepsilon\wedge E(x)\vdash 1\geq 0\wedge f(c)=f(c)}{0\leq t\leq\varepsilon\wedge E(x)\vdash (t'\geq 0\wedge x'=f(c))^{1\,f(c)}_{t'\,x'}}\,\text{subst}}{0\leq t\leq\varepsilon\wedge E(x)\vdash (t\geq 0\wedge x=S_{x_0,c}(t))'^{1\,f(c)}_{t'\,x'}}\,\text{def of }S_{x_0,c}(0)}{}\,\text{DI}}{H(x)\wedge I\vdash\langle[t:=0;x_0:=x]\rangle[x'=f(c),t'=1\,\&\,0\leq t\leq\varepsilon\wedge E(x)](t\geq 0\wedge x=S_{x_0,c}(t))}$$

$$\dfrac{\dfrac{\ldots}{H(x)\wedge I\vdash\langle[t:=0;x_0:=x]\rangle[x'=f(c),t'=1\,\&\,t\leq\varepsilon\wedge E(x)](t\geq 0\wedge x=S_{x_0,c}(t))}\,\text{DC}}{H(x)\wedge I\vdash\forall_c(\langle[t:=0;x_0:=x]\rangle[x'=f(c),t'=1\,\&\,t\leq\varepsilon\wedge E(x)](t\geq 0\wedge x=S_{x_0,c}(t)))}\,\forall\text{r}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{H(\overline{x_0})\wedge I\wedge 0\leq t\leq\varepsilon\wedge E(\overline{x})\wedge\overline{x}=S_{\overline{x_0},c}(t)\vdash H(\overline{x})}{H(x)\wedge I\wedge\overline{x_0}=x\vdash(0\leq t\leq\varepsilon\wedge E(\overline{x})\wedge\overline{x}=S_{\overline{x_0},c}(t))\to H(\overline{x})}\,\to\text{r, subst}}{H(x)\wedge I\wedge\overline{x_0}=x\vdash\forall_{x,t}((0\leq t\leq\varepsilon\wedge E(x)\wedge x=S_{\overline{x_0},c}(t))\to H(x))}\,\forall\text{r}}{H(x)\wedge I\vdash\forall_{\overline{x_0}}(\overline{x_0}=x\to\forall_{x,t}((0\leq t\leq\varepsilon\wedge E(x)\wedge x=S_{\overline{x_0},c}(t))\to H(x)))}\,\forall\text{r},\to\text{r}}{H(x)\wedge I\vdash\langle[t:=0;x_0:=x]\rangle\forall_{x,t}((0\leq t\leq\varepsilon\wedge E(x)\wedge x=S_{x_0,c}(t))\to H(x))}\,\text{US}}{H(x)\wedge I\vdash\langle[t:=0;x_0:=x]\rangle\forall_{x,t}((x'=f(c)\wedge t'=1\wedge 0\leq t\leq\varepsilon\wedge E(x)\wedge x=S_{x_0,c}(t))\to(x'=f(c)\wedge t'=1\wedge H(x)\wedge E(x)))}\,\text{simplify}}{H(x)\wedge I\vdash\langle[t:=0;x_0:=x]\rangle((x'=f(c),t'=1\,\&\,0\leq t\leq\varepsilon\wedge E(x)\wedge x=S_{x_0,c}(t))\leq(x'=f(c),t'=1\,\&\,H(x)\wedge E(x)))}\,\text{DR}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\ldots}{H(x)\wedge I\vdash\langle[t:=0;x_0:=x]\rangle((x'=f(c),t'=1\,\&\,t\leq\varepsilon\wedge E(x))\leq(x'=f(c),t'=1\,\&\,H(x)\wedge E(x)))}\,\text{DC}}{H(x)\wedge I\vdash\langle[t:=0;x_0:=x]\rangle((x'=f(c),t'=1\,\&\,t\leq\varepsilon\wedge E(x))\leq(((x'=f(c),t'=1\,\&\,H(x)\wedge E(x))\cup(x'=f(c),t'=1\,\&\,\sim H(x)\wedge E(x))))}\,\cup_r}{\dfrac{\checkmark}{H(x)\wedge I\vdash(t:=0;x_0:=x)\leq(t:=0;x_0:=x)}\,\text{refl} \quad H(x)\wedge I\vdash[t:=0;x_0:=x]((x'=f(c),t'=1\,\&\,t\leq\varepsilon\wedge E(x))\leq(((x'=f(c),t'=1\,\&\,H(x)\wedge E(x))\cup(x'=f(c),t'=1\,\&\,\sim H(x)\wedge E(x))))}\,[:=]}{H(x)\wedge I\vdash t:=0;x_0:=x;(x'=f(c),t'=1\,\&\,t\leq\varepsilon\wedge E(x))\leq t:=0;x_0:=x;(((x'=f(c),t'=1\,\&\,H(x)\wedge E(x))\cup(x'=f(c),t'=1\,\&\,\sim H(x)\wedge E(x))))}\,;}{H(x)\wedge I\vdash[a:=c](t:=0;x_0:=x;(x'=f(a),t'=1\,\&\,t\leq\varepsilon\wedge E(x))\leq t:=0;x_0:=x;(((x'=f(a),t'=1\,\&\,H(x)\wedge E(x))\cup(x'=f(a),t'=1\,\&\,\sim H(x)\wedge E(x))))}\,[:=]}{H(x)\wedge I\vdash[a:=c](dyn_t\leq dyn_{Ev})}\,\text{subst}$$

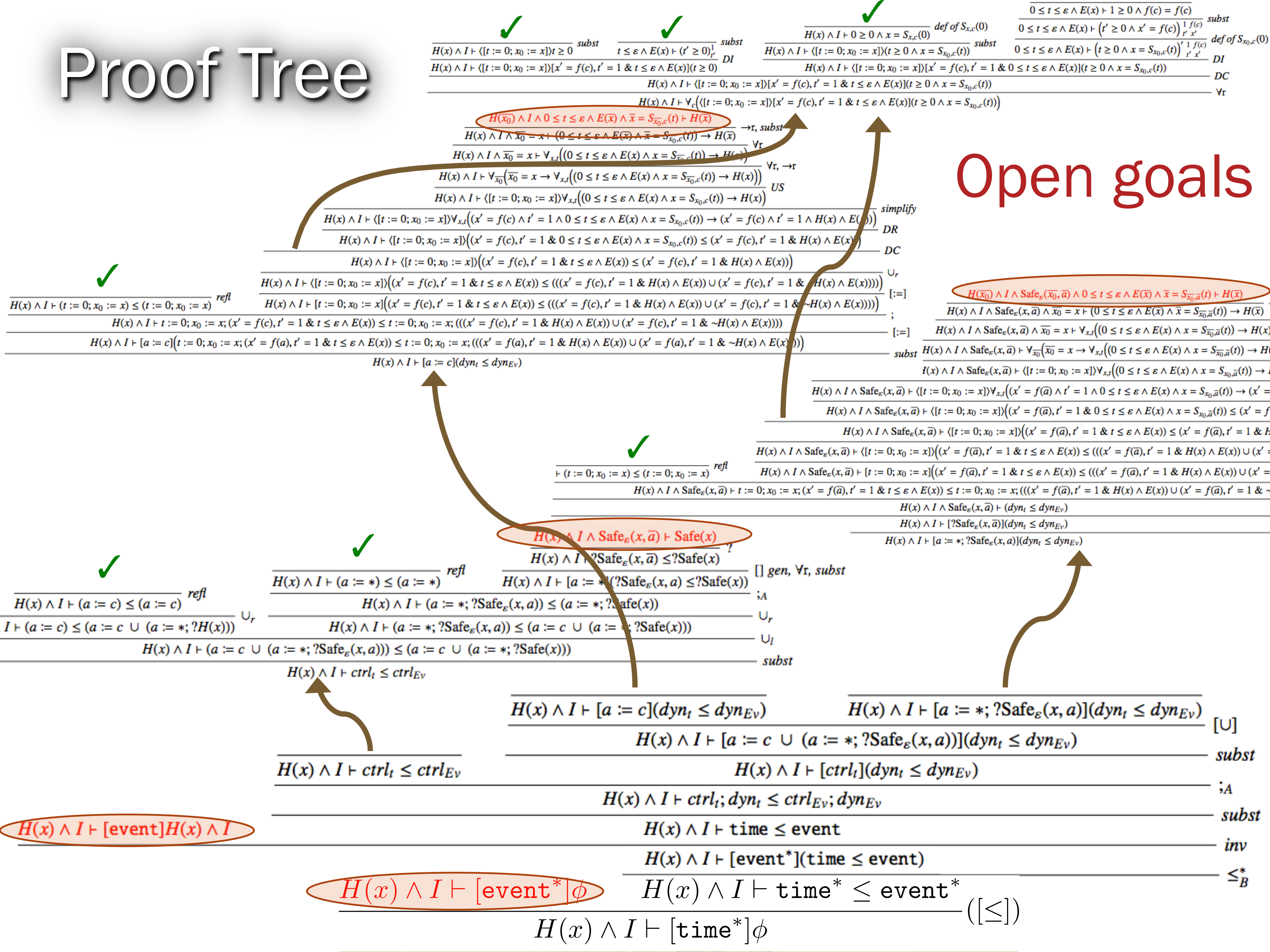
$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{H(\overline{x_0})\wedge I\wedge \text{Safe}_\varepsilon(\overline{x_0},\overline{a})\wedge 0\leq t\leq\varepsilon\wedge E(\overline{x})\wedge\overline{x}=S_{\overline{x_0},\overline{a}}(t)\vdash H(\overline{x})}{H(x)\wedge I\wedge\text{Safe}_\varepsilon(x,\overline{a})\wedge\overline{x_0}=x\vdash(0\leq t\leq\varepsilon\wedge E(\overline{x})\wedge\overline{x}=S_{\overline{x_0},\overline{a}}(t))\to H(\overline{x})}}{H(x)\wedge I\wedge\text{Safe}_\varepsilon(x,\overline{a})\wedge\overline{x_0}=x\vdash\forall_{x,t}((0\leq t\leq\varepsilon\wedge E(x)\wedge x=S_{\overline{x_0},\overline{a}}(t))\to H(x))}}{H(x)\wedge I\wedge\text{Safe}_\varepsilon(x,\overline{a})\vdash\forall_{\overline{x_0}}(\overline{x_0}=x\to\forall_{x,t}((0\leq t\leq\varepsilon\wedge E(x)\wedge x=S_{\overline{x_0},\overline{a}}(t))\to H(x)))}}{H(x)\wedge I\wedge\text{Safe}_\varepsilon(x,\overline{a})\vdash\langle[t:=0;x_0:=x]\rangle\forall_{x,t}((0\leq t\leq\varepsilon\wedge E(x)\wedge x=S_{x_0,\overline{a}}(t))\to\ldots}}{H(x)\wedge I\wedge\text{Safe}_\varepsilon(x,\overline{a})\vdash\langle[t:=0;x_0:=x]\rangle\forall_{x,t}((x'=f(\overline{a})\wedge t'=1\wedge 0\leq t\leq\varepsilon\wedge E(x)\wedge x=S_{x_0,\overline{a}}(t))\to(x'=\ldots}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\ldots}{H(x)\wedge I\wedge\text{Safe}_\varepsilon(x,\overline{a})\vdash\langle[t:=0;x_0:=x]\rangle((x'=f(\overline{a}),t'=1\,\&\,0\leq t\leq\varepsilon\wedge E(x)\wedge x=S_{x_0,\overline{a}}(t))\leq(x'=f\ldots}}{H(x)\wedge I\wedge\text{Safe}_\varepsilon(x,\overline{a})\vdash\langle[t:=0;x_0:=x]\rangle((x'=f(\overline{a}),t'=1\,\&\,t\leq\varepsilon\wedge E(x))\leq(x'=f(\overline{a}),t'=1\,\&\,H\ldots}}{H(x)\wedge I\wedge\text{Safe}_\varepsilon(x,\overline{a})\vdash\langle[t:=0;x_0:=x]\rangle((x'=f(\overline{a}),t'=1\,\&\,t\leq\varepsilon\wedge E(x))\leq(((x'=f(\overline{a}),t'=1\,\&\,H(x)\wedge E(x))\cup(x'\ldots}}{\dfrac{\checkmark}{\vdash(t:=0;x_0:=x)\leq(t:=0;x_0:=x)}\,\text{refl}\quad H(x)\wedge I\wedge\text{Safe}_\varepsilon(x,\overline{a})\vdash[t:=0;x_0:=x]((x'=f(\overline{a}),t'=1\,\&\,t\leq\varepsilon\wedge E(x))\leq(((x'=f(\overline{a}),t'=1\,\&\,H(x)\wedge E(x))\cup(x'\ldots}}{H(x)\wedge I\wedge\text{Safe}_\varepsilon(x,\overline{a})\vdash t:=0;x_0:=x;(x'=f(\overline{a}),t'=1\,\&\,t\leq\varepsilon\wedge E(x))\leq t:=0;x_0:=x;(((x'=f(\overline{a}),t'=1\,\&\,H(x)\wedge E(x))\cup(x'=f(\overline{a}),t'=1\,\&\,\sim\ldots}}{H(x)\wedge I\wedge\text{Safe}_\varepsilon(x,\overline{a})\vdash(dyn_t\leq dyn_{Ev})}}{H(x)\wedge I\vdash[?\text{Safe}_\varepsilon(x,\overline{a})](dyn_t\leq dyn_{Ev})}}{H(x)\wedge I\vdash[a:=*;?\text{Safe}_\varepsilon(x,a)](dyn_t\leq dyn_{Ev})}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\checkmark}{H(x)\wedge I\vdash(a:=c)\leq(a:=c)}\,\text{refl}}{I\vdash(a:=c)\leq(a:=c\,\cup\,(a:=*;?H(x)))}\,\cup_r \quad \dfrac{\dfrac{\dfrac{\checkmark}{H(x)\wedge I\vdash(a:=*)\leq(a:=*)}\,\text{refl}\quad\dfrac{\dfrac{H(x)\wedge I\wedge\text{Safe}_\varepsilon(x,\overline{a})\vdash\text{Safe}(x)}{H(x)\wedge I\vdash ?\text{Safe}_\varepsilon(x,\overline{a})\leq ?\text{Safe}(x)}\,?}{H(x)\wedge I\vdash[a:=*](?\text{Safe}_\varepsilon(x,a)\leq ?\text{Safe}(x))}\,[]\text{ gen},\forall\text{r, subst}}{H(x)\wedge I\vdash(a:=*;?\text{Safe}_\varepsilon(x,a))\leq(a:=*;?\text{Safe}(x))}\,;A}{H(x)\wedge I\vdash(a:=*;?\text{Safe}_\varepsilon(x,a))\leq(a:=c\,\cup\,(a:=*;?\text{Safe}(x)))}\,\cup_r}{H(x)\wedge I\vdash(a:=c\,\cup\,(a:=*;?\text{Safe}_\varepsilon(x,a)))\leq(a:=c\,\cup\,(a:=*;?\text{Safe}(x)))}\,\cup_l}{H(x)\wedge I\vdash ctrl_t\leq ctrl_{Ev}}\,\text{subst}$$

$$\dfrac{H(x)\wedge I\vdash[\texttt{event}^*]\phi \qquad \dfrac{\dfrac{H(x)\wedge I\vdash[\texttt{event}]H(x)\wedge I \qquad \dfrac{\dfrac{\dfrac{H(x)\wedge I\vdash ctrl_t\leq ctrl_{Ev}\qquad\dfrac{\dfrac{H(x)\wedge I\vdash[a:=c](dyn_t\leq dyn_{Ev})\qquad H(x)\wedge I\vdash[a:=*;?\text{Safe}_\varepsilon(x,a)](dyn_t\leq dyn_{Ev})}{H(x)\wedge I\vdash[a:=c\,\cup\,(a:=*;?\text{Safe}_\varepsilon(x,a))](dyn_t\leq dyn_{Ev})}\,[\cup]}{H(x)\wedge I\vdash[ctrl_t](dyn_t\leq dyn_{Ev})}\,\text{subst}}{H(x)\wedge I\vdash ctrl_t;dyn_t\leq ctrl_{Ev};dyn_{Ev}}\,;A}{H(x)\wedge I\vdash\texttt{time}\leq\texttt{event}}\,\text{subst}}{H(x)\wedge I\vdash[\texttt{event}^*](\texttt{time}\leq\texttt{event})}\,\text{inv}}{H(x)\wedge I\vdash\texttt{time}^*\leq\texttt{event}^*}\,\leq^*_B}{H(x)\wedge I\vdash[\texttt{time}^*]\phi}\,([\leq])$$

# Proof Tree



Open goals
H(x) ∧ I ⊢ [a := c](dyn_t ≤ dyn_Ev)
H(x) ∧ I ⊢ [a := ∗; ?Safe_ε(x, a)](dyn_t ≤ dyn_Ev)
[∪]
H(x) ∧ I ⊢ [a := c ∪ (a := ∗; ?Safe_ε(x, a))](dyn_t ≤ dyn_Ev)
subst
H(x) ∧ I ⊢ ctrl_t ≤ ctrl_Ev
H(x) ∧ I ⊢ [ctrl_t](dyn_t ≤ dyn_Ev)
;A
H(x) ∧ I ⊢ ctrl_t; dyn_t ≤ ctrl_Ev; dyn_Ev
subst
H(x) ∧ I ⊢ [event]H(x) ∧ I
H(x) ∧ I ⊢ time ≤ event
inv
H(x) ∧ I ⊢ [event*](time ≤ event)
≤*_B
H(x) ∧ I ⊢ [event*]φ
H(x) ∧ I ⊢ time* ≤ event*
([≤])
H(x) ∧ I ⊢ [time*]φ
H(x) ∧ I ⊢ ctrl_t ≤ ctrl_Ev
H(x) ∧ I ⊢ (a := c) ≤ (a := c)
refl
H(x) ∧ I ⊢ (a := ∗) ≤ (a := ∗)
refl
H(x) ∧ I ∧ Safe_ε(x, a) ⊢ Safe(x)
H(x) ∧ I ⊢ [a := c](dyn_t ≤ dyn_Ev)
H(x) ∧ I ⊢ [a := ∗; ?Safe_ε(x, a)](dyn_t ≤ dyn_Ev)

# Proof Tree

"Braking" is safe for $\varepsilon$ time

$H(S_c(0)) \land 0 \le t \le \varepsilon \vdash H(S_c(t))$

"Accelerating" is safe for $\varepsilon$ time

$\texttt{Safe}_\varepsilon(S_a(0)) \land 0 \le t \le \varepsilon \vdash H(S_a(t))$

Controllers satisfy refinement

$\vdash \ \texttt{Safe}_\varepsilon \to \texttt{Safe}$

Event-triggered is safe

d$\mathcal{L}$

$H(x) \land I \vdash [\texttt{event}]H(x) \land I$

---

Time-triggered is safe

$H(x) \land I \vdash [\texttt{time}^*]\phi$

# Proof Tree

$\mathrm{FOL}_{\mathbb{R}}$

"Braking" is safe for $\varepsilon$ time

$H(S_c(0)) \land 0 \leq t \leq \varepsilon \vdash H(S_c(t))$

$\mathrm{FOL}_{\mathbb{R}}$

"Accelerating" is safe for $\varepsilon$ time

$\mathtt{Safe}_\varepsilon(S_a(0)) \land 0 \leq t \leq \varepsilon \vdash H(S_a(t))$

Controllers satisfy refinement

$\vdash \ \mathtt{Safe}_\varepsilon \to \mathtt{Safe}$

$\mathrm{FOL}_{\mathbb{R}}$

Event-triggered is safe

$H(x) \land I \vdash [\mathtt{event}]H(x) \land I$

$\mathrm{d}\mathcal{L}$

---

Time-triggered is safe

$H(x) \land I \vdash [\mathtt{time}^*]\phi$

# dRL Proof Rules: Partial Order

Reflexive:

$$\frac{}{\Gamma \vdash \alpha \leq \alpha, \Delta}(\leq_{refl})$$

Transitive:

$$\frac{\Gamma \vdash \alpha \leq \beta, \Delta \qquad \Gamma \vdash \beta \leq \gamma, \Delta}{\Gamma \vdash \alpha \leq \gamma, \Delta}(\leq_{trans})$$

Antisymmetric:

$$\frac{\Gamma \vdash \alpha \leq \beta, \Delta \qquad \Gamma \vdash \beta \leq \alpha, \Delta}{\Gamma \vdash \alpha = \beta, \Delta}(\leq_{antisym})^1$$

# dRL Proof Rules: KAT

$$\frac{}{\Gamma \vdash \alpha \cup (\beta \cup \gamma) = (\alpha \cup \beta) \cup \gamma, \Delta}(\cup_{assoc}) \qquad \frac{}{\Gamma \vdash \alpha \cup \beta = \beta \cup \alpha, \Delta}(\cup_{comm})$$

$$\frac{}{\Gamma \vdash \alpha \cup ?\bot = \alpha, \Delta}(\cup_{id}) \qquad \frac{}{\Gamma \vdash (\alpha \cup \alpha) = \alpha, \Delta}(\cup_{idemp})$$

$$\frac{}{\Gamma \vdash \alpha; (\beta; \gamma) = (\alpha; \beta); \gamma, \Delta}(;_{assoc}) \qquad \frac{}{\Gamma \vdash (?\top; \alpha) = \alpha, \Delta}(;_{id-l}) \qquad \frac{}{\Gamma \vdash (\alpha; ?\top) = \alpha, \Delta}(;_{id-r})$$

$$\frac{}{\Gamma \vdash \alpha; (\beta \cup \gamma) = ((\alpha; \beta) \cup (\alpha; \gamma)), \Delta}(dist\text{-}l) \qquad \frac{}{\Gamma \vdash (\alpha \cup \beta); \gamma = ((\alpha; \gamma) \cup (\beta; \gamma)), \Delta}(dist - r)$$

$$\frac{}{\Gamma \vdash (\alpha; ?\bot) = ?\bot, \Delta}(;_{annih-r}) \qquad \frac{}{\Gamma \vdash (?\bot; \alpha) = ?\bot, \Delta}(;_{annih-l})$$

$$\frac{}{\Gamma \vdash (?\top \cup (\alpha; \alpha^*)) = \alpha^*, \Delta}(unroll_l) \qquad \frac{}{\Gamma \vdash (?\top \cup (\alpha^*; \alpha)) = \alpha^*, \Delta}(unroll_r)$$

$$\frac{\Gamma \vdash [\alpha^*](\alpha; \gamma) \le \gamma, \Delta \qquad \Gamma \vdash [\alpha^*]\beta \le \gamma, \Delta}{\Gamma \vdash \alpha^*; \beta \le \gamma, \Delta}(loop_l) \qquad \frac{\Gamma \vdash \beta \le \gamma, \Delta \qquad \Gamma \vdash (\gamma; \alpha) \le \gamma, \Delta}{\Gamma \vdash \beta; \alpha^* \le \gamma, \Delta}(loop_r)$$

# dRL Proof Rules: Differential Equations

$$\frac{\Gamma \vdash [x' = \theta \,\&\, H_1]H_2, \Delta}{\Gamma \vdash (x' = \theta \,\&\, H_1) = (x' = \theta \,\&\, H_1 \wedge H_2), \Delta}(DC)$$

$$\frac{\Gamma \vdash \forall x\, (H_1 \rightarrow H_2), \Delta}{\Gamma \vdash (x' = \theta \,\&\, H_1) \leq (x' = \theta \,\&\, H_2), \Delta}(DR)$$

$$\frac{\Gamma \vdash \forall x \Big(\theta_1 ||\theta_2|| = \theta_2 ||\theta_1|| \wedge (||\theta_1||^2 = 0 \leftrightarrow ||\theta_2||^2 = 0)\Big), \Delta}{\Gamma \vdash (x' = \theta_1) = (x' = \theta_2), \Delta} \,(\textit{match direction field})^2$$

$$\frac{\Gamma \vdash \forall x \left(\frac{\theta_1}{||\theta_1||} = \frac{\theta_2}{||\theta_2||} \wedge (||\theta_1|| = 0 \leftrightarrow ||\theta_2|| = 0)\right), \Delta}{\Gamma \vdash (x' = \theta_1) = (x' = \theta_2), \Delta} \,(\textit{mdf})^2$$

# dRL Proof Rules: Structural

$$\frac{\Gamma \vdash \alpha \leq \gamma \wedge \beta \leq \gamma, \Delta}{\Gamma \vdash \alpha \cup \beta \leq \gamma, \Delta}(\cup_l)$$

$$\frac{\Gamma \vdash \alpha \leq \beta \vee \alpha \leq \gamma, \Delta}{\Gamma \vdash \alpha \leq \beta \cup \gamma, \Delta}(\cup_r)$$

$$\frac{\Gamma \vdash [\alpha^*](\alpha \leq \beta), \Delta}{\Gamma \vdash \alpha^* \leq \beta^*, \Delta}(unloop)$$

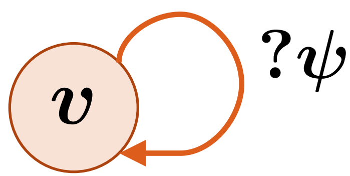
$$\frac{\Gamma \vdash \alpha_1 \leq \alpha_2, \Delta \qquad \Gamma \vdash [\alpha_1]\,(\beta_1 \leq \beta_2), \Delta}{\Gamma \vdash (\alpha_1;\beta_1) \leq (\alpha_2;\beta_2), \Delta}(\,;\,)$$

$$\frac{}{\Gamma \vdash (x := \theta) \leq (x := *), \Delta}(:= *)$$

$$\frac{\Gamma \vdash \phi \rightarrow \psi, \Delta}{\Gamma \vdash ?\phi \leq ?\psi, \Delta}(?)$$

# Test

$$\frac{\Gamma \vdash \phi \rightarrow \psi, \Delta}{\Gamma \vdash ?\phi \leq ?\psi, \Delta}(?)$$

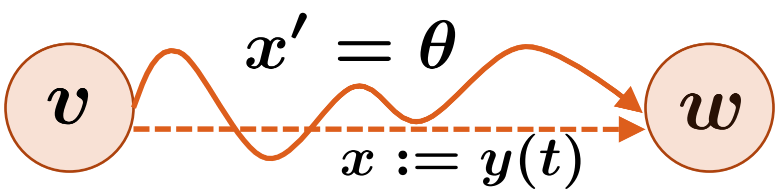



Iff $\psi$ holds in state $v$

$$\rho(?\psi) = \{(v, v) : v \models \psi\}$$

# Differential Refinement

$$\frac{\Gamma \vdash \forall x\,(H_1 \rightarrow H_2), \Delta}{\Gamma \vdash (x' = \theta \,\&\, H_1) \leq (x' = \theta \,\&\, H_2), \Delta}\,(DR)$$

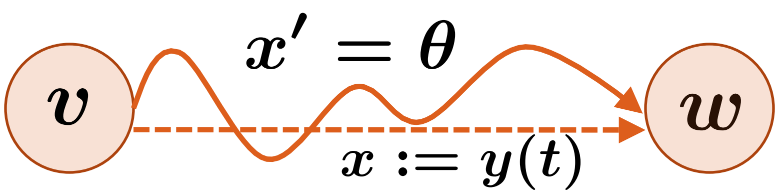



If $y(t)$ solves $x' = \theta$

$$\rho(x' = \theta) = \{(\varphi(0), \varphi(t)) : \varphi(s) \models x' = \theta \text{ for all } 0 \leq s \leq t\}$$

# dRL Proof Rules: Differential Equations

$$\frac{\Gamma \vdash [x' = \theta \,\&\, H_1]H_2, \Delta}{\Gamma \vdash (x' = \theta \,\&\, H_1) = (x' = \theta \,\&\, H_1 \wedge H_2), \Delta}(DC)$$



# Kleene Algebra with Tests (KAT)

- Kleene algebra with tests is a system for manipulating programs that are equivalent.
- KAT doesn't have continuous dynamics, but we can see that it is still relevant to hybrid programs



[Kozen97]

# Verifying a specific local lane controller

$$
\begin{aligned}
\texttt{llc} &\equiv (ctrl; dyn)^* \\
ctrl &\equiv \ell_{ctrl} \,||\, f_{ctrl}; \\
\ell_{ctrl} &\equiv (a_\ell := *;\ \ ?(-B \le a_\ell \le A)) \\
f_{ctrl} &\equiv \texttt{brake} \cup \texttt{safe}_* \cup \texttt{stopped} \\
\texttt{brake} &\equiv (a_f := *;\ \ ?(-B \le a_f \le -b)) \\
\texttt{safe}_* &\equiv (?\mathbf{Safe}_\varepsilon;\ \ a_f := *;\ \ ?(-B \le a_f \le A)) \\
\texttt{stopped} &\equiv (?(v_f = 0);\ a_f := 0) \\
\mathbf{Safe}_\varepsilon &\equiv x_f + \frac{v_f^2}{2b} + \left(\frac{A}{b} + 1\right)\left(\frac{A}{2}\varepsilon^2 + \varepsilon v_f\right) < x_\ell + \frac{v_\ell^2}{2B} \\
dyn &\equiv (t := 0;\ x_f' = v_f,\ v_f' = a_f,\ x_\ell' = v_\ell,\ v_\ell' = a_\ell, t' = 1 \\
&\qquad \&\ v_f \ge 0 \ \wedge\ v_\ell \ge 0 \ \wedge\ t \le \varepsilon)
\end{aligned}
$$

# Verifying a specific local lane controller

$$
\begin{aligned}
\mathtt{llc}_\theta &\equiv (ctrl_\theta; dyn)^* \\
ctrl_\theta &\equiv \ell_{ctrl} \parallel f_{ctrl_\theta}; \\
\ell_{ctrl} &\equiv (a_\ell := *;\ \ ?(-B \le a_\ell \le A)) \\
f_{ctrl_\theta} &\equiv \mathtt{brake} \cup \mathtt{safe}_\theta \cup \mathtt{stopped} \\
\mathtt{brake} &\equiv (a_f := *;\ \ ?(-B \le a_f \le -b)) \\
\mathtt{safe}_\theta &\equiv a_f := \theta(x_f, x_\ell, v_f, v_\ell) \\
\mathtt{stopped} &\equiv (?(v_f = 0);\ a_f := 0) \\
&\cancel{\mathbf{Safe}_\varepsilon \equiv x_f + \frac{v_f^2}{2b} + \left(\frac{A}{b} + 1\right)\left(\frac{A}{2}\varepsilon^2 + \varepsilon v_f\right) < x_\ell + \frac{v_\ell^2}{2B}} \\
dyn &\equiv (t := 0;\ x_f' = v_f,\ v_f' = a_f,\ x_\ell' = v_\ell,\ v_\ell' = a_\ell, t' = 1 \\
&\quad \&\, v_f \ge 0\ \wedge\ v_\ell \ge 0\ \wedge\ t \le \varepsilon)
\end{aligned}
$$

# Additional dRL applications

- Designing proof search heuristics that exploit refinement to automatically create more hierarchical proof structures.
- Shifting the proof responsibility completely to determining refinement.
- Code synthesis – verifying that refinement relation is satisfied with each transformation step.

# Event-triggered vs. Time-triggered

## Event-triggered

- Continuous sensing
- Unrealistic, hard to implement
- Easier to design controllers
- Easier to verify

## Time-triggered

- Discrete sensing
- Realistic, easy to implement
- Difficult to design controllers
- Challenging to verify

$$\Big(\underbrace{\texttt{ctrl}}_{\text{discrete controller}};\underbrace{\texttt{dyn}}_{\text{continuous dynamics}}\Big)^*$$

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$$(\texttt{ctrl};\ x' = \theta \,\&\, H)^*$$

discrete
controller

?

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discrete
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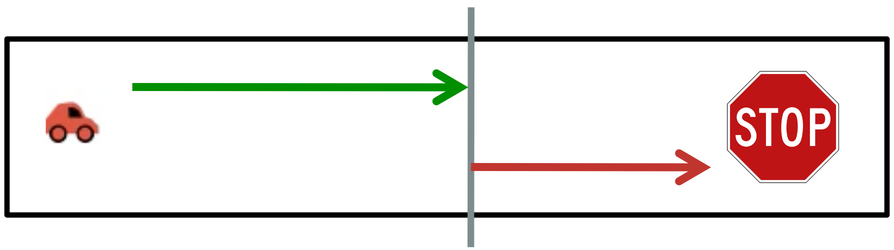


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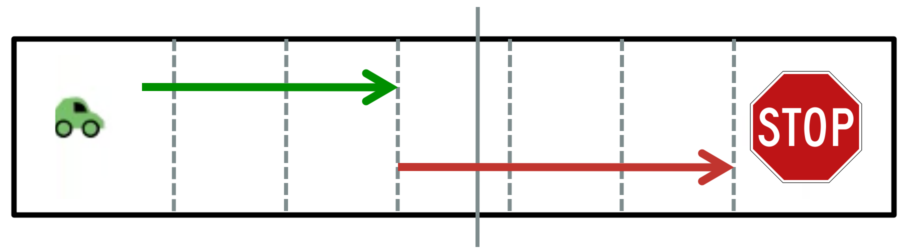


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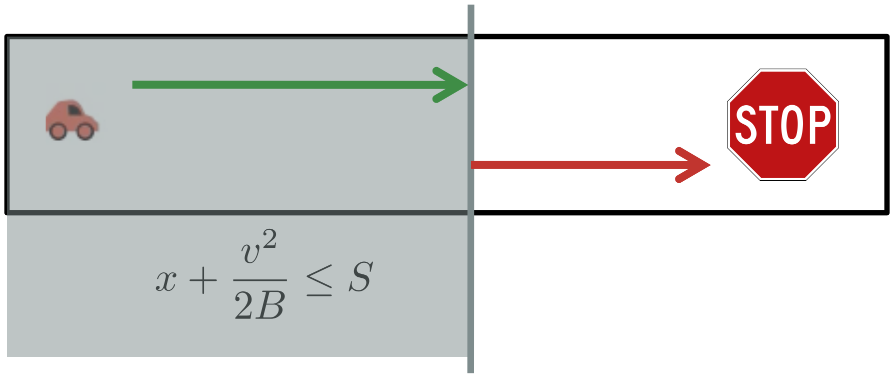


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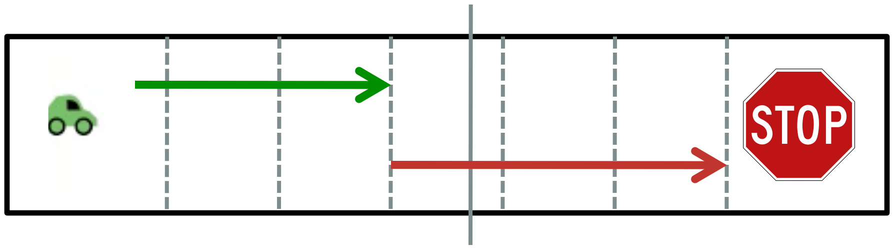


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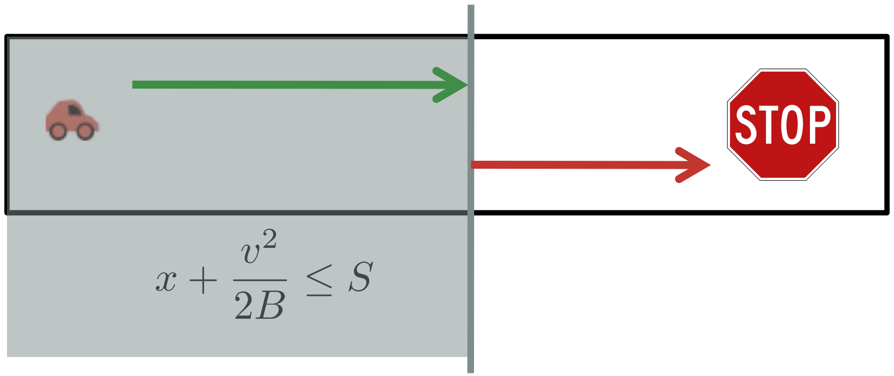


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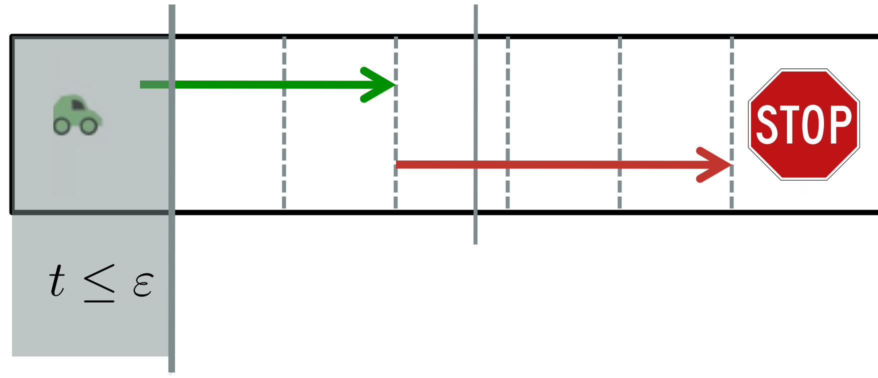


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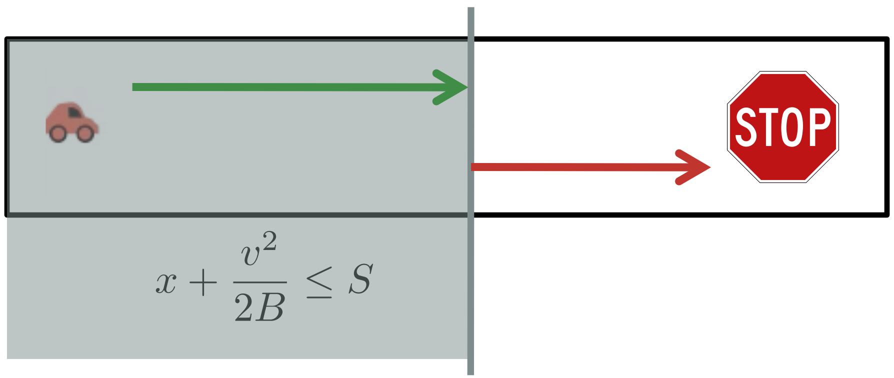


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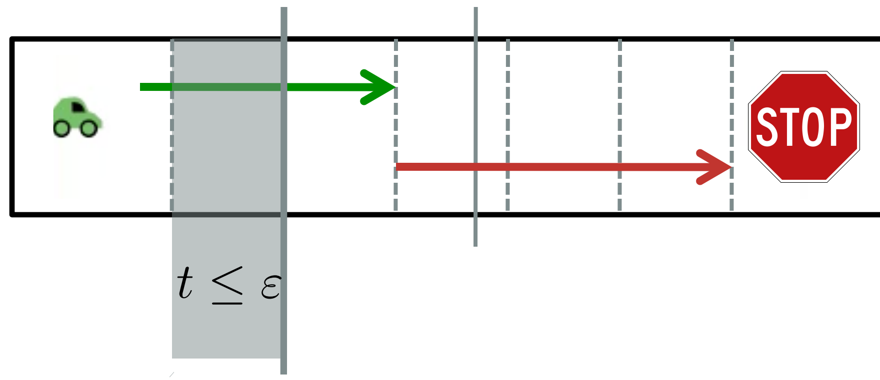


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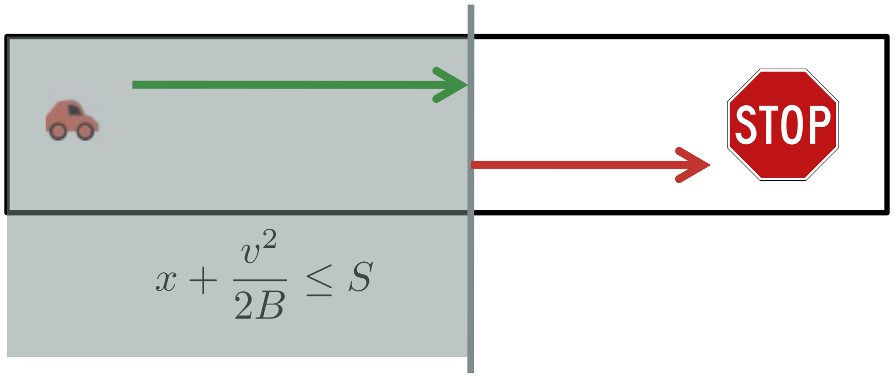


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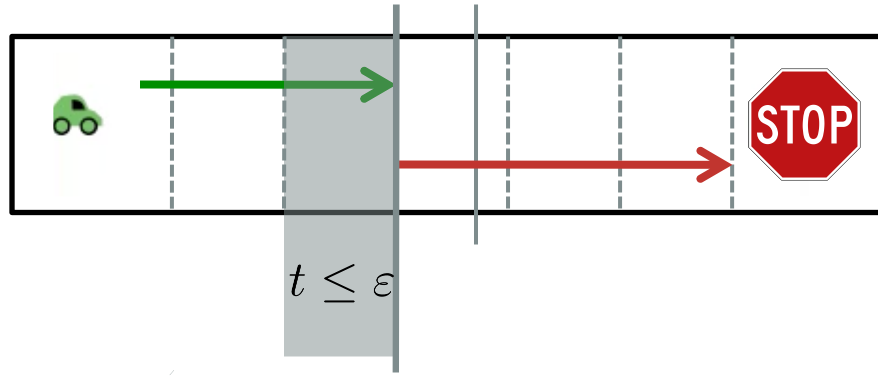



$$(\mathtt{ctrl}_{\mathtt{t}};\, x' = \theta \,\&\, t \leq \varepsilon)^*$$

# Event-triggered vs. Time-triggered

## event-triggered

$$((?\mathtt{Safe}; a := *) \cup a := c;$$
$$x' = \theta \,\&\, E(x))^*$$

## time-triggered

$$((?\mathtt{Safe}_\varepsilon; a := *) \cup a := c;$$
$$x' = \theta \,\&\, t \leq \varepsilon)^*$$

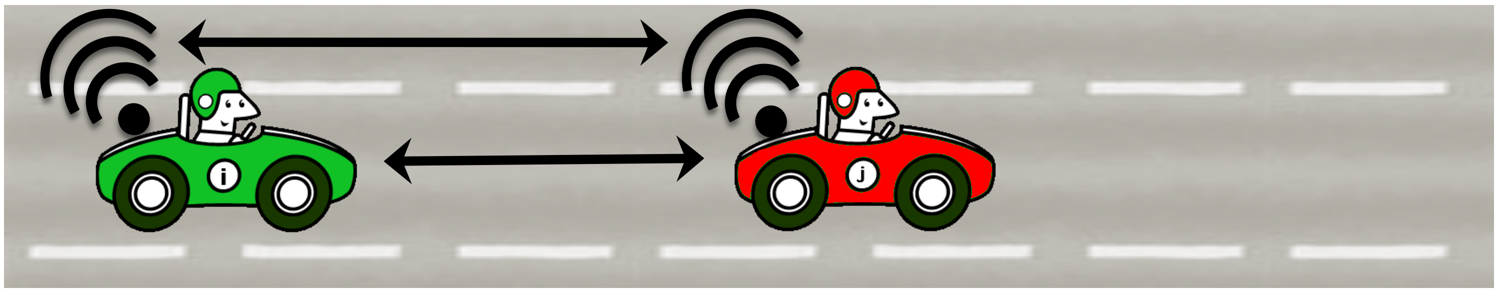
# dRL Proof Rules: Independence

$$\frac{}{\vdash (x := \theta_1; y := \theta_2) = (y := \theta_2; x := \theta_1)}(indep_{:=})$$

$$\frac{}{\vdash (x' = \theta_1; y' = \theta_2) = (y' = \theta_2; x' = \theta_1)}(indep')$$

$$\frac{}{\vdash (x := \theta_1; y' = \theta_2) = (y' = \theta_2; x := \theta_1)}(indep'_{:=})$$

# Motivation: Adaptive Cruise Control



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Low packet loss, small margin for error.

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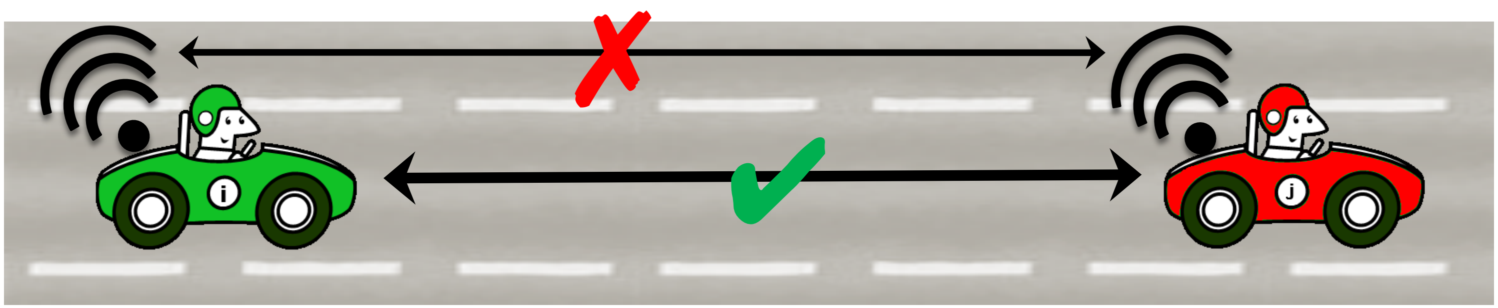
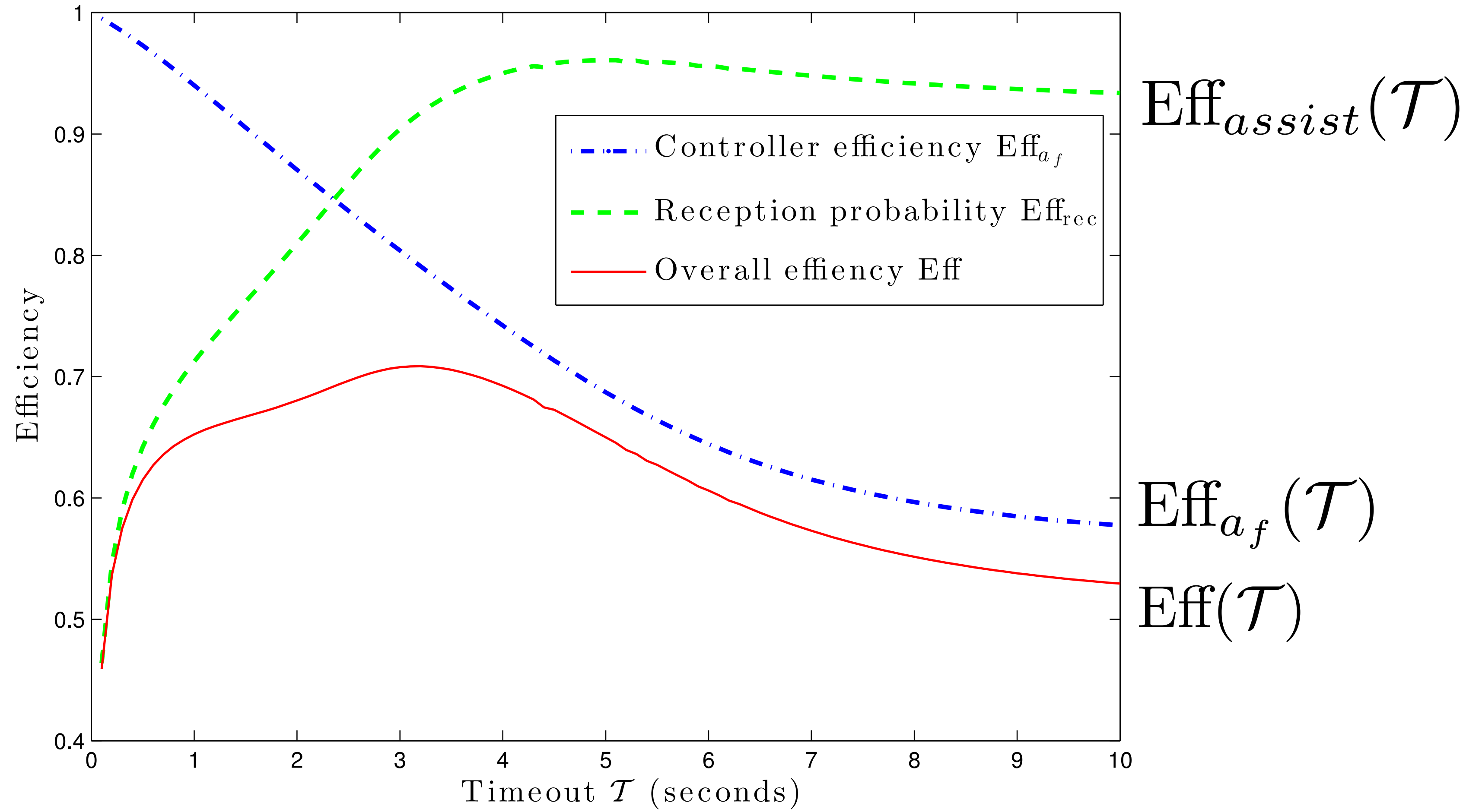



High packet loss, large margin for error.

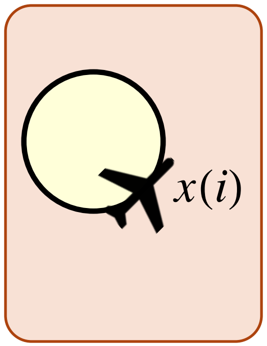
# Efficiency Analysis of ACC



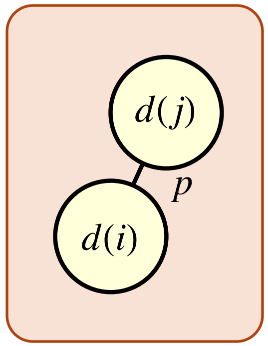
# Modular Proof for Distributed Aircraft

**To Prove:**

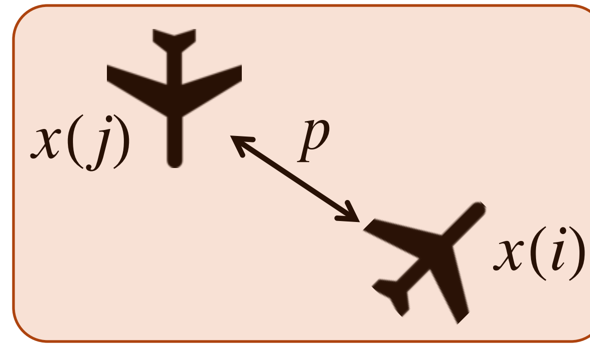
Safe separation of aircraft.




$\wedge$




$\Longrightarrow$




$$\forall i : \mathbb{A} \quad \|x(i) - d(i)\| \leq r \quad \wedge \quad \forall i \neq j : \mathbb{A} \quad \|d(i) - d(j)\| \geq 2r + p \quad \Longrightarrow \quad \forall i \neq j : \mathbb{A} \quad \|x(i) - x(j)\| \geq p$$

“How can we provide people with cyber-physical systems they can *bet their lives on*?”

-- Jeanette Wing

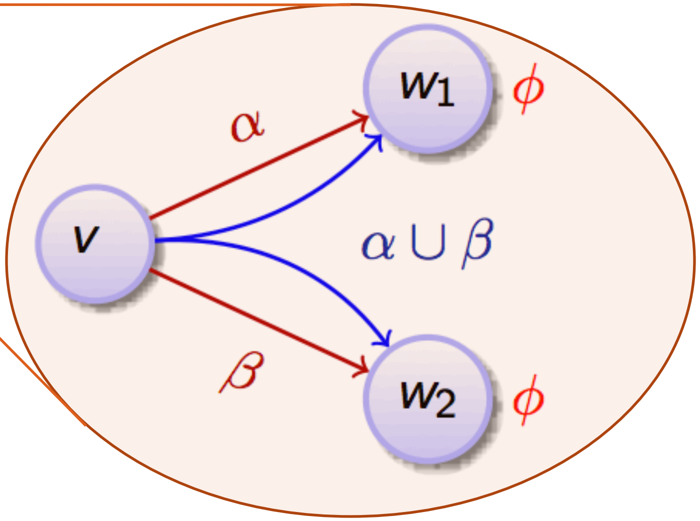
# Differential Dynamic Logic: Axiomatization

[:=] $[x := \theta]\phi(x) \leftrightarrow \phi(\theta)$

[?] $[?H]\phi \leftrightarrow (H \rightarrow \phi)$

['] $[x' = f(x)]\phi \leftrightarrow \forall t{\geq}0\, [x := y(t)]\phi \qquad (y'(t) = f(y))$

[∪] $[\alpha \cup \beta]\phi \leftrightarrow [\alpha]\phi \wedge [\beta]\phi$




[;] $[\alpha; \beta]\phi \leftrightarrow [\alpha][\beta]\phi$

[*] $[\alpha^*]\phi \leftrightarrow \phi \wedge [\alpha][\alpha^*]\phi$

K $[\alpha](\phi \rightarrow \psi) \rightarrow ([\alpha]\phi \rightarrow [\alpha]\psi)$

I $[\alpha^*](\phi \rightarrow [\alpha]\phi) \rightarrow (\phi \rightarrow [\alpha^*]\phi)$

[Platzer08]

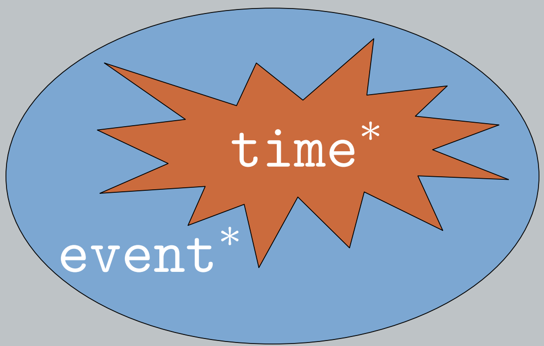
# Roadmap

## Differential Refinement Logic (dRL)

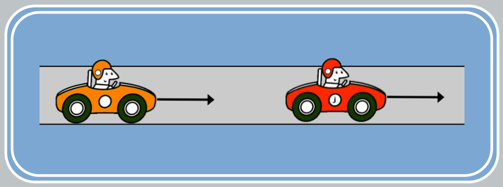
$$\alpha \leq \beta$$

- Proof rules
- Examples

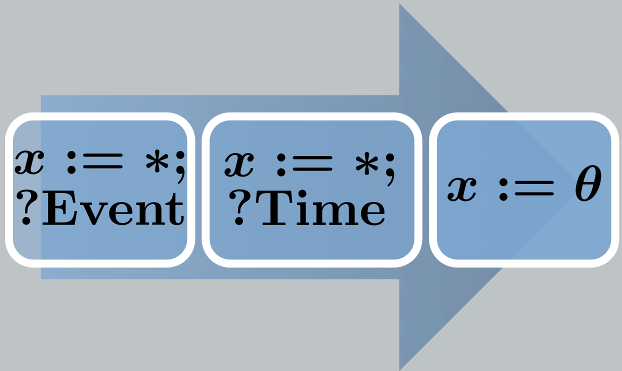
### Time-triggered vs. Event-triggered




### Verified Car Control



### Iterative System Design



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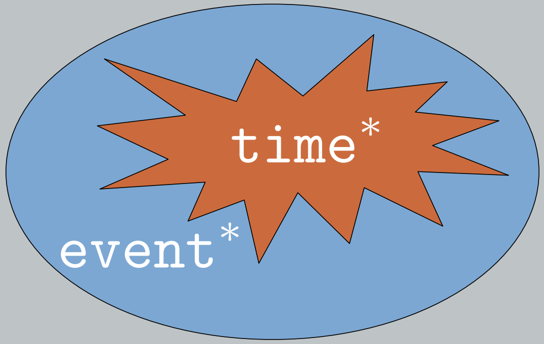


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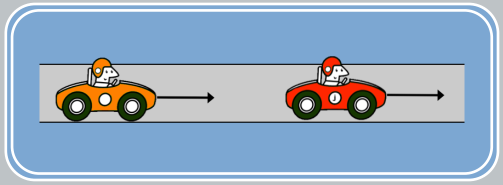

### Iterative System Design

$x := *;$ $?\mathrm{Event}$ → $x := *;$ $?\mathrm{Time}$ → $x := \theta$

# Verifying a specific local lane controller

$$\texttt{safe}_* \equiv (?\mathbf{Safe}_\varepsilon;\ \ a_f := *;\ \ ?(-B \leq a_f \leq A))$$

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$$\texttt{safe}_* \equiv (?\mathbf{Safe}_\varepsilon;\ a_f := *;\ ?(-B \le a_f \le A))$$

$$\texttt{safe}_\theta \equiv$$

$$a_f := K_p\left((x_l - x_f) - \left(\frac{\overline{v}^2}{2b} - \frac{\underline{v}^2}{2b} + (\frac{A}{b} + 1)(\frac{A}{2}\varepsilon^2 + \varepsilon\overline{v})\right)\right)$$

$$+ K_i(\overline{z}) + K_d(v_l - v_f)$$

# Verifying a specific local lane controller

$$\mathtt{safe}_* \equiv (?\mathbf{Safe}_\varepsilon;\ a_f := *;\ ?(-B \le a_f \le A))$$

$$\mathtt{safe}_{\boldsymbol{\theta}} \equiv$$

$$\boldsymbol{a_f := \theta}$$

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$$\mathtt{safe}_* \equiv (?\mathbf{Safe}_\varepsilon;\ a_f := *;\ ?(-B \le a_f \le A))$$

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# Verifying a specific local lane controller

$$\mathtt{safe}_* \equiv (?\mathbf{Safe}_\varepsilon;\ a_f := *;\ ?(-B \le a_f \le A))$$

$$-B \le \theta \le A \qquad \le \qquad (\theta > -b) \to \mathbf{Safe}_\varepsilon$$

$$\mathtt{safe}_{\boldsymbol{\theta}} \equiv \boldsymbol{a_f} := \boldsymbol{\theta}$$

# Roadmap

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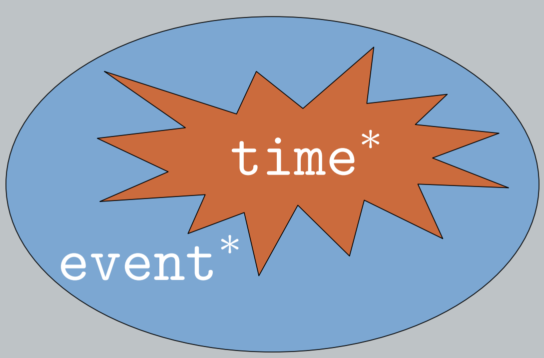


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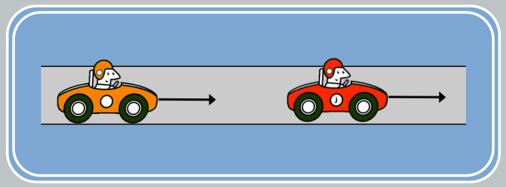

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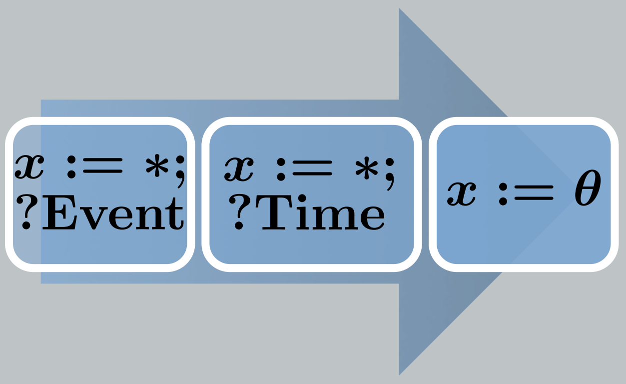
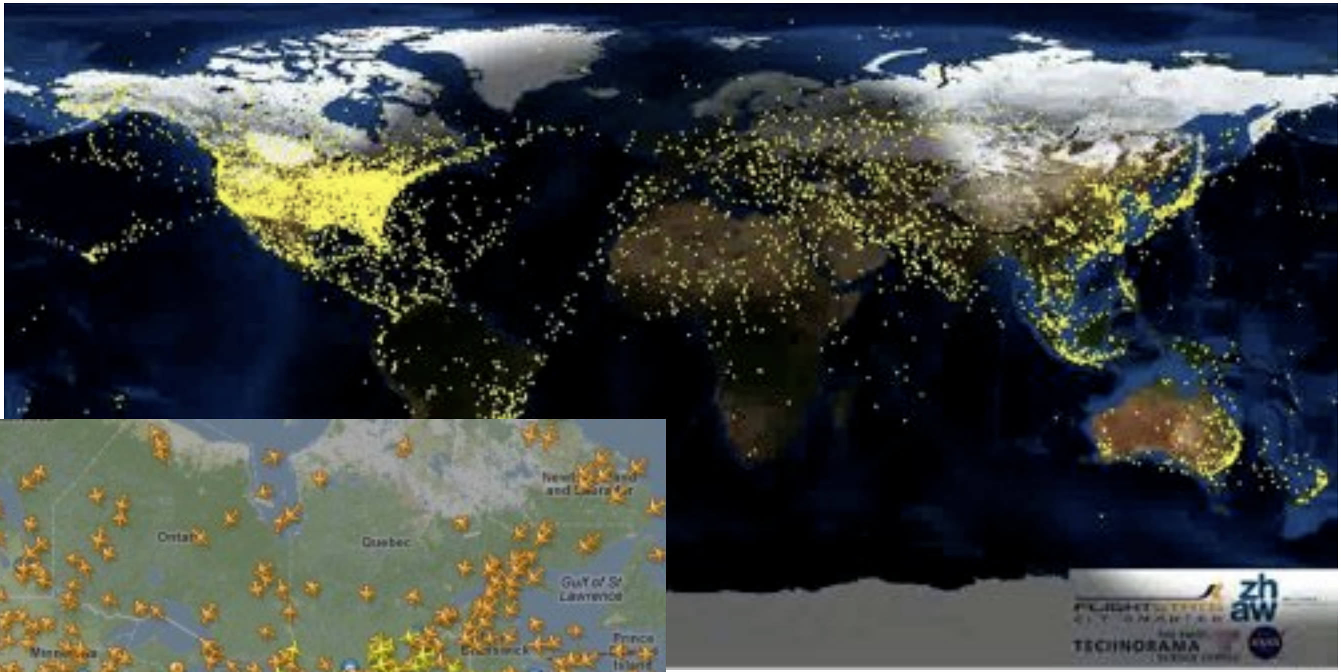


# How Can We Prove Distributed Airspace?



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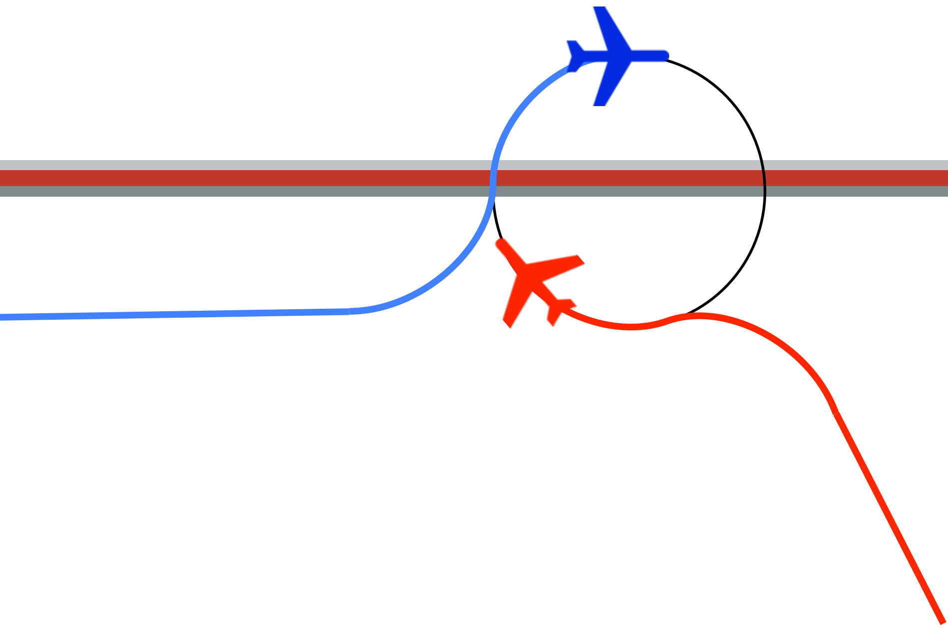

Sensor limits on aircraft are local.

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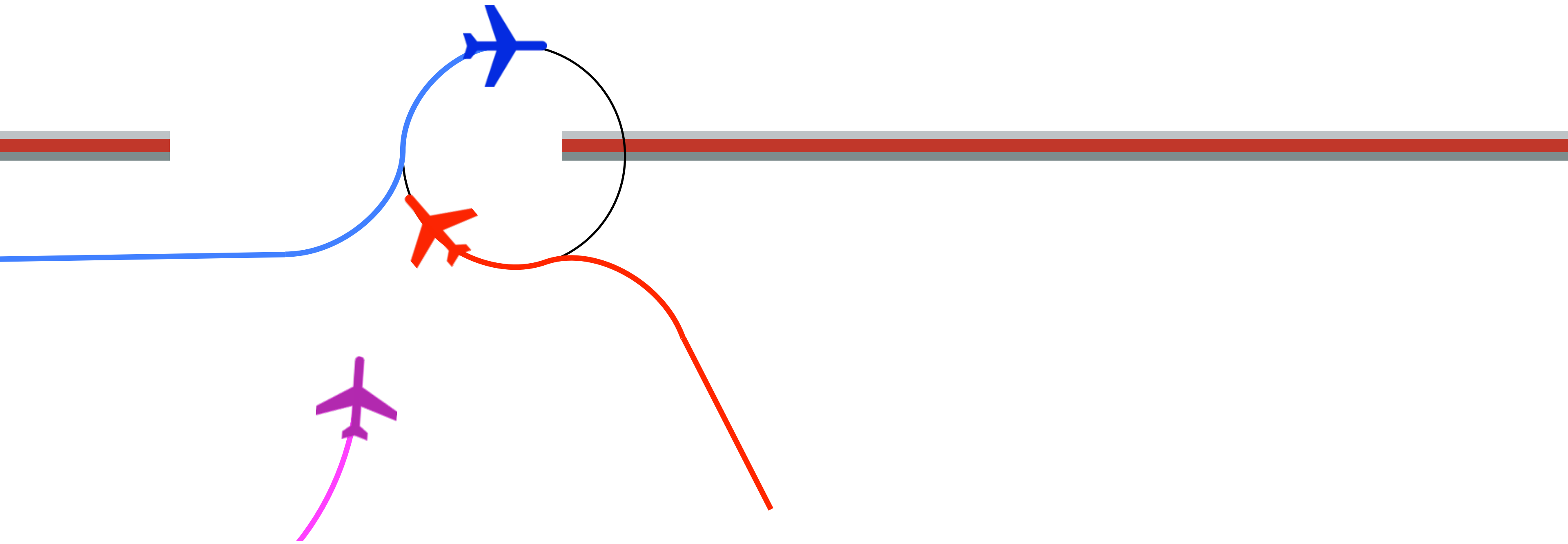

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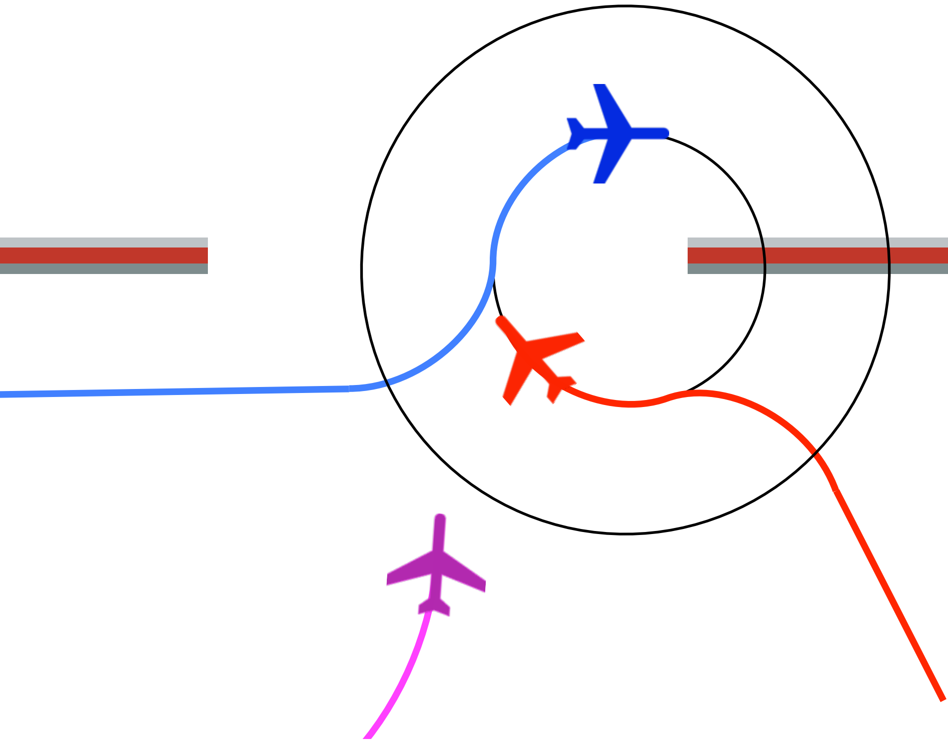

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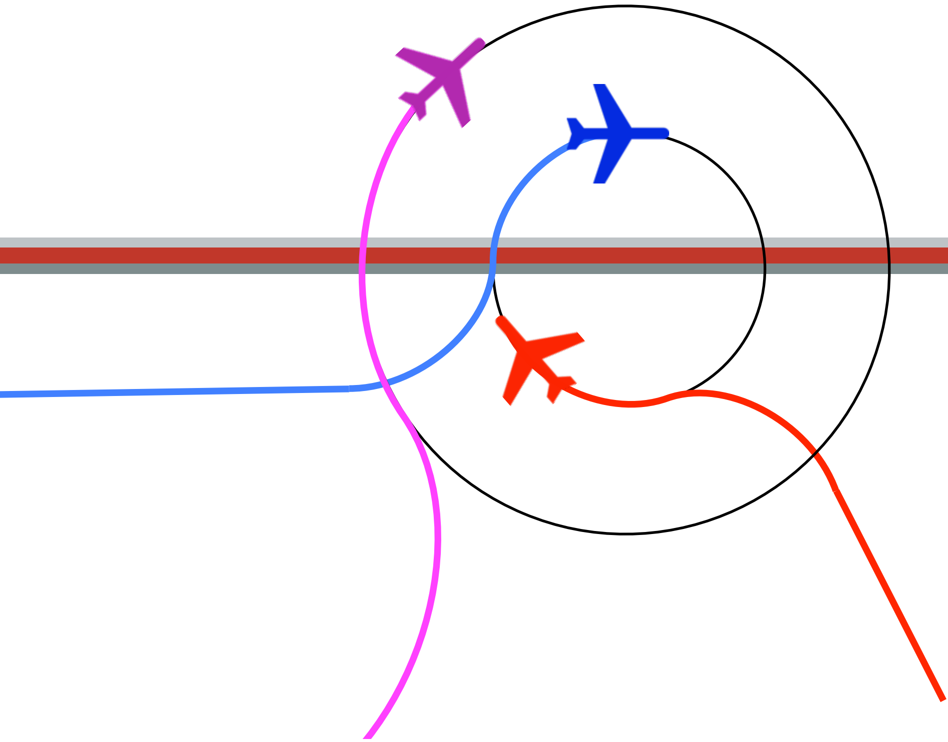

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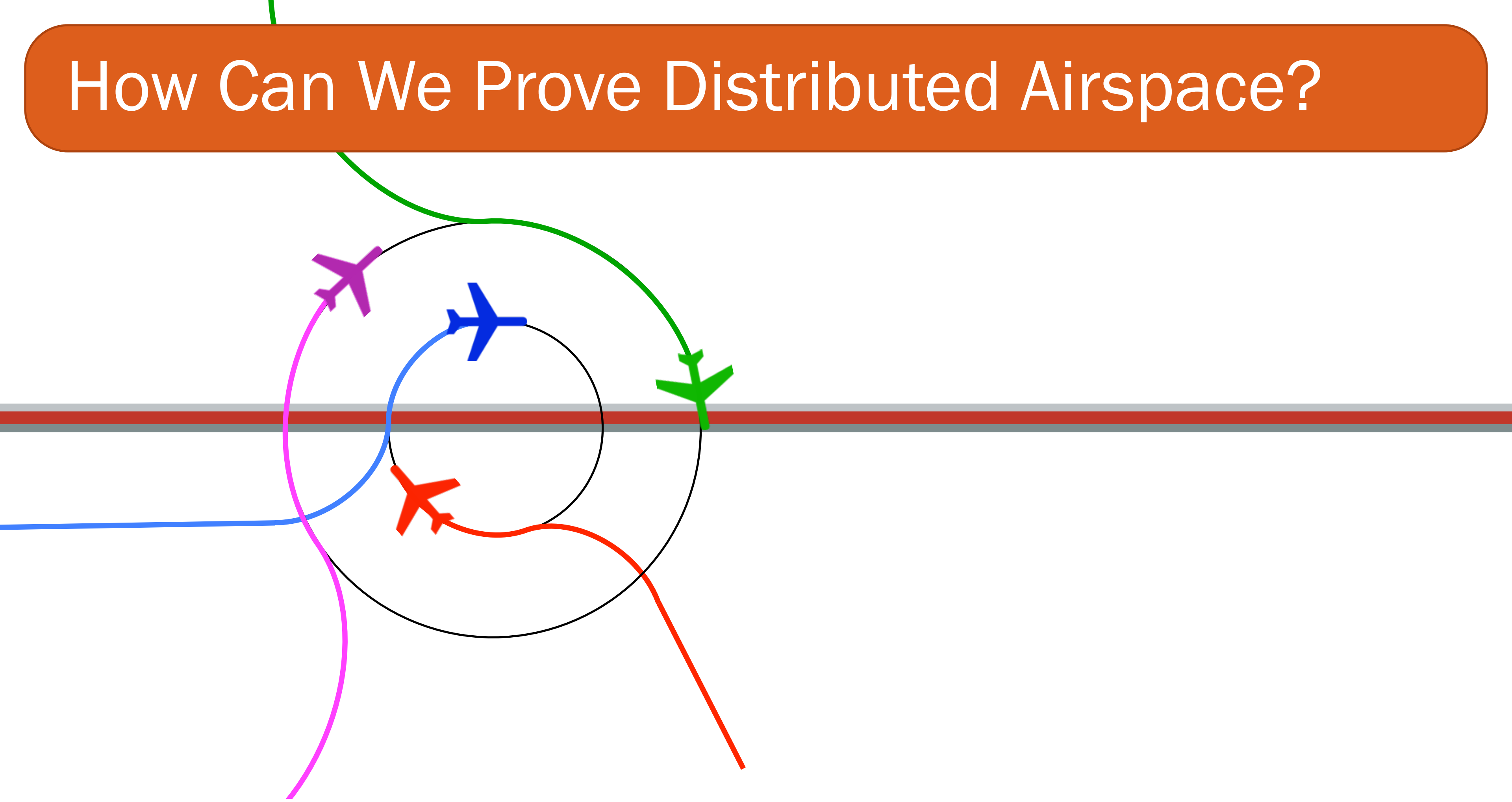

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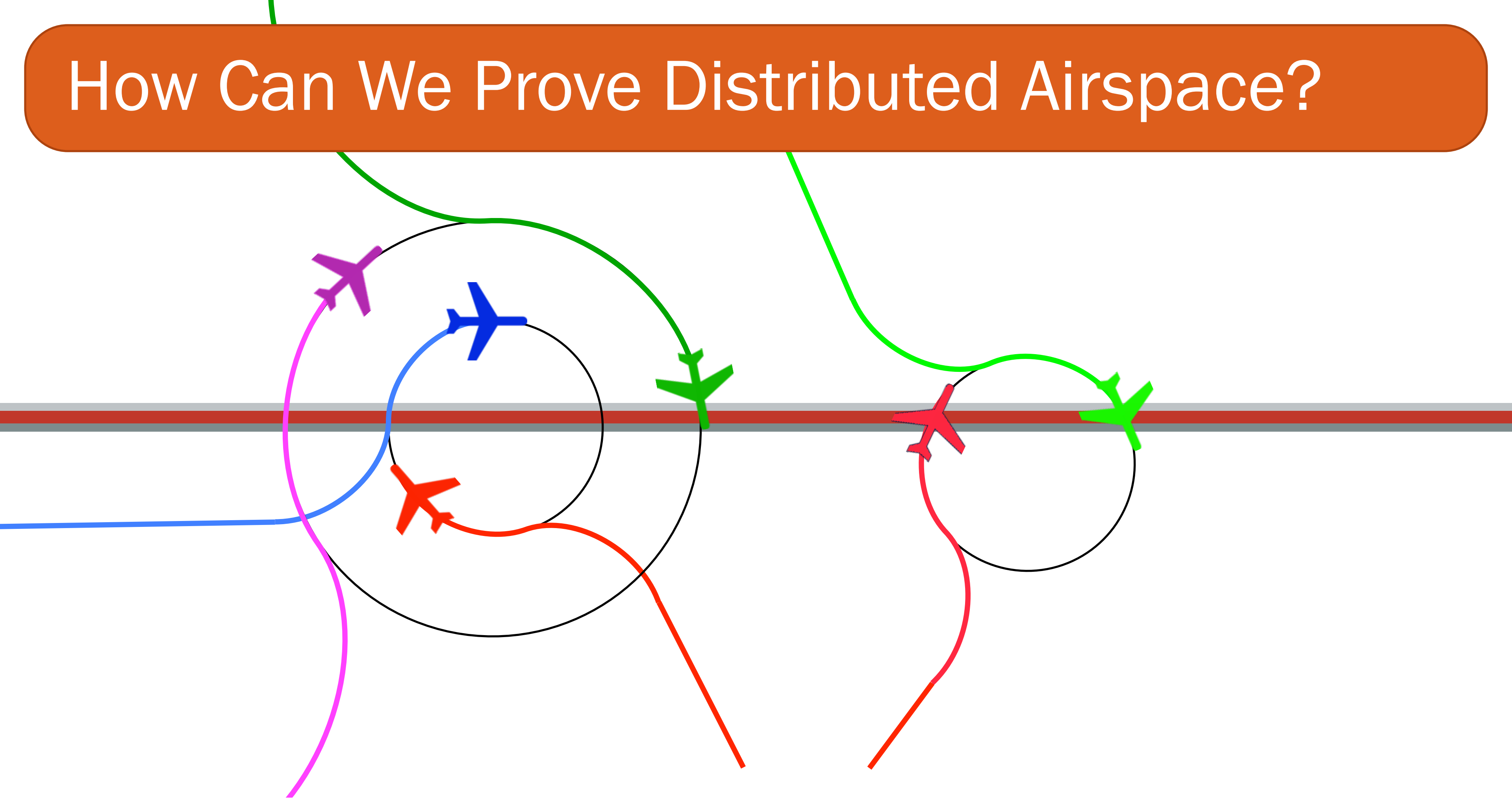

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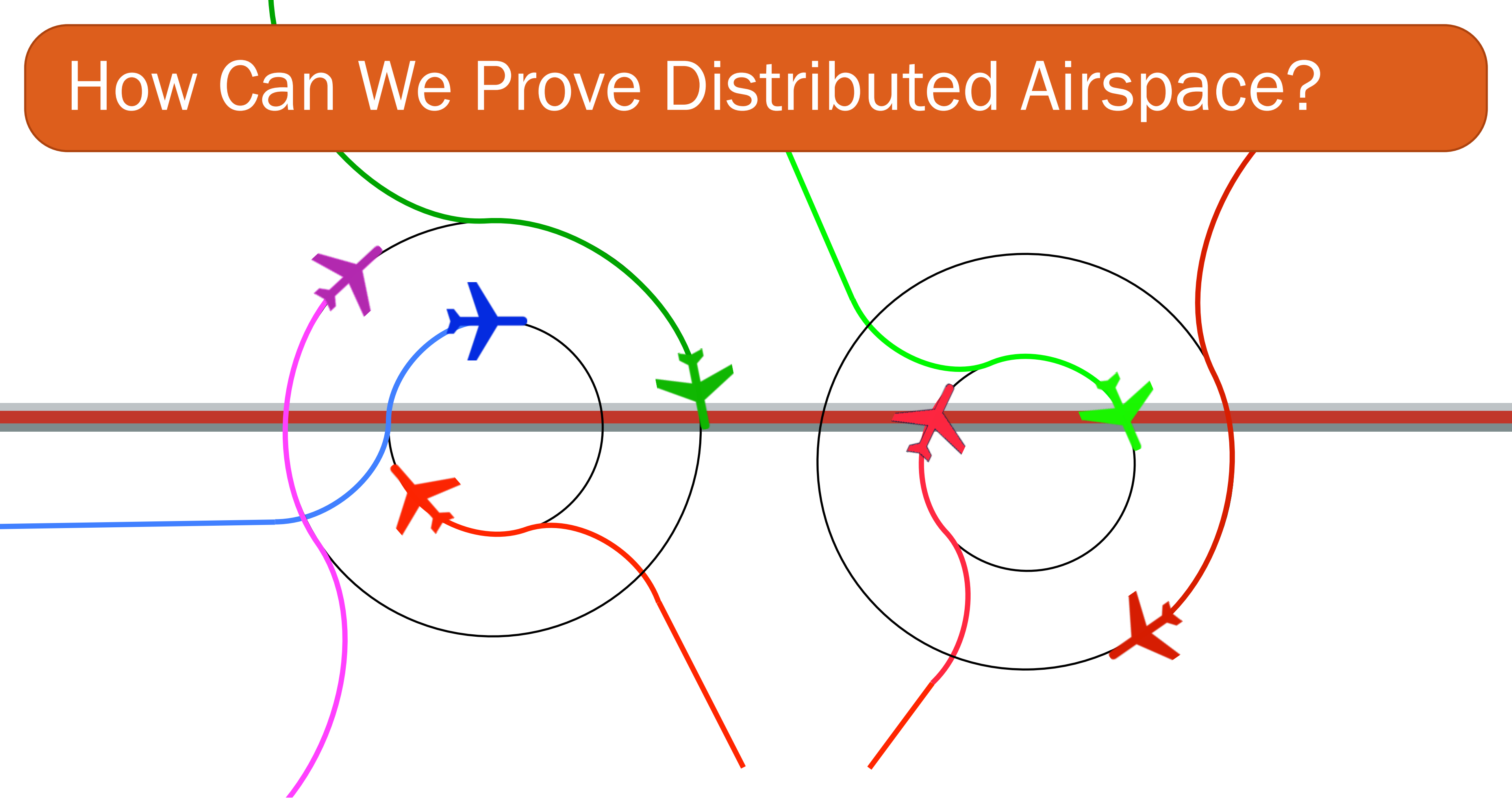

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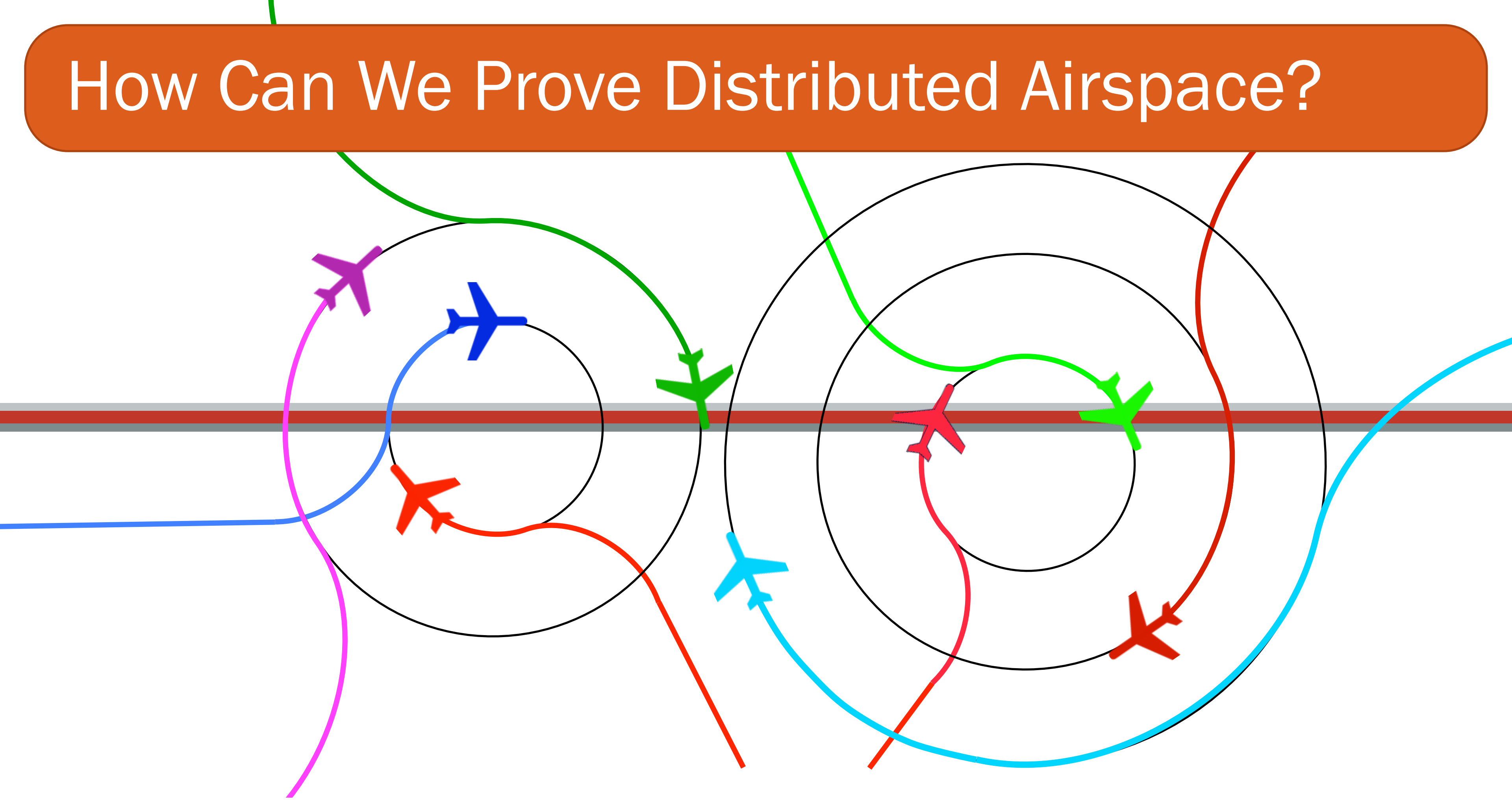

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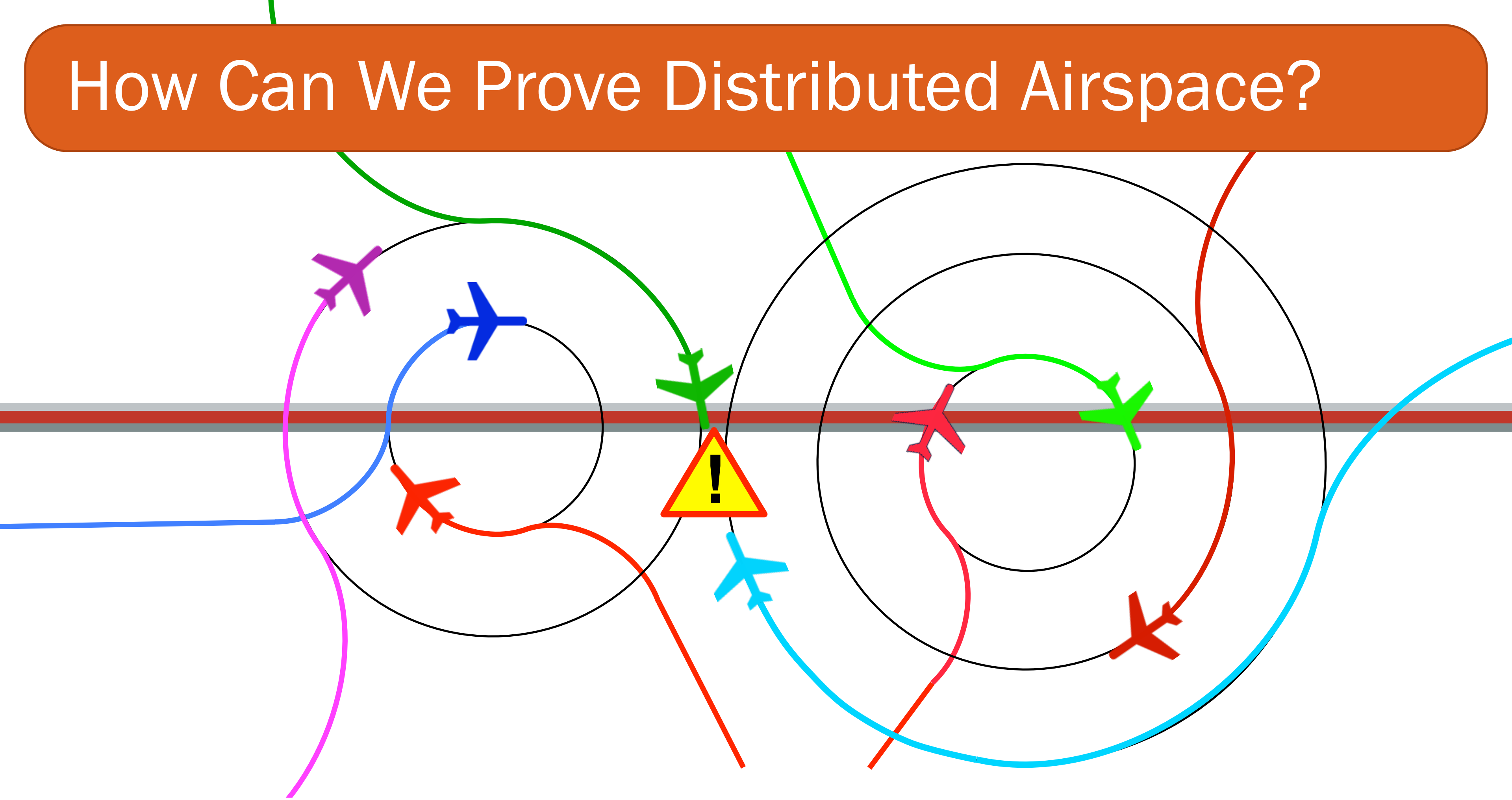
# How Can We Prove Distributed Airspace?



Sensor limits on aircraft are local.

Sometimes a maneuver may look safe locally…

But is a terrible idea when implemented globally.

# Assumptions and Requirements

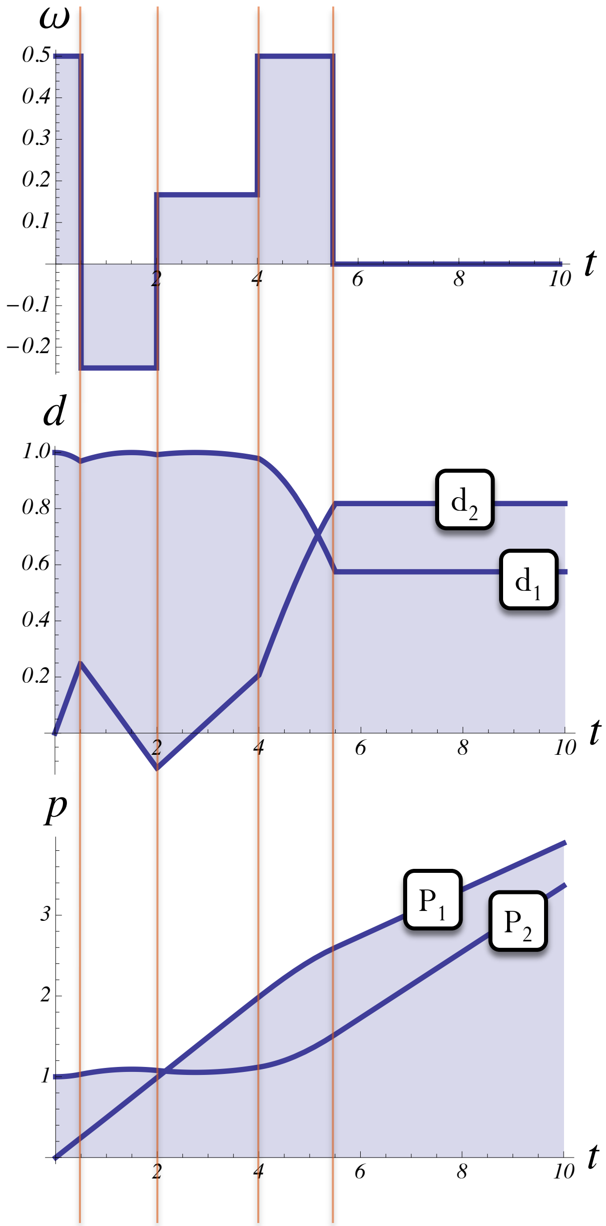
## Requirements

- **Safety**: At all times, the aircraft must be separated by distance greater than $p$.
- Aircraft trajectories must always be **flyable**.
- An **arbitrary number** of aircraft may enter the maneuver at any time.

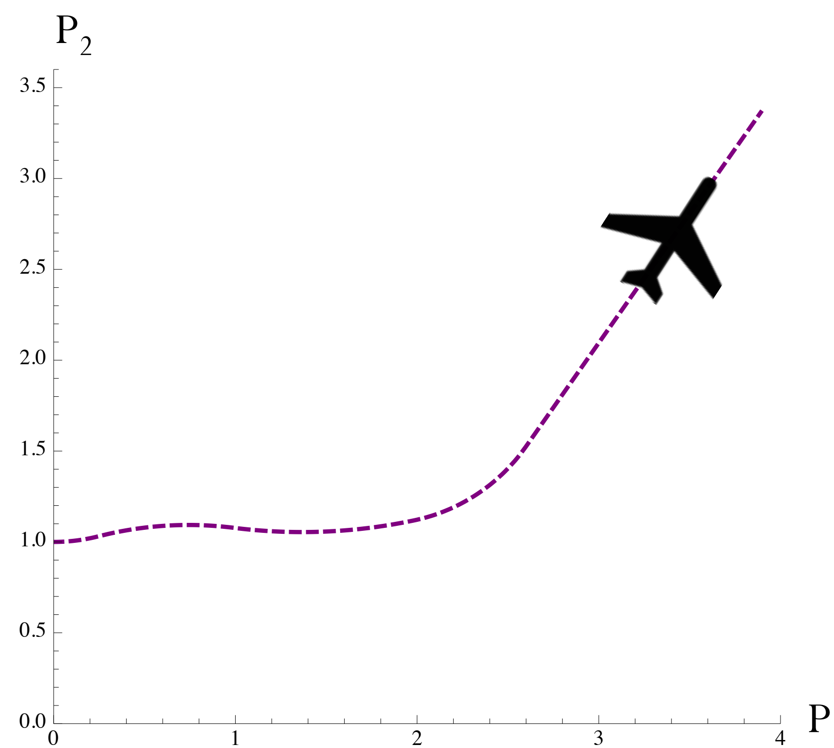
## Assumptions

- Aircraft maintain constant velocity.
- Sensors are accurate and have no delay.
- Collision avoidance maneuvers are executed on the 2D plane.

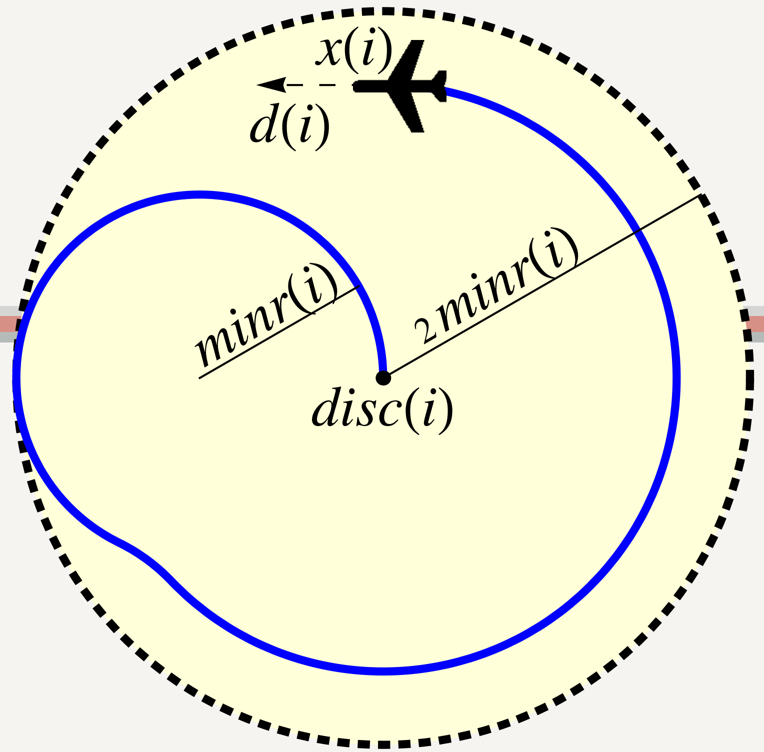
# Hybrid Dynamics




Aircraft are controlled by steering, through discrete changes in angular velocity $\omega$.



# Big Disc Control

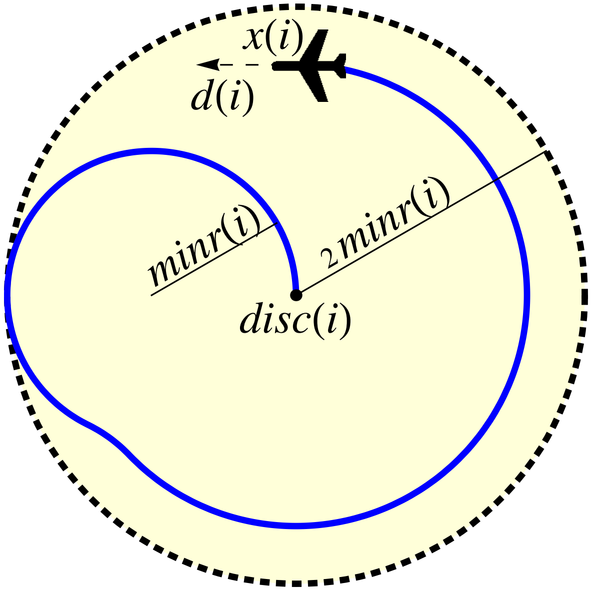



- Leaves maneuverability to pilot discretion.
- Requires large buffer disc.
- Requires aircraft to return to the center of the disc before completing avoidance maneuver.

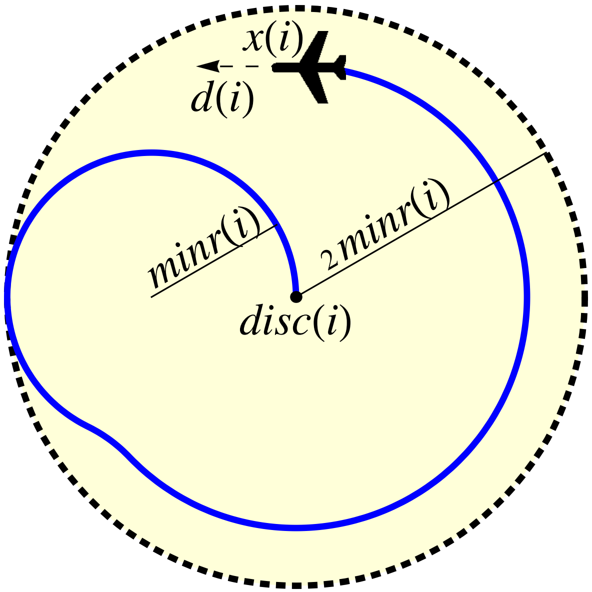
[LoosRP13]

# To Prove:

$$\mathtt{Init} \rightarrow [\mathtt{BigDisc}]\mathtt{Safe}$$



# Big Disc Control

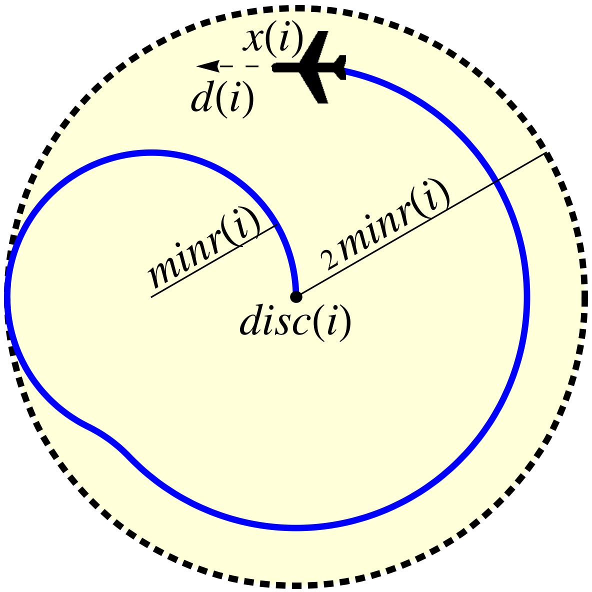



## To Prove:

$$\mathtt{Init} \rightarrow [\mathtt{BigDisc}]\mathtt{Safe}$$

$$\mathtt{Safe} \equiv$$
$$(\forall i, j : \mathbb{A} \;\; i \neq j \rightarrow$$
$$\|x(i) - x(j)\| \geq p)$$

# Big Disc Control $\quad \mathtt{Init} \rightarrow [\mathtt{BigDisc}]\mathtt{Safe}$




$$
\begin{aligned}
\mathtt{BigDisc} &\equiv (\mathtt{Control} \cup \mathtt{Plant})^* \\
\mathtt{Control} &\equiv k := *_{\mathbb{A}};\ (\mathtt{CA} \cup \mathtt{NotCA}) \\
\mathtt{CA} &\equiv ?(ca(k) = 1);\ (\mathtt{Steer} \cup \mathtt{Exit}) \\
\mathtt{NotCA} &\equiv ?(ca(k) = 0);\ (\mathtt{Steer} \cup \mathtt{Flip} \cup \mathtt{Enter}) \\
\mathtt{Steer} &\equiv \omega(k) := *_{\mathbb{R}};\ ?(-\Omega(k) \leq \omega(k) \leq \Omega(k)) \\
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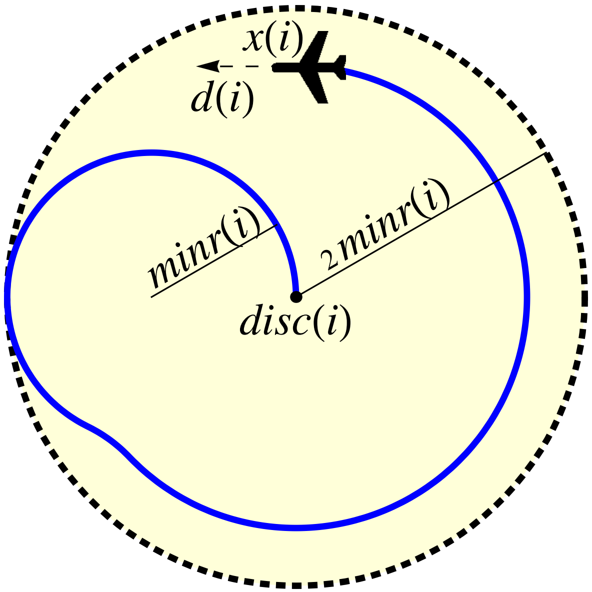
\mathtt{Enter} &\equiv \omega(k) := side(k) \cdot \Omega(k);\ ca(k) := 1 \\
\mathtt{Flip} &\equiv side(k) := -side(k) \\
\mathtt{Plant} &\equiv \forall i : \mathbb{A} \Big( x(i)' = v(i) \cdot d(i),\ d(i)' = \omega(i) \cdot d(i)^{\perp}, \\
&\qquad disc(i)' = (1 - ca(i)) \cdot v(i) \cdot d(i)\ \&\ \mathtt{EvDom} \Big) \\
\mathtt{EvDom} &\equiv \forall j : \mathbb{A} \\
&\quad ((j \neq i \land (ca(i) = 0 \lor ca(j) = 0)) \rightarrow \mathsf{Sep}(i, j) \\
&\quad \land \|disc(i) - (x(i) + minr(i) \cdot side(i) \cdot d(i)^{\perp})\| \\
&\qquad \leq minr(i)) \\
\mathsf{Sep}(i, j) &\equiv \|disc(i) - disc(j)\| \geq 2minr(i) + 2minr(j) + p
\end{aligned}
$$

# Big Disc Control $\mathtt{Init} \to [\mathtt{BigDisc}]\mathtt{Safe}$



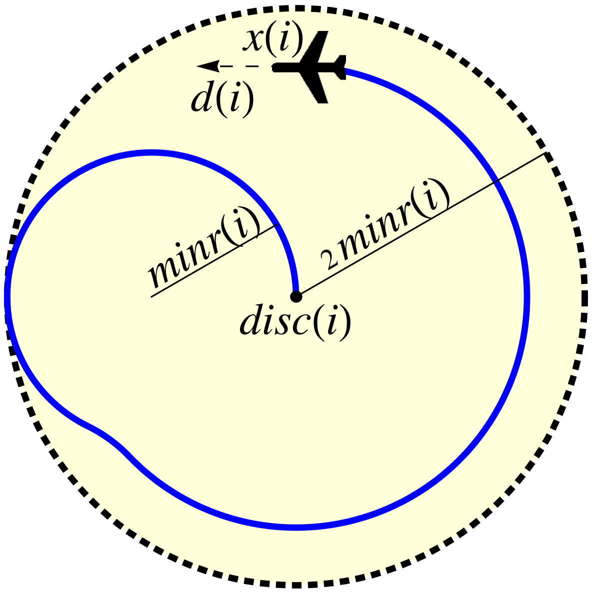

$$\mathtt{Plant} \equiv \forall i : \mathbb{A} \left( \boxed{x(i)' = v(i) \cdot d(i),\ d(i)' = \omega(i) \cdot d(i)^{\perp},} \right.$$
$$\left. disc(i)' = (1 - ca(i)) \cdot v(i) \cdot d(i)\ \&\ \mathtt{EvDom} \right)$$

Dubins Model
for 2D motion

[Dubins57]

# Big Disc Control

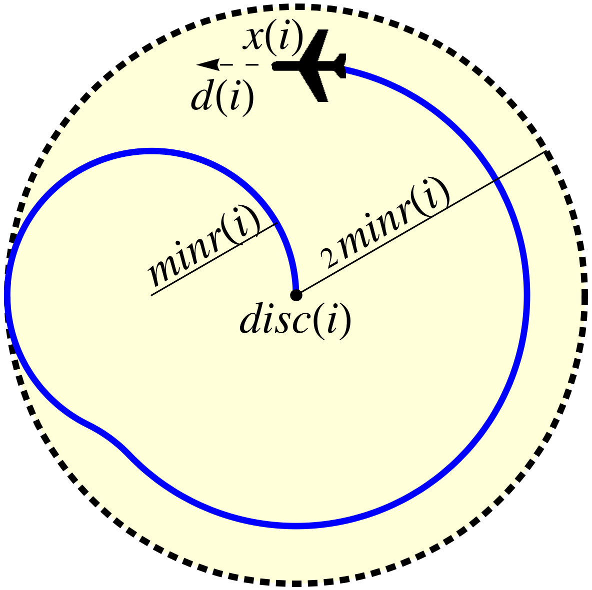
$\mathtt{Init} \to [\mathtt{BigDisc}]\mathtt{Safe}$




$$\mathtt{Plant} \equiv \forall i : \mathbb{A}\Big(x(i)' = v(i)\cdot d(i),\ d(i)' = \omega(i)\cdot d(i)^{\perp},$$
$$\boxed{disc(i)' = (1 - ca(i))\cdot v(i)\cdot d(i)}\ \&\ \mathtt{EvDom}\Big)$$

The disc does not move when in a collision avoidance maneuver

# Big Disc Control $\texttt{Init} \rightarrow [\texttt{BigDisc}]\texttt{Safe}$



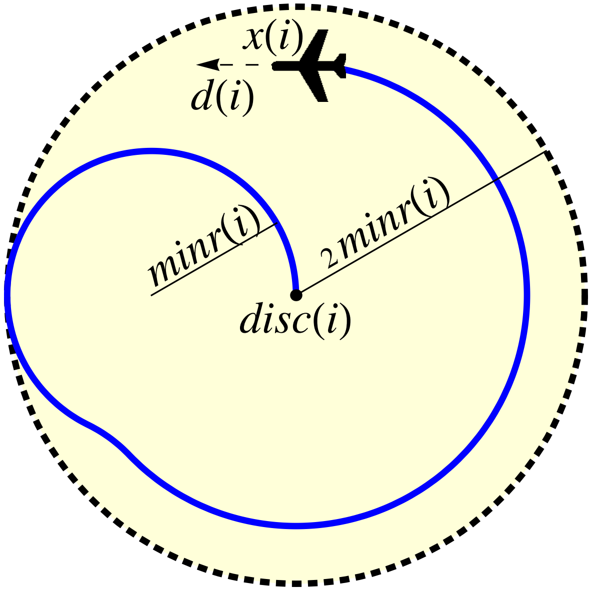

$$\texttt{Plant} \equiv \boxed{\forall i : \mathbb{A}} \Big( x(i)' = v(i) \cdot d(i),\ d(i)' = \omega(i) \cdot d(i)^{\perp},\ disc(i)' = (1 - ca(i)) \cdot v(i) \cdot d(i)\ \&\ \texttt{EvDom} \Big)$$

All aircraft evolve simultaneously

# Big Disc Control

$\text{Init} \rightarrow [\text{BigDisc}]\text{Safe}$

$\text{BigDisc} \equiv (\text{Control} \cup \text{Plant})^*$

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# Big Disc Control $\texttt{Init} \rightarrow [\texttt{BigDisc}]\texttt{Safe}$

$$\begin{aligned}
\texttt{BigDisc} &\equiv (\texttt{Control} \cup \texttt{Plant})^* \\
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\texttt{Steer} &\equiv \omega(k) := *_{\mathbb{R}};\ ?(-\Omega(k) \leq \omega(k) \leq \Omega(k)) \\
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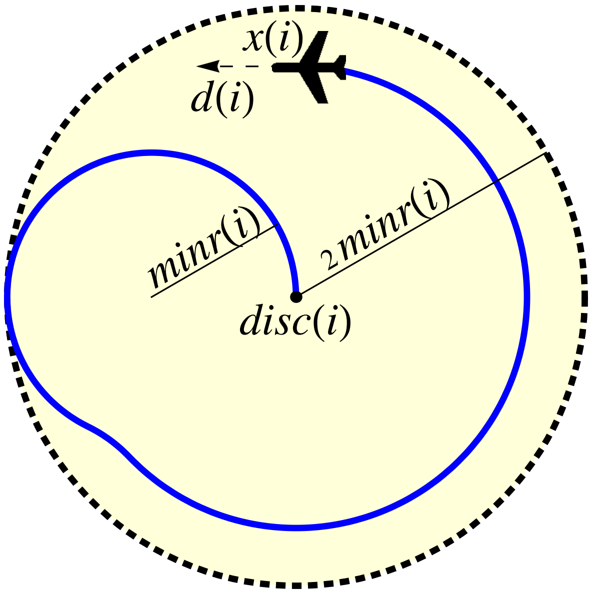
\end{aligned}$$



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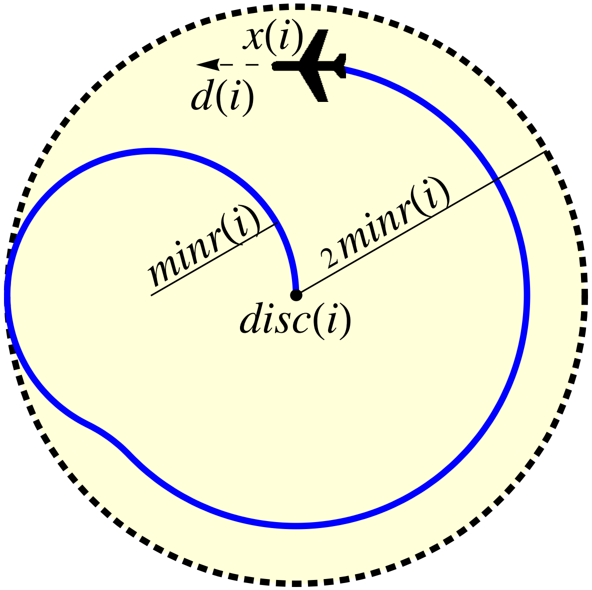
$$\mathtt{EvDom} \equiv \forall j : \mathbb{A}\ ((j \neq i \land (ca(i) = 0 \lor ca(j) = 0)) \rightarrow \mathsf{Sep}(i, j) \land \|disc(i) - (x(i) + minr(i) \cdot side(i) \cdot d(i)^{\perp})\| \leq minr(i))$$
$$\mathsf{Sep}(i, j) \equiv \|disc(i) - disc(j)\| \geq 2minr(i) + 2minr(j) + p$$




-Ensures aircraft control can engage CA maneuver.

- Aircraft can flyably remain within disc

# Big Disc Control $\texttt{Init} \to [\texttt{BigDisc}]\texttt{Safe}$

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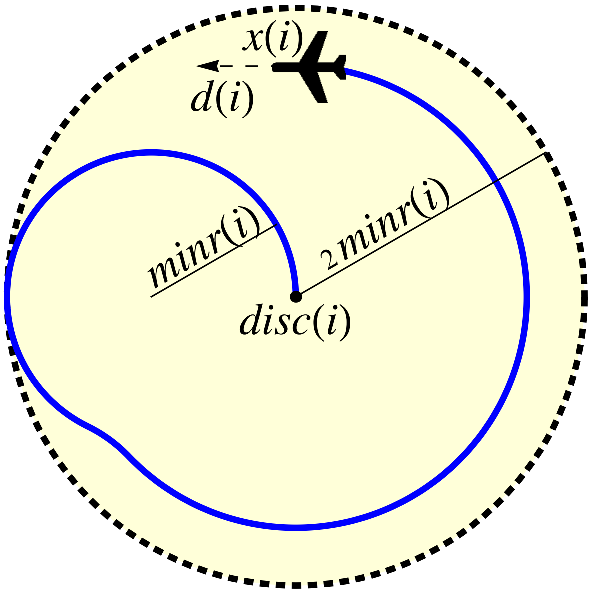
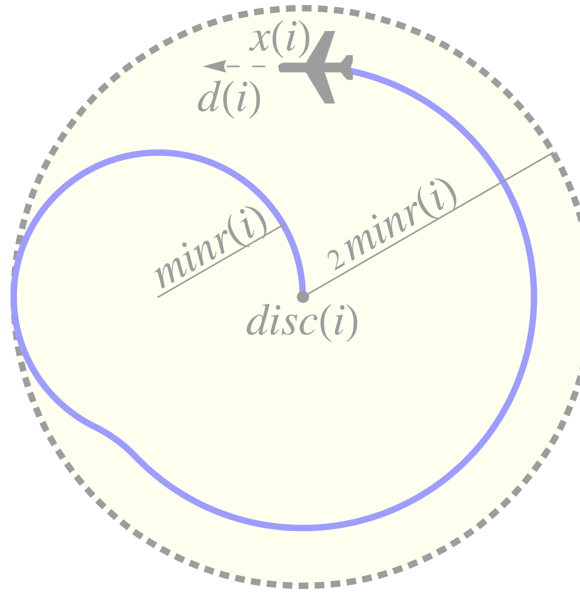


# Big Disc Control

$\mathtt{Init} \rightarrow [\mathtt{BigDisc}]\mathtt{Safe}$

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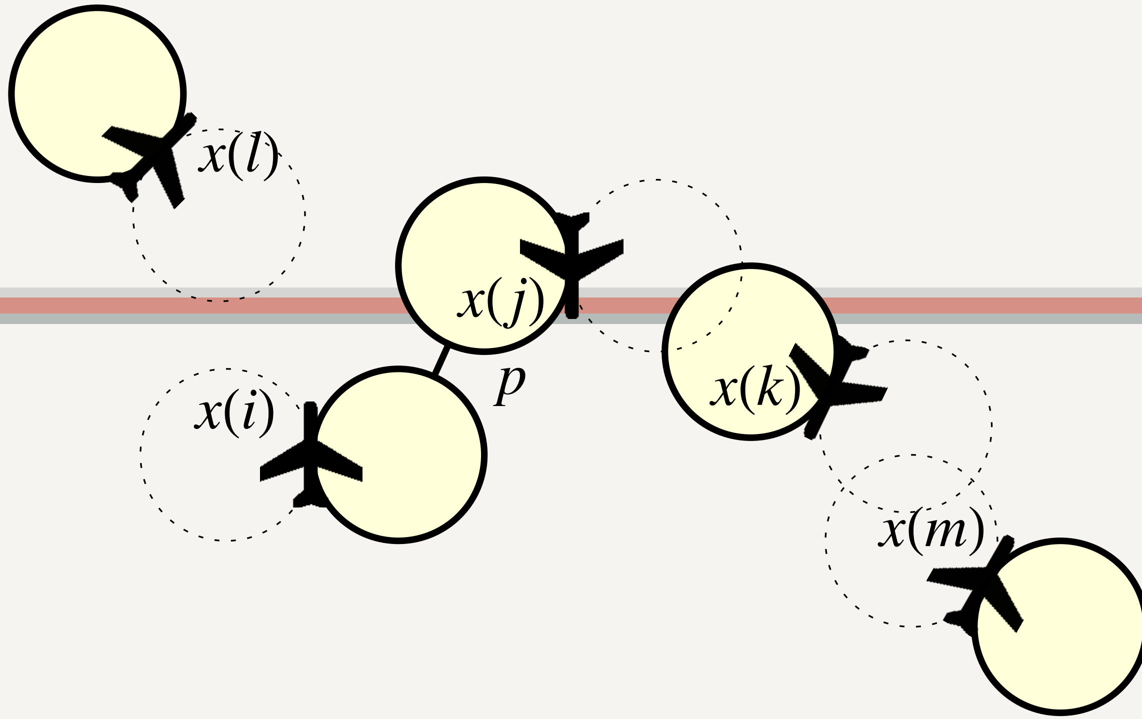
\end{aligned}
$$

**✔ Verified in KeYmaeraD**

# Small Discs Control




- Deterministic control makes it well suited for UAVs.
- Smaller discs allow aircraft to fly closer together.
- Aircraft may exit maneuver as soon as it is safe to do so.

[PallottinoSBF07, LoosRP13]

# Small Discs Control

$$\texttt{SmallDiscs} \equiv (\texttt{Control} \cup \texttt{Plant})^*$$

$$\texttt{Control} \equiv k := *_{\mathbb{A}};\ (\texttt{CA} \cup \texttt{NotCA})$$

$$\texttt{CA} \equiv ?(ca(k) = 1);\ (\texttt{Exit} \cup \texttt{Skip})$$

$$\texttt{NotCA} \equiv ?(ca(k) = 0);\ (\texttt{Steer} \cup \texttt{Flip} \cup \texttt{Enter})$$

$$\texttt{Skip} \equiv ?true$$

$$\texttt{Steer} \equiv \omega(k) := *_{\mathbb{R}};\ ?(-\Omega(k) \le \omega(k) \le \Omega(k))$$

$$\texttt{Exit} \equiv ca(k) := 0$$

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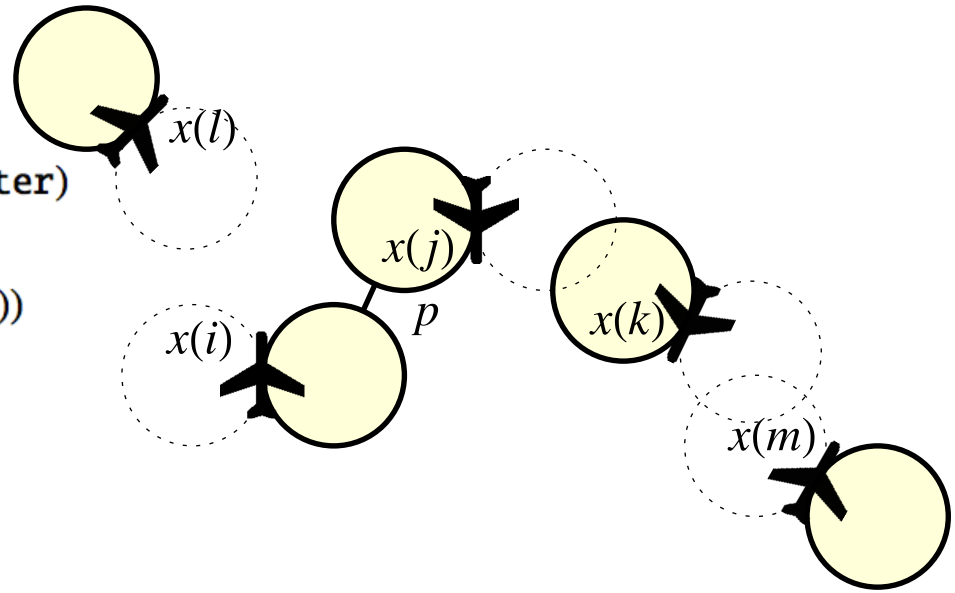
$$\texttt{Flip} \equiv ?(\forall j : \mathbb{A}\ (j \ne k \to \ \mathsf{FlipSep}(j,k)));\ side(k) := -side(k)$$

$$\mathsf{FlipSep}(i,j) \equiv \|(x(i) + minr(i) \cdot side(i) \cdot d(i)^{\perp}) - (x(j) - minr(j) \cdot side(j) \cdot d(j)^{\perp})\| \ge minr(i) + minr(j) + p$$

$$\texttt{Plant} \equiv \forall i : \mathbb{A}\ \Big(x(i)' = v(i) \cdot d(i),\ d(i)' = \omega(i) d(i)^{\perp}\ \&\ \forall j : \mathbb{A}\ ((j \ne i \land (ca(i) = 0 \lor ca(j) = 0)) \to \mathsf{Sep}(i,j))\Big)$$

$$\mathsf{Sep}(i,j) \equiv \|(x(i) + minr(i) \cdot side(i) \cdot d(i)^{\perp}) - (x(j) + minr(j) \cdot side(j) \cdot d(j)^{\perp})\| \ge minr(i) + minr(j) + p$$



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&\quad\ side(k) := -side(k) \\
\mathsf{FlipSep}(i, j) &\equiv \|(x(i) + minr(i) \cdot side(i) \cdot d(i)^{\perp}) \\
&\quad - (x(j) - minr(j) \cdot side(j) \cdot d(j)^{\perp})\| \\
&\quad \ge minr(i) + minr(j) + p \\
\texttt{Plant} &\equiv \forall i : \mathbb{A}\ \Big(x(i)' = v(i) \cdot d(i),\ d(i)' = \omega(i) d(i)^{\perp} \\
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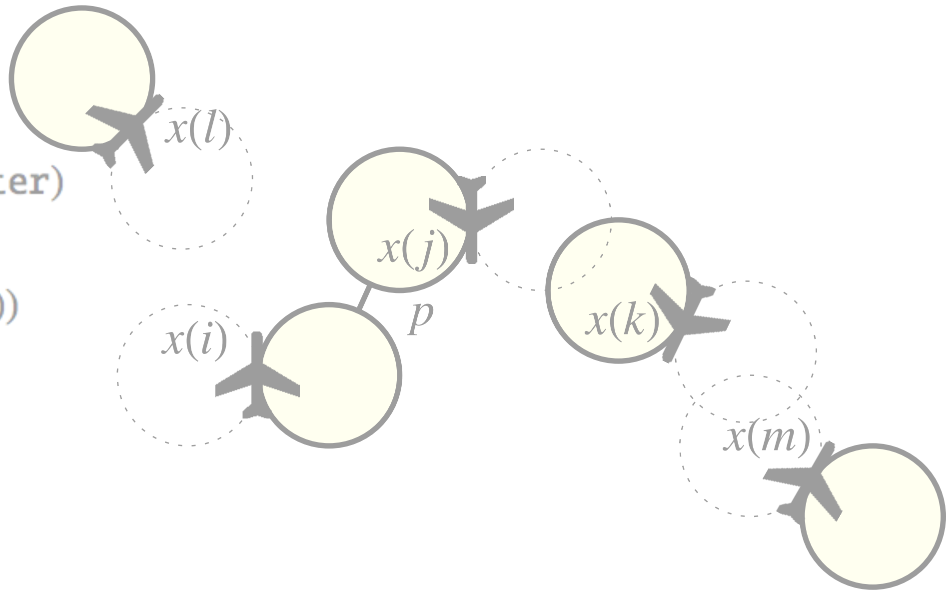
&\qquad\qquad \rightarrow \mathsf{Sep}(i, j))\Big) \\
\mathsf{Sep}(i, j) &\equiv \|(x(i) + minr(i) \cdot side(i) \cdot d(i)^{\perp}) \\
&\quad - (x(j) + minr(j) \cdot side(j) \cdot d(j)^{\perp})\| \\
&\quad \ge minr(i) + minr(j) + p
\end{aligned}
$$

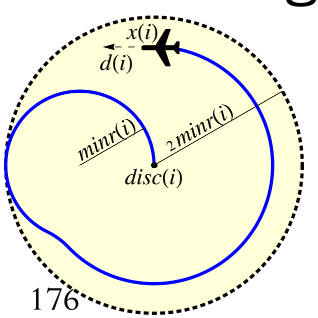


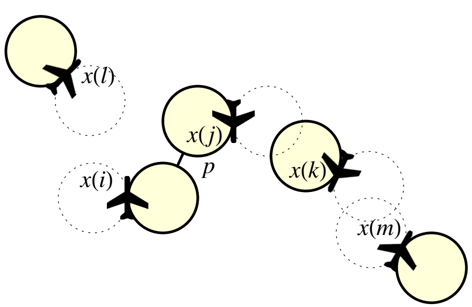

✔ **Verified in KeYmaeraD**

# Conclusions

## Challenges

- CPS needs verification
- Infinite, continuous, and evolving state space, $\mathbb{R}^\infty$
- Continuous dynamics
- Discrete control decisions
- Distributed dynamics
- Arbitrary number of aircraft
- Emergent behaviors




## Contributions

- Theorem proving is powerful for verifying distributed dynamics
- Non-linear flight paths and flyable maneuvers
- Compositionality – using small problems to solve the big ones
- Hierarchical proofs
- Undergraduates can understand and verify hybrid systems!

# Complete Proof Theory of Hybrid Systems

**Theorem (Continuous Relative Completeness) (J.Autom.Reas. 2008)**

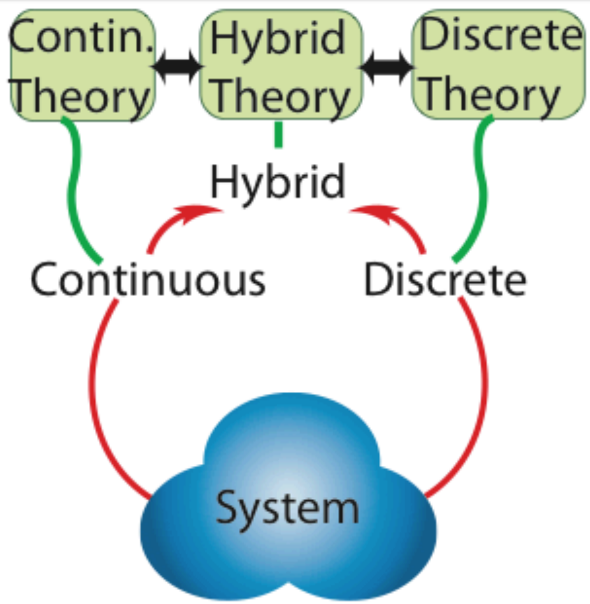
*d$\mathcal{L}$ calculus is a sound & complete axiomatization of hybrid systems relative to differential equations.*

Proof 15pp

**Theorem (Discrete Relative Completeness) (LICS'12)**

*d$\mathcal{L}$ calculus is a sound & complete axiomatization of hybrid systems relative to discrete dynamics.*

Proof +10pp



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