Uniform Substitution for Dynamic Logic with Communicating Hybrid Programs

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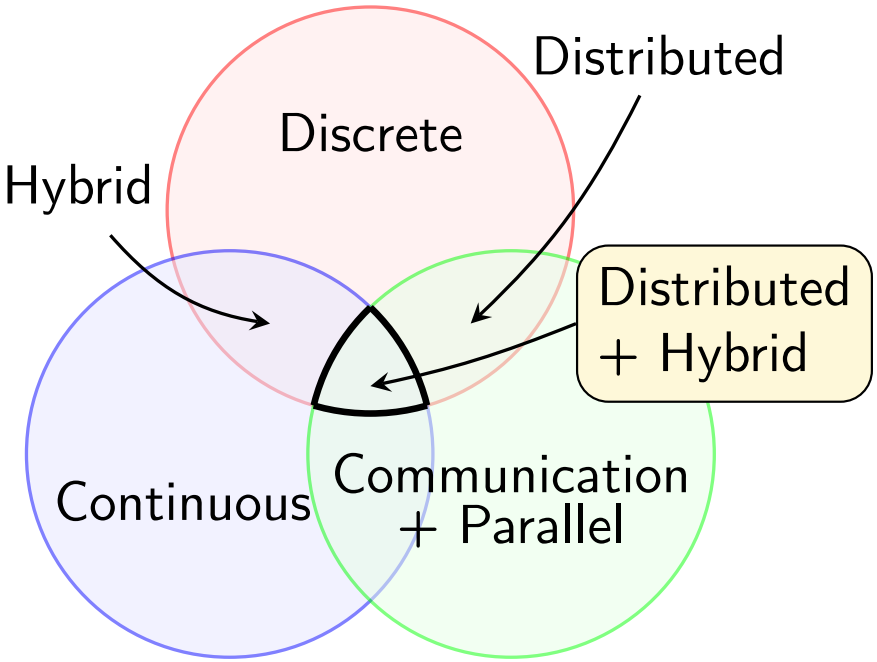
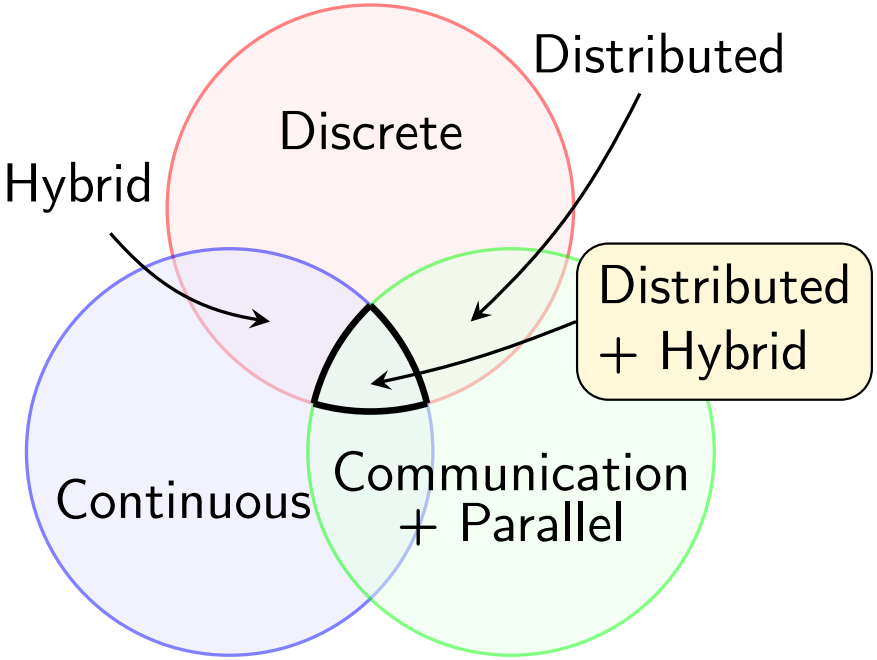
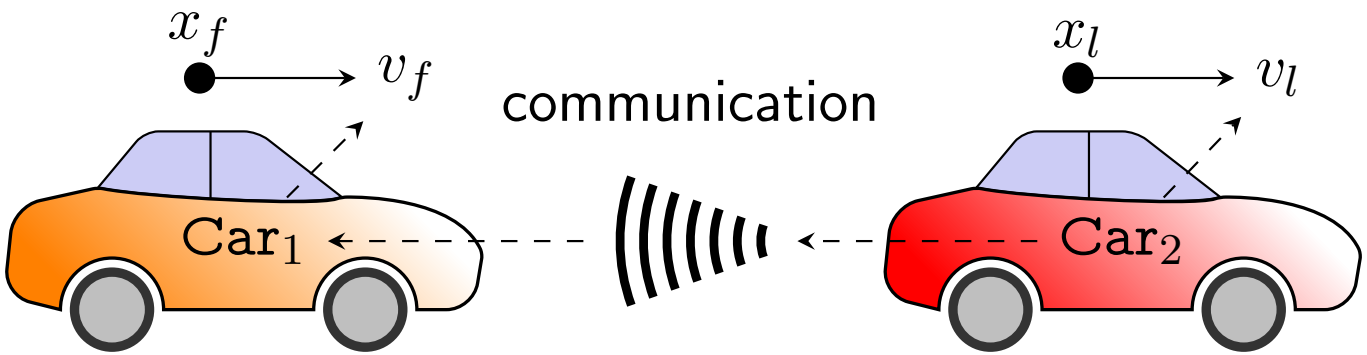
Challenge: Distributed Cyber-physical Systems



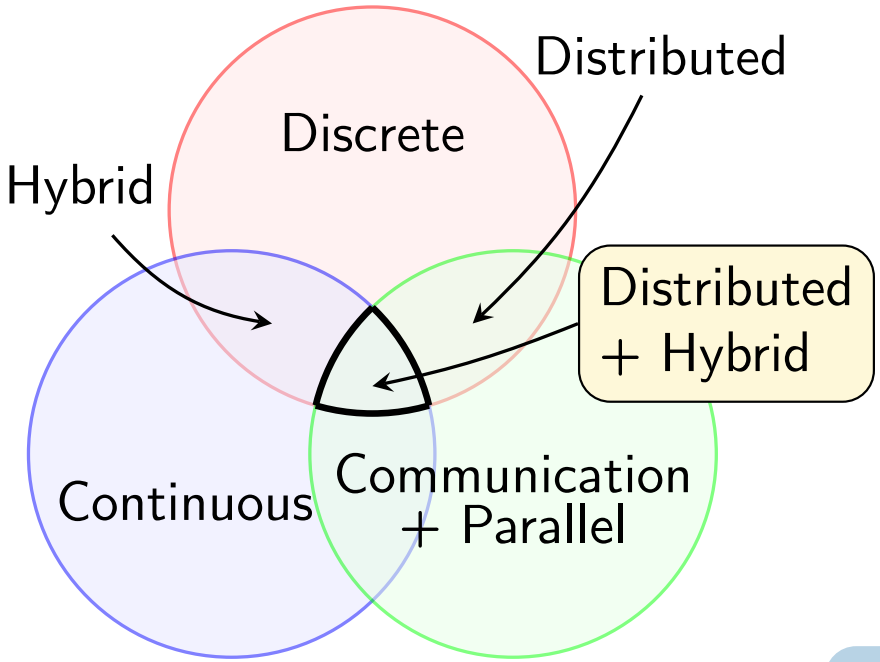
Challenge: Distributed Cyber-physical Systems

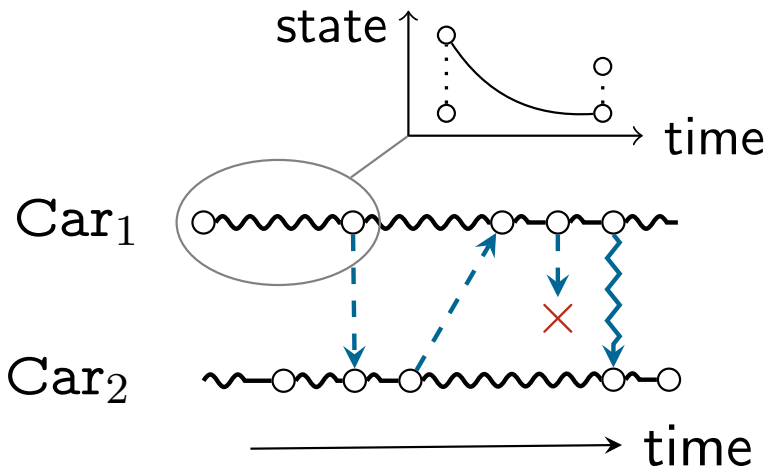




Many real-world systems are distributed hybrid systems

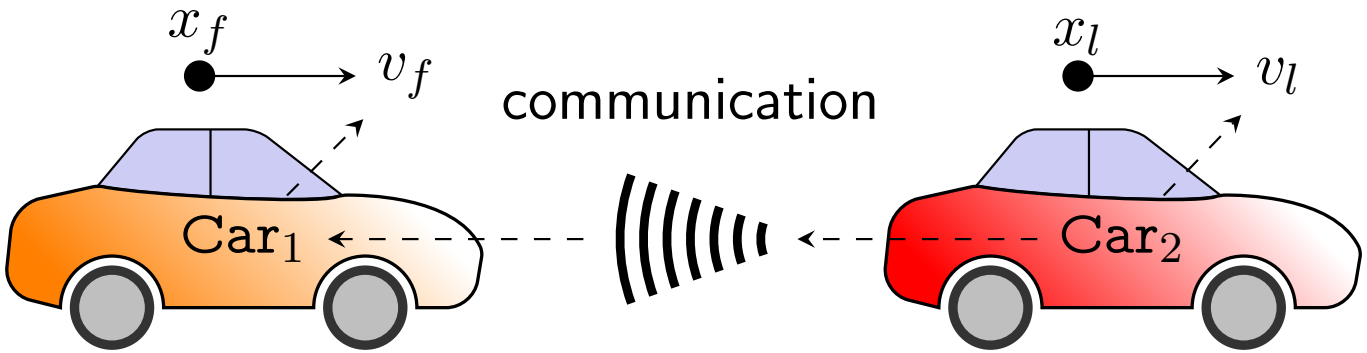
Challenge: Distributed Cyber-physical Systems





Challenge: Capture the truly simultaneous continuous dynamics of physics

Requires: Reflection in semantics and reasoning



Many real-world systems are distributed hybrid systems

The Downside of Schematic Proof Calculi

$$\frac{[\alpha]\phi \qquad [\beta]\psi}{[\alpha \parallel \beta](\phi \wedge \psi)}$$

- Compositional reasoning reduces complex systems to their building blocks

The Downside of Schematic Proof Calculi

$$\frac{[\alpha]\phi \qquad [\beta]\psi}{[\alpha \parallel \beta](\phi \wedge \psi)}\ (\star\star)$$

- Compositional reasoning reduces complex systems to their building blocks
- In case of parallel (hybrid) systems, side conditions $(\star\star)$ are extensive and subtle

$$\mathrm{FV}(\phi) \cap \mathrm{BV}(\beta) \subseteq \emptyset$$
$$\mathrm{CN}(\phi) \cap \mathrm{CN}(\beta) \subseteq \emptyset$$
$$\mathrm{FV}(\psi) \cap \mathrm{BV}(\alpha) \subseteq \emptyset$$
$$\mathrm{CN}(\psi) \cap \mathrm{CN}(\alpha) \subseteq \emptyset$$

Free variables $\mathrm{FV}(\cdot)$, bound variables $\mathrm{BV}(\cdot)$, read or written channels $\mathrm{CN}(\cdot)$, and globally synchronized variables V_G

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- In case of parallel (hybrid) systems, side conditions $(\star\star)$ are extensive and subtle

$$\begin{aligned}
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\mathrm{CN}(\phi) \cap \mathrm{CN}(\beta) &\subseteq \emptyset \\
\mathrm{FV}(\psi) \cap \mathrm{BV}(\alpha) &\subseteq \emptyset \\
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$$\mathrm{FV}(\phi) \cap \mathrm{BV}(\beta) \subseteq V_G$$
$$\mathrm{CN}(\phi) \cap \mathrm{CN}(\beta) \subseteq \mathrm{CN}(\alpha)$$
$$\mathrm{FV}(\psi) \cap \mathrm{BV}(\alpha) \subseteq V_G$$
$$\mathrm{CN}(\psi) \cap \mathrm{CN}(\alpha) \subseteq \mathrm{CN}(\beta)$$

V_G = globally synchronized behavior

Free variables $\mathrm{FV}(\cdot)$, bound variables $\mathrm{BV}(\cdot)$, read or written channels $\mathrm{CN}(\cdot)$, and globally synchronized variables V_G

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$V_G =$ globally synchronized behavior
$+$ incl. global time

Free variables $\mathrm{FV}(\cdot)$, bound variables $\mathrm{BV}(\cdot)$, read or written channels $\mathrm{CN}(\cdot)$, and globally synchronized variables V_G

The Downside of Schematic Proof Calculi

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- In case of parallel (hybrid) systems, side e and subtle

$$\frac{[\alpha]\phi \qquad [\beta]\psi}{[\alpha \parallel \beta](\phi \wedge \psi)}\,(\star\star)$$

Schematic proof rules with side conditions cause large soundness-critical prover kernels

$$\mathsf{FV}(\phi) \cap \mathsf{BV}(\beta) \subseteq V_G$$

$$\mathsf{CN}(\phi) \cap \mathsf{CN}($$

ized behavior

$$\mathsf{FV}(\psi) \cap \mathsf{BV}(\alpha) \subseteq V_G$$

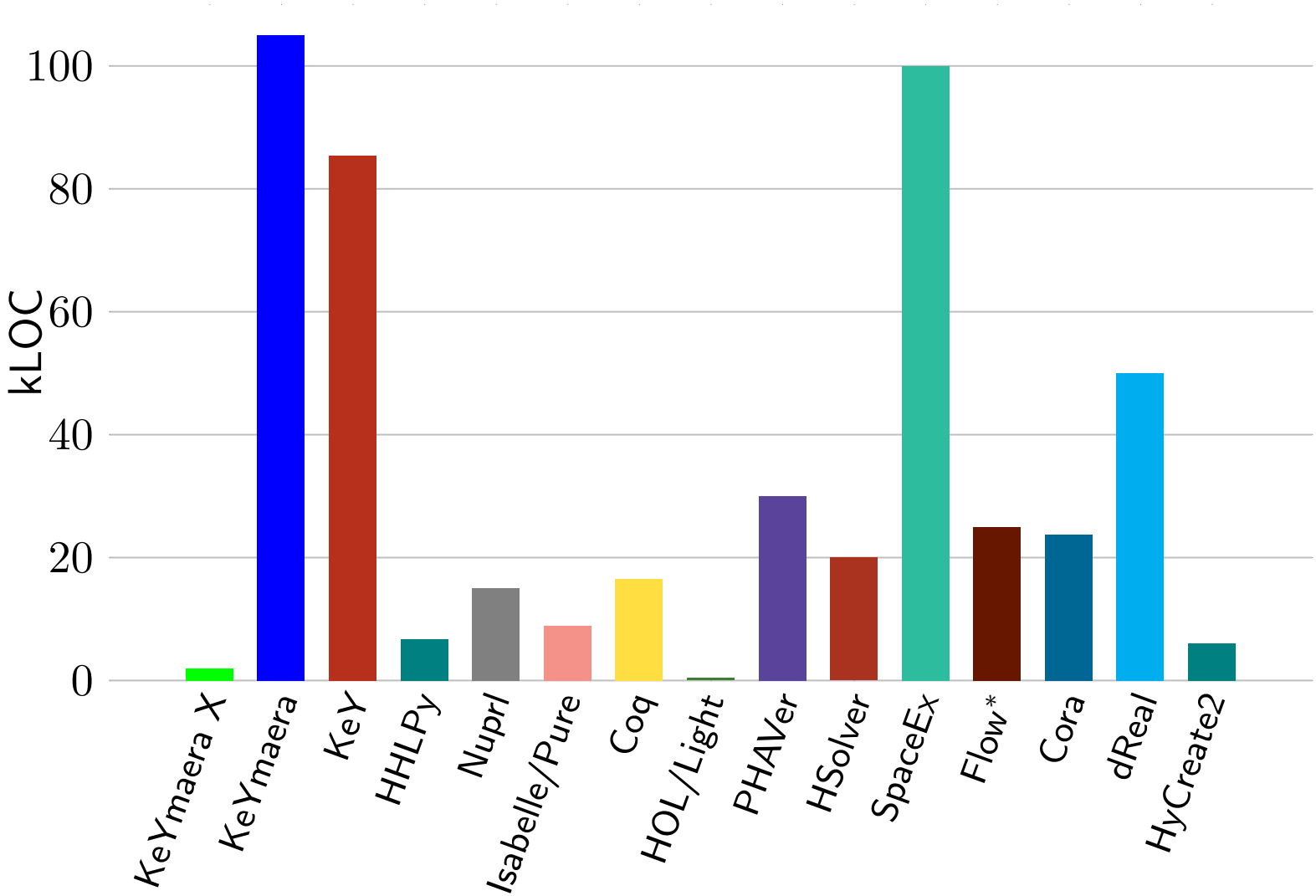
le

$$\mathsf{CN}(\psi) \cap \mathsf{CN}(\alpha) \subseteq \mathsf{CN}(\beta)$$

Uniform substitution is to the rescue

Free variables $\mathsf{FV}(\cdot)$, bound variables $\mathsf{BV}(\cdot)$, read or written channels $\mathsf{CN}(\cdot)$, and globally synchronized variables V_G

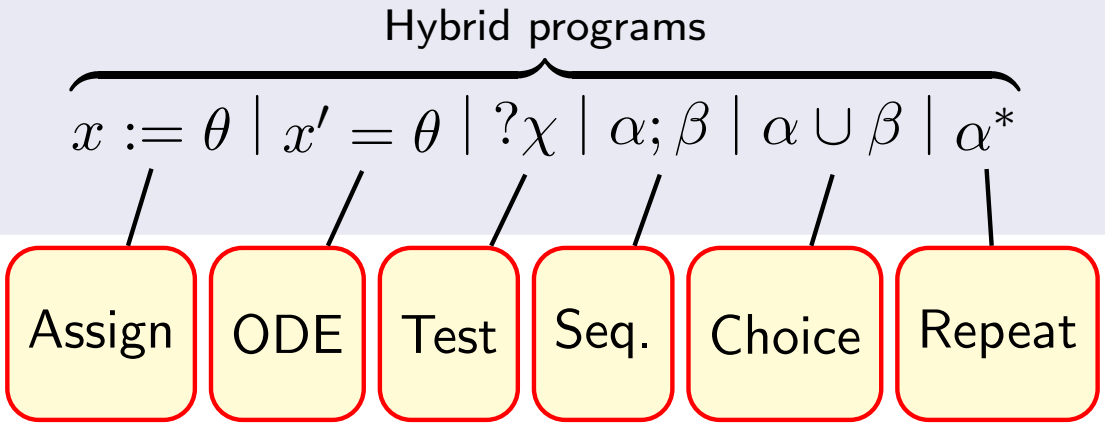
Sound Microkernels for Theorem Provers



Disclaimer: self-reported estimates

Dynamic Logic of Communicating Hybrid Programs $d\mathcal{L}_{\text{CHP}}$

Definition: Communicating hybrid programs



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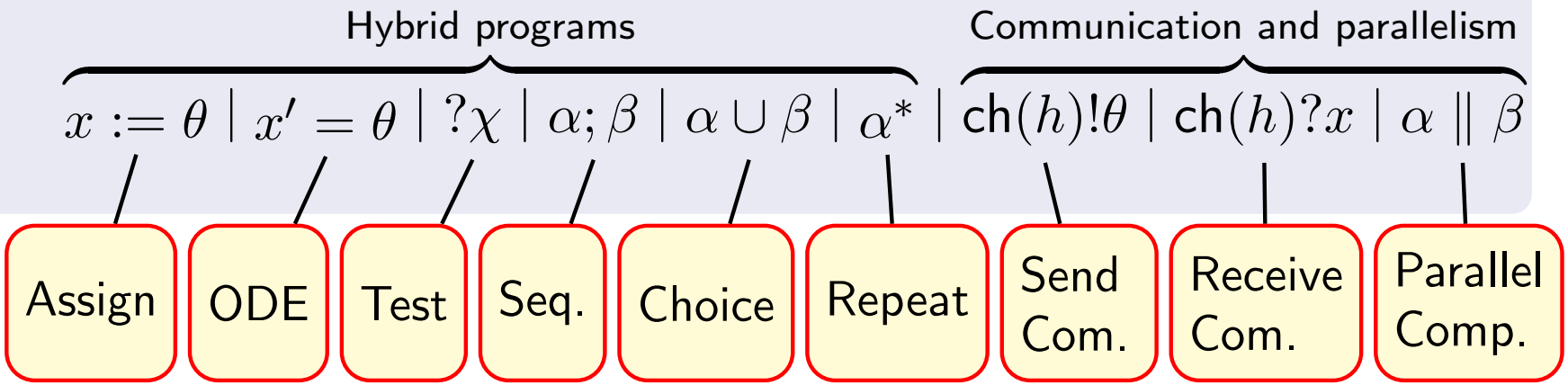

Dynamic Logic of Communicating Hybrid Programs $d\mathcal{L}_{\text{CHP}}$

Definition: Communicating hybrid programs

Hybrid programs

Communication and parallelism

$$a(\!|Y, \bar{z}|\!) \mid x := \theta \mid x' = \theta \mid ?\chi \mid \alpha; \beta \mid \alpha \cup \beta \mid \alpha^* \mid \mathsf{ch}(h)!\theta \mid \mathsf{ch}(h)?x \mid \alpha \parallel \beta$$

- Prog. Const.
- Assign
- ODE
- Test
- Seq.
- Choice
- Repeat
- Send Com.
- Receive Com.
- Parallel Comp.

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Definition: Dynamic assumption-commitment logic

First-order logic

$$\underbrace{e_1 \sim e_2 \mid \neg\varphi \mid \varphi \wedge \psi \mid \forall x\, \varphi}_{\text{First-order logic}}$$

- Forall

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$$\overbrace{e_1 \sim e_2 \mid \neg\varphi \mid \varphi \wedge \psi \mid \forall x\, \varphi}^{\text{First-order logic}} \mid \overbrace{[\alpha]\psi \mid [\alpha]_{\{\mathsf{A},\mathsf{C}\}}\psi}^{\text{Dynamic modalities}}$$

- Forall
- All Prog. Runs
- All Runs in Env.

Dynamic Logic of Communicating Hybrid Programs $d\mathcal{L}_{\text{CHP}}$

Definition: Communicating hybrid programs

$$a(\!|Y, \bar{z}|\!) \mid \overbrace{x := \theta \mid x' = \theta \mid ?\chi \mid \alpha;\beta \mid \alpha \cup \beta \mid \alpha^*}^{\text{Hybrid programs}} \mid \overbrace{\mathsf{ch}(h)!\theta \mid \mathsf{ch}(h)?x \mid \alpha \parallel \beta}^{\text{Communication and parallelism}}$$

Prog. Const. | Assign | ODE | Test | Seq. | Choice | Repeat | Send Com. | Receive Com. | Parallel Comp.

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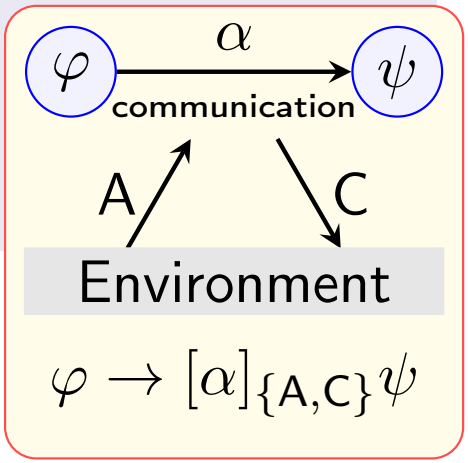
Forall | All Prog. Runs | All Runs in Env.



Dynamic Logic of Communicating Hybrid Programs $d\mathcal{L}_{\text{CHP}}$

Definition: Communicating hybrid programs

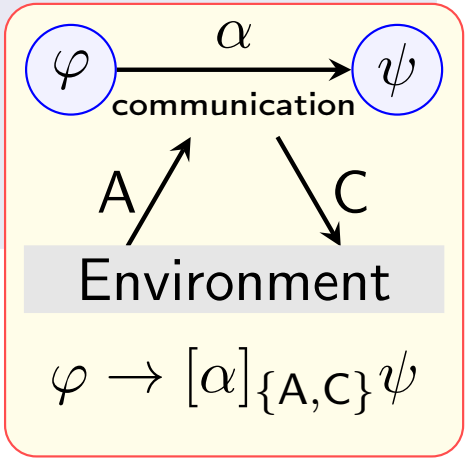
$$a(\!|Y,\bar{z}|\!) \mid \overbrace{x := \theta \mid x' = \theta \mid ?\chi \mid \alpha;\beta \mid \alpha \cup \beta \mid \alpha^*}^{\text{Hybrid programs}} \mid \overbrace{\mathsf{ch}(h)!\theta \mid \mathsf{ch}(h)?x \mid \alpha \parallel \beta}^{\text{Communication and parallelism}}$$

Prog. Const. | Assign | ODE | Test | Seq. | Choice | Repeat | Send Com. | Receive Com. | Parallel Comp.

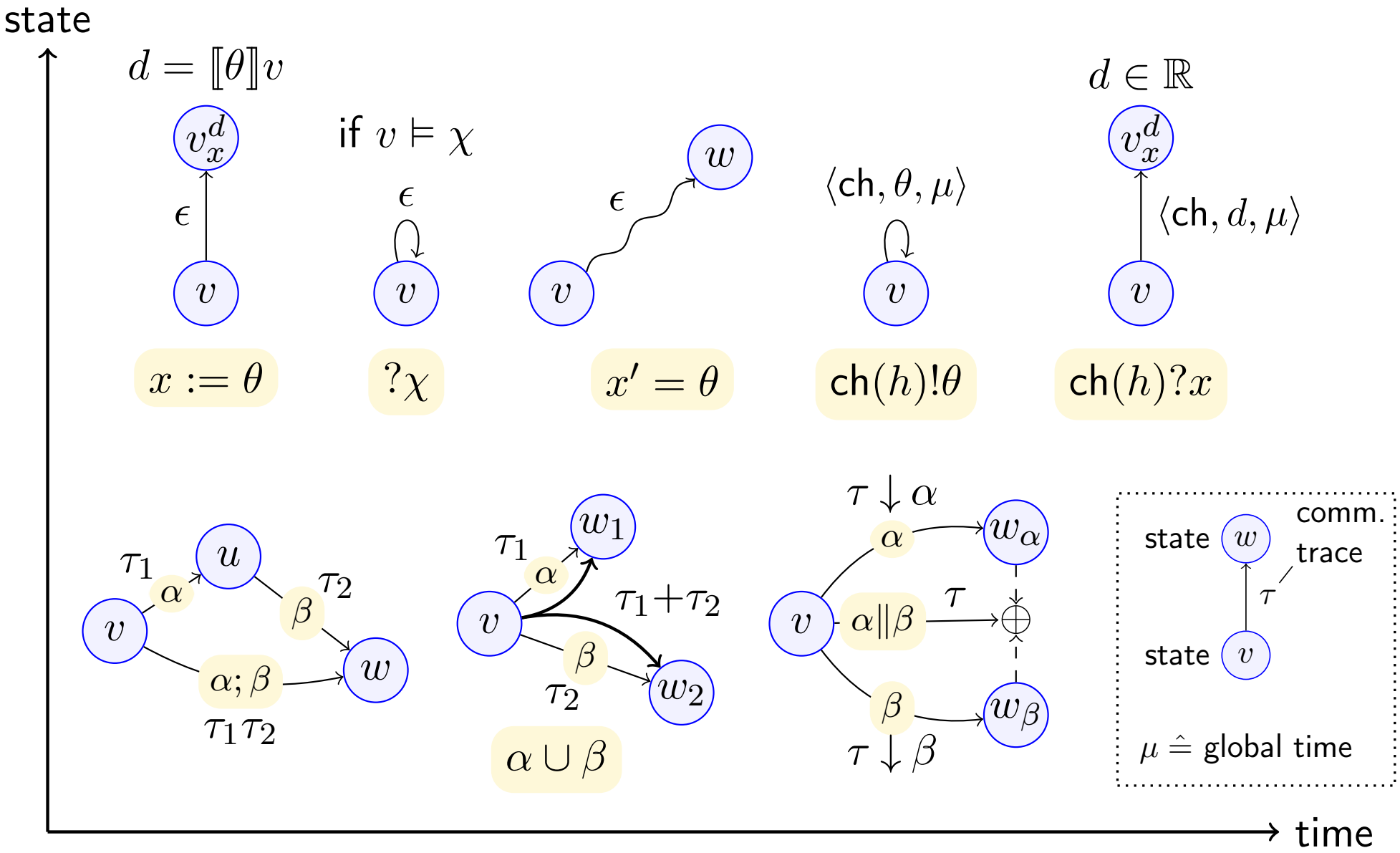
Definition: Dynamic assumption-commitment logic

$$p(Y,\bar{e}) \mid \overbrace{e_1 \sim e_2 \mid \neg\varphi \mid \varphi \wedge \psi \mid \forall x\, \varphi}^{\text{First-order logic}} \mid \overbrace{[\alpha]\psi \mid [\alpha]_{\{\mathsf{A},\mathsf{C}\}}\psi}^{\text{Dynamic modalities}}$$

Pred. Symb. | Forall | All Prog. Runs | All Runs in Env.



Semantics of Communicating Hybrid Programs



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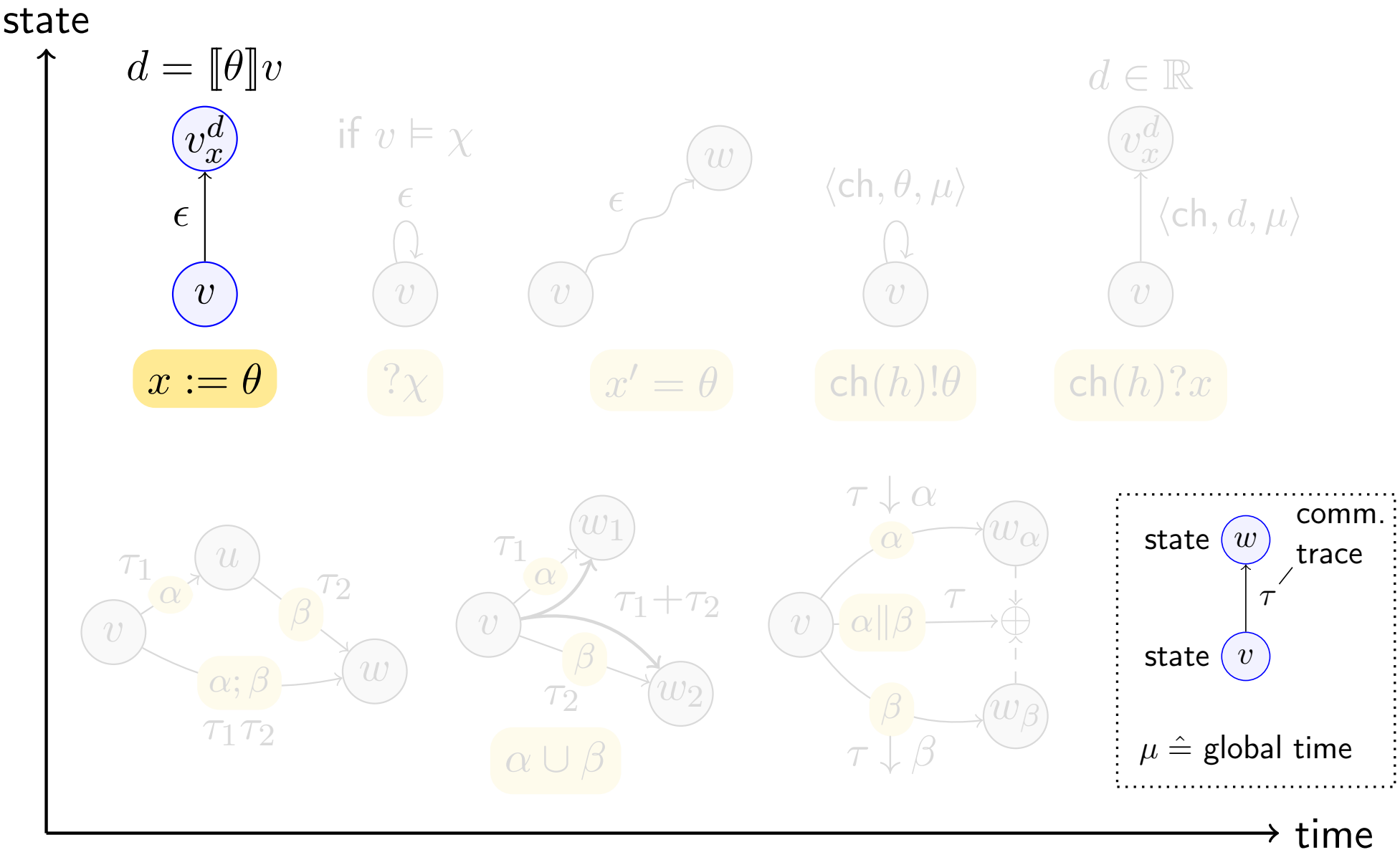

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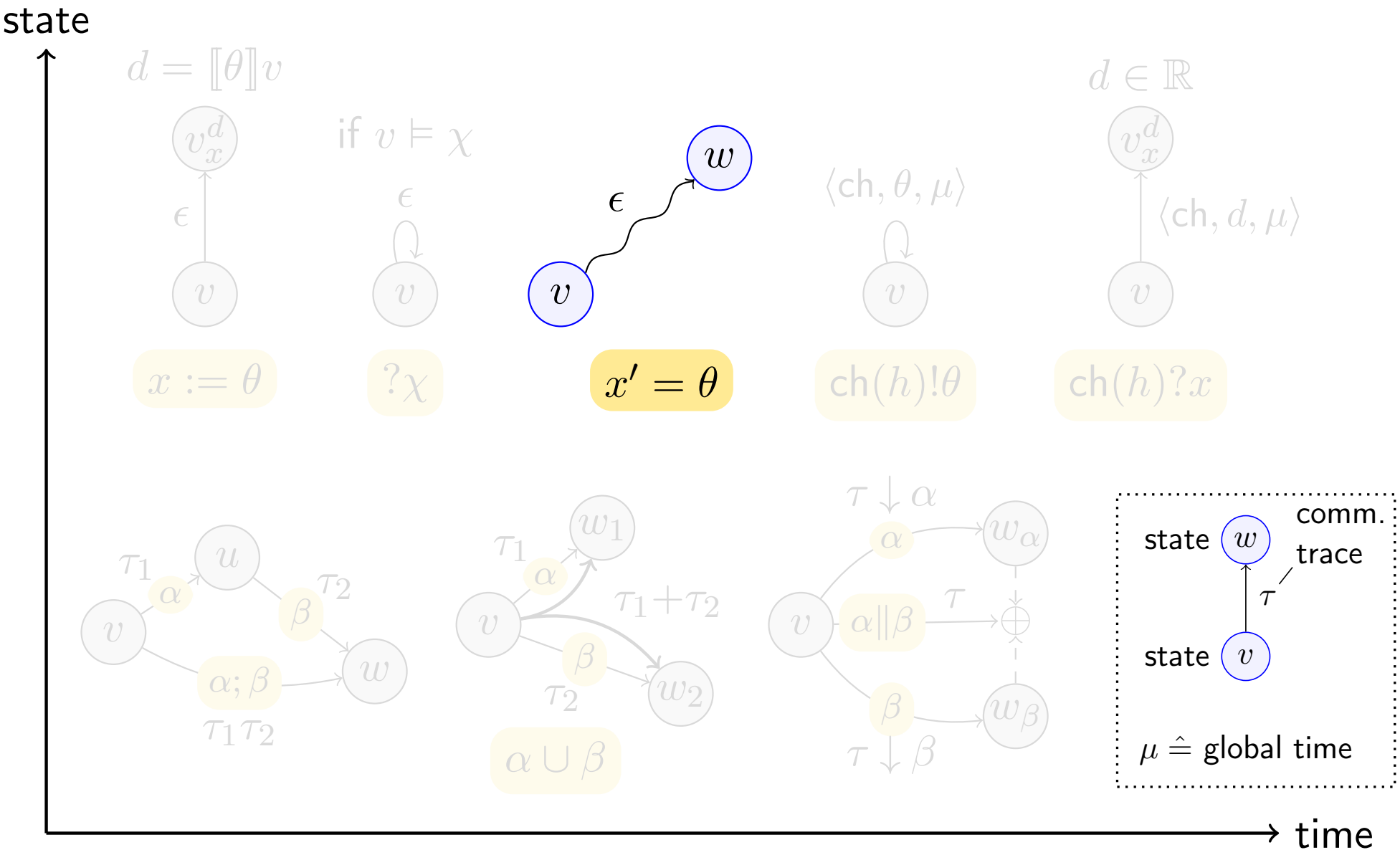

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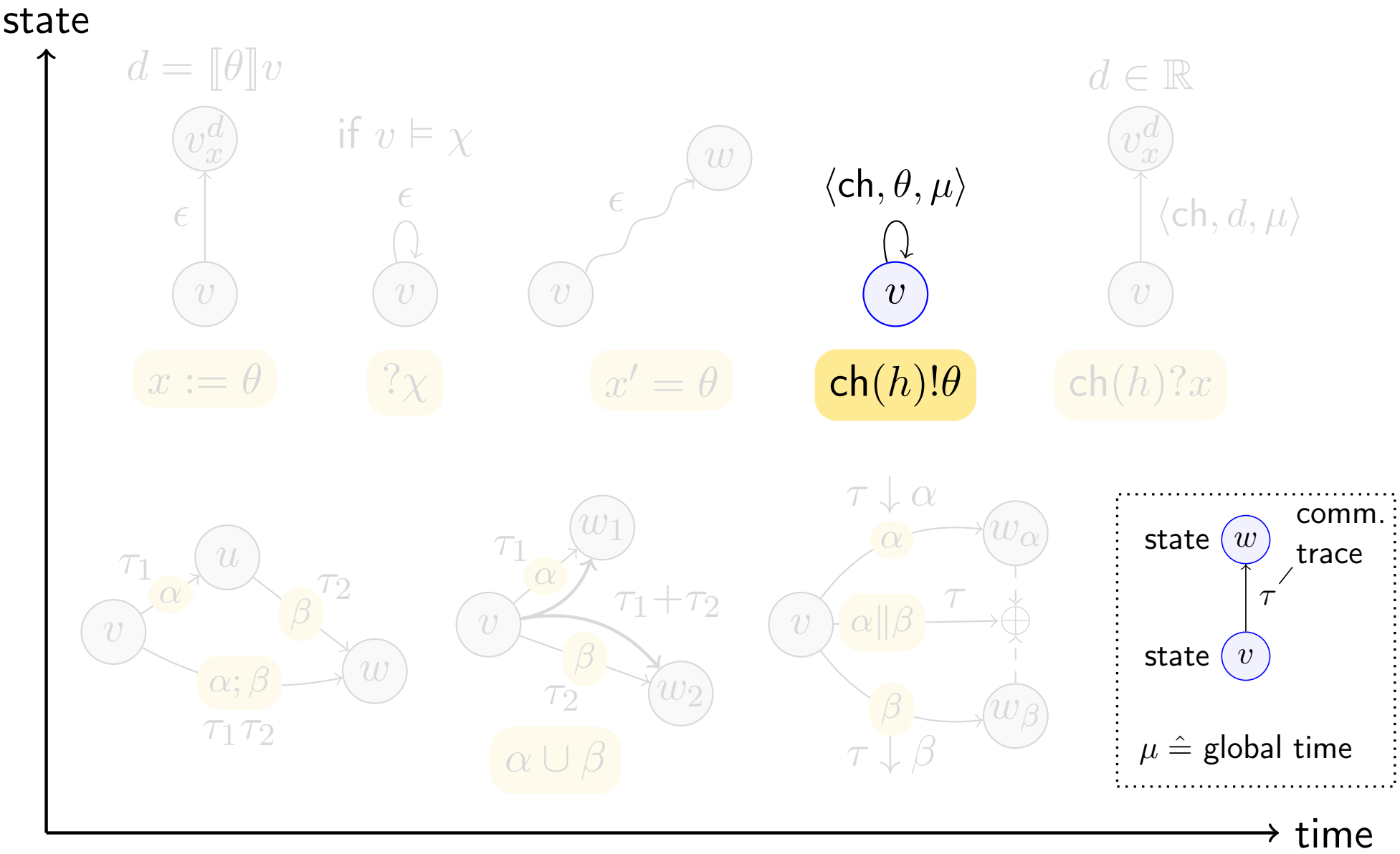

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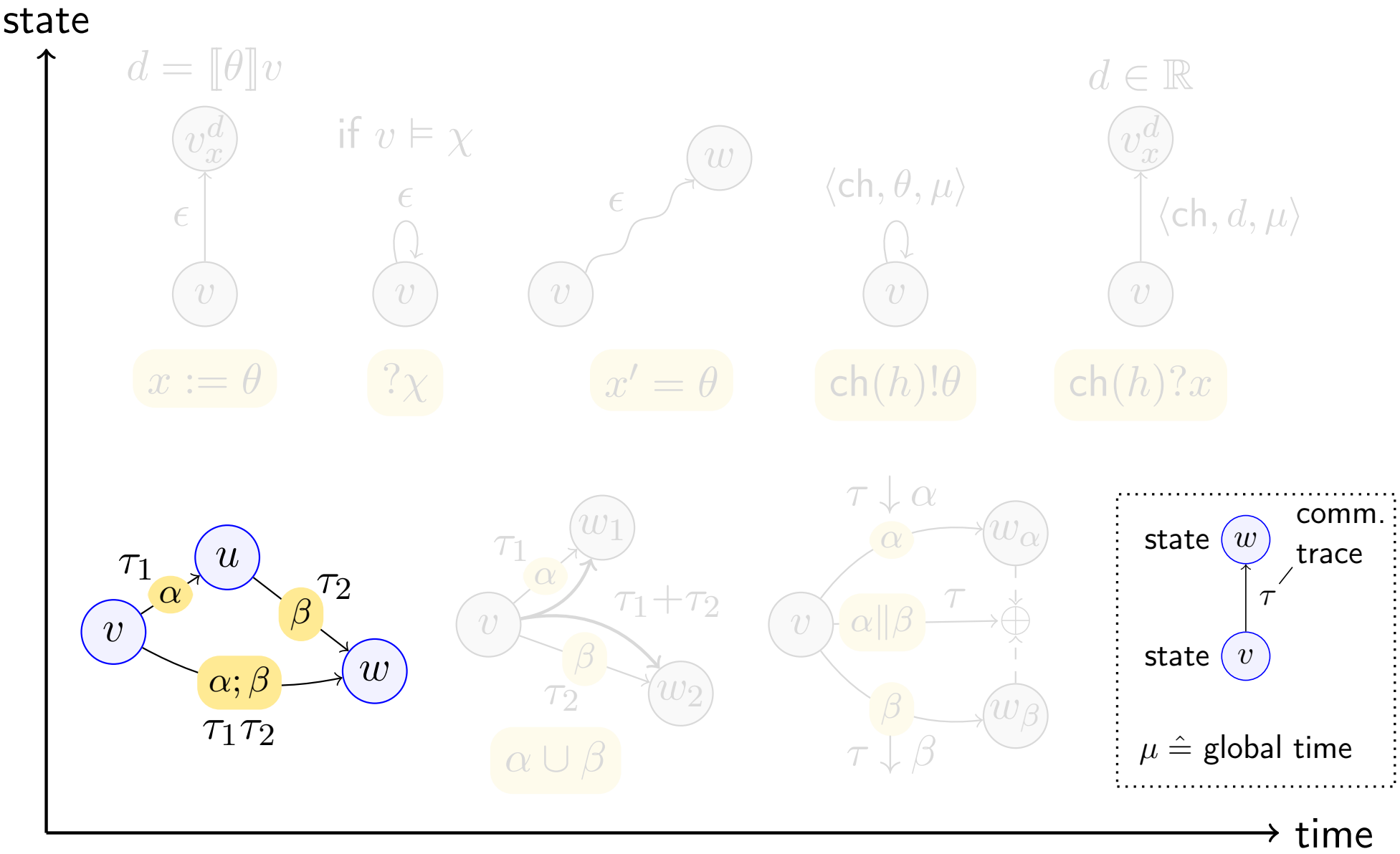

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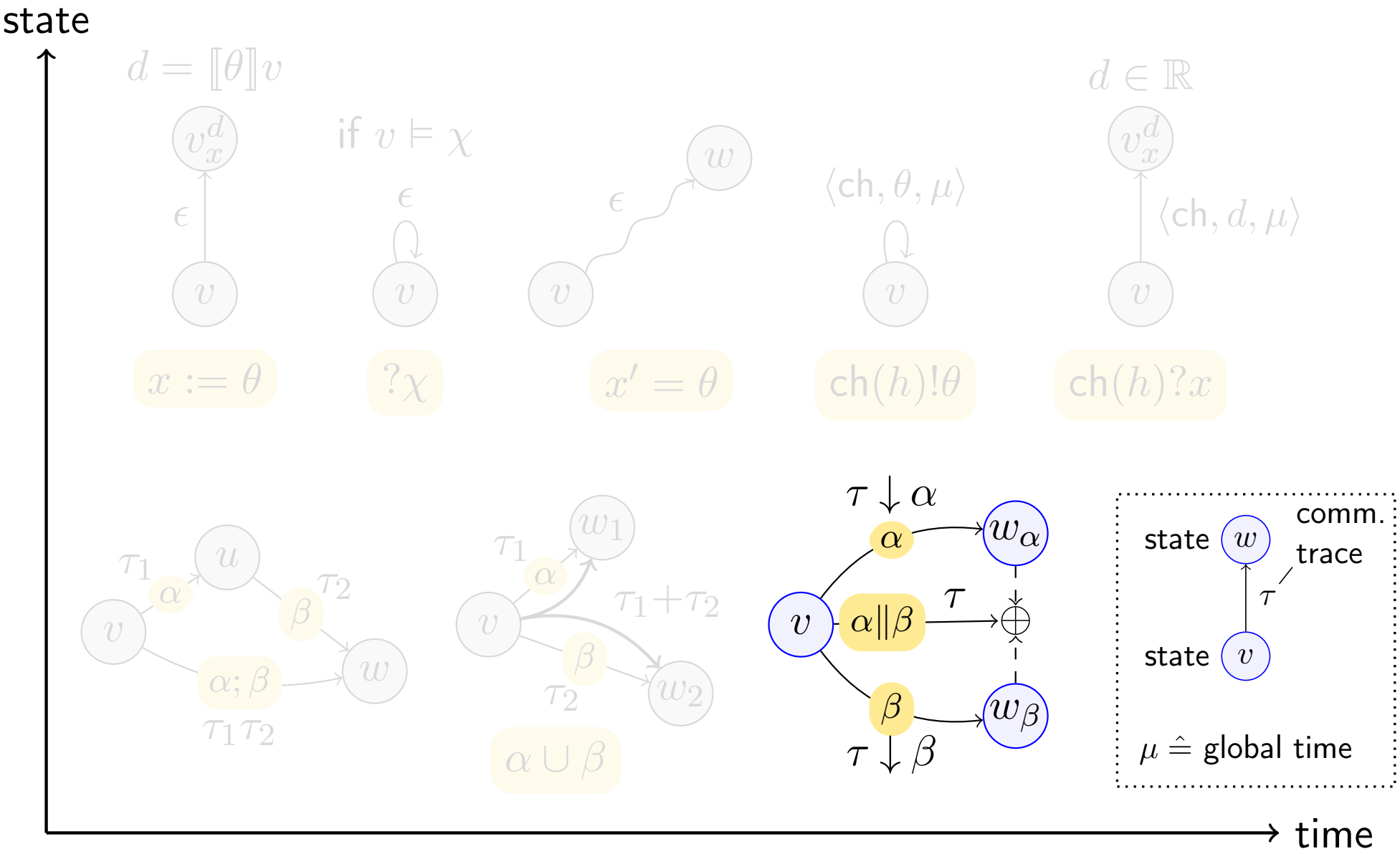

Axiomatization of $d\mathcal{L}_{\mathsf{CHP}}$

Axiom (one formula)

modulo symbolic (co)finite sets

$[x := f]p(x) \leftrightarrow p(f)$

$[?q]p \leftrightarrow (q \rightarrow p)$

$[\mathsf{ch}(h)!\theta]p(\mathsf{ch},h)$

$\quad \leftrightarrow \forall h_0\,(h_0 = h \cdot \langle \mathsf{ch}, \theta, \mu\rangle \rightarrow p(\mathsf{ch},h_0))$

$[\mathsf{ch}(h)?x]_{\{\mathsf{A},\mathsf{C}\}}p(\mathsf{ch},h,x)$

$\quad \leftrightarrow [x := *][\mathsf{ch}(h)!x]_{\{\mathsf{A},\mathsf{C}\}}p(\mathsf{ch},h,x)$

$[a;b]p(Y,\bar{z}) \leftrightarrow [a][b]p(Y,\bar{z})$

Axiom schema (∞ formulas)

for all $x, \theta, \psi, \chi, \alpha, \beta$

$[x := \theta]\psi(x) \leftrightarrow \psi(\theta)$

$[?\chi]\psi \leftrightarrow (\chi \rightarrow \psi)$

$[\mathsf{ch}(h)!\theta]\psi(h)$

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(where h_0 is fresh)

$[\mathsf{ch}(h)?x]_{\{\mathsf{A},\mathsf{C}\}}\psi$

$\quad \leftrightarrow [x := *][\mathsf{ch}(h)!x]_{\{\mathsf{A},\mathsf{C}\}}\psi$

$[\alpha;\beta]\psi \leftrightarrow [\alpha][\beta]\psi$

Axiomatization of $d\mathcal{L}_{\mathsf{CHP}}$

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Axiomatization of $d\mathcal{L}_{\mathsf{CHP}}$

Axiom (one formula)		Axiom schema (∞ formulas)
modulo symbolic (co)finite sets		for all $x, \theta, \psi, \chi, \alpha, \beta$
$[x := f]p(x) \leftrightarrow p(f)$	uniform substitution	$[x := \theta]\psi(x) \leftrightarrow \psi(\theta)$
$[?q]p \leftrightarrow (q \to p)$		$[?\chi]\psi \leftrightarrow (\chi \to \psi)$
$[\mathsf{ch}(h)!\theta]p(\mathsf{ch}, h)$		$[\mathsf{ch}(h)!\theta]\psi(h)$
$\leftrightarrow \forall h_0\,(h_0 = h \cdot \langle \mathsf{ch}, \theta, \mu\rangle \to p(\mathsf{ch}, h_0))$		$\leftrightarrow \forall h_0\,(h_0 = h \cdot \langle \mathsf{ch}, \theta, \mu\rangle \to \psi(h_0))$
		(where h_0 is fresh)
$[\mathsf{ch}(h)?x]_{\{\mathsf{A},\mathsf{C}\}}p(\mathsf{ch}, h, x)$		$[\mathsf{ch}(h)?x]_{\{\mathsf{A},\mathsf{C}\}}\psi$
$\leftrightarrow [x := *][\mathsf{ch}(h)!x]_{\{\mathsf{A},\mathsf{C}\}}p(\mathsf{ch}, h, x)$		$\leftrightarrow [x := *][\mathsf{ch}(h)!x]_{\{\mathsf{A},\mathsf{C}\}}\psi$
$[a; b]p(Y, \bar{z}) \leftrightarrow [a][b]p(Y, \bar{z})$		$[\alpha; \beta]\psi \leftrightarrow [\alpha][\beta]\psi$

Axiomatization of $d\mathcal{L}_{\mathsf{CHP}}$

Axiom (one formula)		Axiom schema (∞ formulas)
modulo symbolic (co)finite sets		for all $x, \theta, \psi, \chi, \alpha, \beta$
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$[?q]p \leftrightarrow (q \to p)$		$[?\chi]\psi \leftrightarrow (\chi \to \psi)$
$[\mathsf{ch}(h)!\theta]p(\mathsf{ch}, h)$ $\leftrightarrow \forall h_0\,(h_0 = h \cdot \langle \mathsf{ch}, \theta, \mu\rangle \to p(\mathsf{ch}, h_0))$		$[\mathsf{ch}(h)!\theta]\psi(h)$ $\leftrightarrow \forall h_0\,(h_0 = h \cdot \langle \mathsf{ch}, \theta, \mu\rangle \to \psi(h_0))$ (where h_0 is fresh)
$[\mathsf{ch}(h)?x]_{\{\mathsf{A},\mathsf{C}\}}p(\mathsf{ch}, h, x)$ $\leftrightarrow [x := *][\mathsf{ch}(h)!x]_{\{\mathsf{A},\mathsf{C}\}}p(\mathsf{ch}, h, x)$		$[\mathsf{ch}(h)?x]_{\{\mathsf{A},\mathsf{C}\}}\psi$ $\leftrightarrow [x := *][\mathsf{ch}(h)!x]_{\{\mathsf{A},\mathsf{C}\}}\psi$
$[a; b]p(Y, \bar{z}) \leftrightarrow [a][b]p(Y, \bar{z})$		$[\alpha; \beta]\psi \leftrightarrow [\alpha][\beta]\psi$

Axiomatization of $d\mathcal{L}_{\mathsf{CHP}}$

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modulo symbolic (co)finite sets

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$$\leftrightarrow [x := *][\mathsf{ch}(h)!x]_{\{\mathsf{A},\mathsf{C}\}}p(\mathsf{ch}, h, x)$$

$$[a; b]p(Y, \bar{z}) \leftrightarrow [a][b]p(Y, \bar{z})$$

uniform substitution

Axiom schema (∞ formulas)

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$$[?\chi]\psi \leftrightarrow (\chi \to \psi)$$

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$$\leftrightarrow \forall h_0\,(h_0 = h \cdot \langle \mathsf{ch}, \theta, \mu\rangle \to \psi(h_0))$$

(where h_0 is fresh)

$$[\mathsf{ch}(h)?x]_{\{\mathsf{A},\mathsf{C}\}}\psi$$

$$\leftrightarrow [x := *][\mathsf{ch}(h)!x]_{\{\mathsf{A},\mathsf{C}\}}\psi$$

$$[\alpha; \beta]\psi \leftrightarrow [\alpha][\beta]\psi$$

implementation by object instances

kLOC

algorithmic implementation

Uniform Substitution for $d\mathcal{L}_{\text{CHP}}$

Theorem (substitution σ maps symbols to terms, formulas, or programs)

$$\frac{\phi}{\sigma\phi}\ \text{US}$$

provided for each operation $\otimes(e)$
and program constant $a(\!|Y,\bar{z}|\!)$ *in* ϕ*:*

(B I) $\mathsf{FV}(\sigma|_{\Sigma(e)}) \cap \mathsf{BV}(\otimes(\cdot)) = \emptyset$ *and* $\mathsf{CN}(\sigma|_{\Sigma(e)}) \cap \mathsf{CN}(\otimes(\cdot)) = \emptyset$

(B II) $\mathsf{BV}(\sigma a) \subseteq \mathsf{BV}(a(\!|Y,\bar{z}|\!))$ *and* $\mathsf{CN}(\sigma a) = \mathsf{CN}(a(\!|Y,\bar{z}|\!))$

Uniform Substitution for $d\mathcal{L}_{\text{CHP}}$

Theorem (substitution σ maps symbols to terms, formulas, or programs)

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(B II) $\mathsf{BV}(\sigma a) \subseteq \mathsf{BV}(a(\!|Y,\bar{z}|\!))$ *and* $\mathsf{CN}(\sigma a) = \mathsf{CN}(a(\!|Y,\bar{z}|\!))$

Uniform substitution is sound if [FOL: Church, $d\mathcal{L}$: Platzer]

(B I) it never introduces **free variables** or **channel access** into a **context** where the variable or channel is **written**

(B II) it never extends **bound variables** or **writes channels** beyond the **original** scope

Uniform Substitution for $d\mathcal{L}_{\text{CHP}}$

Theorem (substitution σ maps symbols to terms, formulas, or programs)

$$\frac{\phi}{\sigma\phi}\ \text{US}$$ *provided for each operation* $\otimes(e)$ *and program constant* $a(\!|Y,\bar{z}|\!)$ *in* ϕ*:*

(B I) $\mathrm{FV}(\sigma|_{\Sigma(e)}) \cap \mathrm{BV}(\otimes(\cdot)) = \emptyset$ *and* $\mathrm{CN}(\sigma|_{\Sigma(e)}) \cap \mathrm{CN}(\otimes(\cdot)) = \emptyset$

(B II) $\mathrm{BV}(\sigma a) \subseteq \mathrm{BV}(a(\!|Y,\bar{z}|\!))$ *and* $\mathrm{CN}(\sigma a) = \mathrm{CN}(a(\!|Y,\bar{z}|\!))$

Uniform substitution is sound if

(B I) it never ... **ariables** or **channel access** into a **c** ...riable or channel is **written**

(B II) it never extends **bound variables** or **writes channels** beyond the **original** scope

Don't release the context from synchronization!

Uniform Substitution for $d\mathcal{L}_{\text{CHP}}$

Theorem (substitution σ maps symbols to terms, formulas, or programs)

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provided for each operation $\otimes(e)$
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(B II) $\mathsf{BV}(\sigma a) \subseteq \mathsf{BV}(a(\!|Y,\bar{z}|\!))$ *and* $\mathsf{CN}(\sigma a) = \mathsf{CN}(a(\!|Y,\bar{z}|\!))$

$$\frac{[a;b]p(Y,\bar{z}) \leftrightarrow [a][b]p(Y,\bar{z})}{[\mathsf{ch}(h)?v;\{x'=v\}]x>0 \leftrightarrow [\mathsf{ch}(h)?v][\{x'=v\}]x>0}\ \text{US}$$

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Uniform substitution is sound if

(B I) it never introduces **free variables** or **channel access** into a **context** where the variable or channel is **written**

$$\frac{p(h) \rightarrow [a]p(h)}{|h \downarrow \mathsf{ch}| = 0 \rightarrow [\mathsf{ch}(h)!\theta]|h \downarrow \mathsf{ch}| = 0}\ \text{clash}$$

free in a context where it is bound

Uniform Substitution for $d\mathcal{L}_{\text{CHP}}$

Theorem (substitution σ maps symbols to terms, formulas, or programs)

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Uniform substitution is sound if

(B I) it never introduces **free variables** or **channel access** into a **context** where the variable or channel is **written**

$$\frac{p(h) \to [a]p(h)}{|h\downarrow \mathsf{dh}| = 0 \to [\mathsf{ch}(h)!\theta]|h\downarrow \mathsf{dh}| = 0}\ \text{US}$$

Parallel Injection Axiom

$$\frac{[\alpha]\phi \qquad [\beta]\psi}{[\alpha \parallel \beta](\phi \wedge \psi)}\ (\star\star) \quad \xrightarrow[\text{per branch}]{\text{replace by}} \quad [\alpha]\psi \rightarrow [\alpha \parallel \beta]\psi\ (\star)$$

($\star$) β does not affect ψ

Parallel Injection Axiom

$$\frac{[\alpha]\phi \qquad [\beta]\psi}{[\alpha \parallel \beta](\phi \wedge \psi)}\ (\star\star) \qquad \xrightarrow{\text{replace by per branch}} \qquad [\alpha]\psi \to [\alpha \parallel \beta]\psi\ (\star)$$

Theorem

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The parallel injection axiom is sound:

$$[a(\!|Y_a, \bar{z}_a|\!)]p(Y, \bar{z}) \to [a(\!|Y_a, \bar{z}_a|\!) \parallel_{wf} b(\!|Y_b \cap (Y^{\complement} \cup Y_a), \bar{z}^{\complement}|\!)]p(Y, \bar{z})$$

where $a(\!|Y_a, \bar{z}_a|\!) \parallel_{wf} b(\!|Y_b, \bar{z}_b|\!) \equiv a(\!|Y_a, \bar{z}_a|\!) \parallel b(\!|Y_b, (\bar{z}_b \cap \bar{z}_a^{\complement}) \cup \{\mu, \mu'\} \cup V_{\mathcal{T}}|\!)$

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$$[a(\!|\{\mathsf{ch}\}, h|\!)]p(\mathsf{gh}, h) \rightarrow [a(\!|\{\mathsf{ch}\}, h|\!) \parallel_{\mathsf{wf}} b(\!|\{\mathsf{gh}\} \cap (\{\mathsf{gh}\}^{\complement} \cup \{\mathsf{ch}\}), \{h\}^{\complement}|\!)]p(\mathsf{gh}, h)$$

$$[\mathsf{ch}(h)!1]|h \downarrow \mathsf{gh}| = 1 \rightarrow [\mathsf{ch}(h)!1 \parallel \mathsf{gh}(h)!2]|h \downarrow \mathsf{gh}| = 1$$

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The parallel injection axiom is sound:

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where $a(\!|Y_a, \bar{z}_a|\!) \parallel_{wf} b(\!|Y_b, \bar{z}_b|\!) \equiv a(\!|Y_a, \bar{z}_a|\!) \parallel b(\!|Y_b, (\bar{z}_b \cap \bar{z}_a^{\complement}) \cup \{\mu, \mu'\} \cup V_{\mathcal{T}}|\!)$

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↯ clash due to (B II)

Parallel Injection Axiom

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where $a(\!|Y_a, \bar{z}_a|\!) \parallel_{wf} b(\!|Y_b, \bar{z}_b|\!) \equiv a(\!|Y_a, \bar{z}_a|\!) \parallel b(\!|Y_b, (\bar{z}_b \cap \bar{z}_a^{\complement}) \cup \{\mu, \mu'\} \cup V_{\mathcal{T}}|\!)$

$$\frac{[a(\!|\{\mathsf{ch}\}, h|\!)]p(\mathsf{ch}, h) \to [a(\!|\{\mathsf{ch}\}, h|\!) \parallel_{\mathsf{wf}} b(\!|\{\mathsf{ch}\} \cap (\{\mathsf{ch}\}^{\complement} \cup \{\mathsf{ch}\}), \{h\}^{\complement}|\!)]p(\mathsf{ch}, h)}{[\mathsf{ch}(h)!1]|h \downarrow \mathsf{ch}| = 1 \to [\mathsf{ch}(h)!1 \parallel \mathsf{ch}(h)?x]|h \downarrow \mathsf{ch}| = 1}$$

Parallel Injection Axiom

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The parallel injection axiom is sound:

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$$\frac{[a(\!|\{\mathsf{ch}\}, h|\!)]p(\mathsf{ch}, h) \rightarrow [a(\!|\{\mathsf{ch}\}, h|\!) \parallel_{\mathsf{wf}} b(\!|\{\mathsf{ch}\} \cap (\{\mathsf{ch}\}^{\complement} \cup \{\mathsf{ch}\}), \{h\}^{\complement}|\!)]p(\mathsf{ch}, h)}{[\mathsf{ch}(h)!1]|h \downarrow \mathsf{ch}| = 1 \rightarrow [\mathsf{ch}(h)!1 \parallel \mathsf{ch}(h)?x]|h \downarrow \mathsf{ch}| = 1}$$

Parallel Injection Axiom

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Theorem

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where $a(\!|Y_a, \bar{z}_a|\!) \parallel_{wf} b(\!|Y_b, \bar{z}_b|\!) \equiv a(\!|Y_a, \bar{z}_a|\!) \parallel b(\!|Y_b, (\bar{z}_b \cap \bar{z}_a^{\complement}) \cup \{\mu, \mu'\} \cup V_{\mathcal{T}}|\!)$

$$\frac{[a(\!|\{\mathsf{ch}\}, h|\!)]p(\mathsf{ch}, h) \to [a(\!|\{\mathsf{ch}\}, h|\!) \parallel_{\mathsf{wf}} b(\!|\{\mathsf{ch}\} \cap (\{\mathsf{ch}\}^{\complement} \cup \{\mathsf{ch}\}), \{h\}^{\complement}|\!)]p(\mathsf{ch}, h)}{[?\mathsf{true}]|h \downarrow \mathsf{ch}| = 1 \to [?\mathsf{true} \parallel \mathsf{ch}(h)?x]|h \downarrow \mathsf{ch}| = 1}$$

↯ clash due to (B II)

Parallel Injection Axiom

Say goodbye to schematic parallel proof rules with subtle side conditions!

All parallel systems reasoning reduces to flat **axiom + uniform substitution**!

Theorem

($\star$) β does not affect ψ

The parallel injection axiom is sound:

$$[a(\!|Y_a, \bar{z}_a|\!)]p(Y,\bar{z}) \rightarrow [a(\!|Y_a, \bar{z}_a|\!) \,\|_{wf}\, b(\!|Y_b \cap (Y^{\complement} \cup Y_a), \bar{z}^{\complement}|\!)]p(Y,\bar{z})$$

where $a(\!|Y_a, \bar{z}_a|\!) \,\|_{wf}\, b(\!|Y_b, \bar{z}_b|\!) \equiv a(\!|Y_a, \bar{z}_a|\!) \,\|\, b(\!|Y_b, (\bar{z}_b \cap \bar{z}_a^{\complement}) \cup \{\mu, \mu'\} \cup V_{\mathcal{T}}|\!)$

Uniform Substitution for $d\mathcal{L}_{\text{CHP}}$

Theorem (substitution σ maps symbols to terms, formulas, or programs)

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Uniform substitution is sound if

(B I) it never introduces **free vari** into a **context** wh **cess** **is written**

(B II) it nev **les** or **writes channels** scope

Binding free parameters sentences you to logic jail!

Application of Uniform Substitution

substitution σ $\quad$ $U \subseteq V \cup \Omega$ tabooed variables and channels

$$\begin{aligned}
\sigma^U(e_1 \sim e_2) &\equiv \sigma^U(e_1) \sim \sigma^U(e_2) \\
\sigma^U(p(Y,e)) &\equiv \{\cdot \mapsto \sigma^U(e \downarrow Y)\}^{\emptyset}(\sigma p(\cdot)) \quad \text{if } (\mathsf{FV}(\sigma p(\cdot)) \cup \mathsf{CN}(\sigma p(\cdot))) \cap U = \emptyset \\
\sigma^U(\neg\varphi) &\equiv \neg\sigma^U(\varphi) \\
\sigma^U(\varphi \wedge \psi) &\equiv \sigma^U(\varphi) \wedge \sigma^U(\psi) \\
\sigma^U(\forall z\, \varphi) &\equiv \forall z\, \sigma^{U \cup \{z\}}(\varphi) \\
\sigma^U([\alpha]_{\{\mathsf{A},\mathsf{C}\}}\psi) &\equiv [\sigma_Z^{U,\emptyset}(\alpha)]_{\{\sigma^Z(\mathsf{A}),\sigma^Z(\mathsf{C})\}}\sigma^Z(\psi)
\end{aligned}$$

$$\begin{aligned}
\sigma^U_{U \cup \mathsf{BV}(\sigma a) \cup \mathsf{CN}(\sigma a)}(a(\!|Y,\bar{z}|\!)) &\equiv \sigma a \quad \text{if } \mathsf{BV}(\sigma a) \subseteq \bar{z} \text{ and } \mathsf{CN}(\sigma a) = Y \\
\sigma^U_{U \cup \{x\}}(x := \theta) &\equiv x := \sigma^U(\theta) \\
\sigma^U_Z(\{x' = \theta\}) &\equiv \{x' = \sigma^U(\theta)\} \quad \text{with } Z = U \cup \{x, x', \mu, \mu'\} \\
\sigma^U_{U \cup \{\mathsf{ch},h\}}(\mathsf{ch}(h)!\theta) &\equiv \mathsf{ch}(h)!\sigma^U(\theta) \\
\sigma^U_{U \cup \{\mathsf{ch},h,x\}}(\mathsf{ch}(h)?x) &\equiv \mathsf{ch}(h)?x \\
\sigma^U_{B \cup Z}(\alpha \cup \beta) &\equiv \sigma^U_B(\alpha) \cup \sigma^U_Z(\beta) \\
\sigma^U_Z(\alpha;\beta) &\equiv \sigma^U_B(\alpha); \sigma^B_Z(\beta) \\
\sigma^U_{B \cup Z}(\alpha \parallel \beta) &\equiv \sigma^U_B(\alpha) \parallel \sigma^U_Z(\beta)
\end{aligned}$$

Implementation of

$$\frac{\phi}{\sigma\phi}\ \text{US} \qquad \text{(B I) + (B II)}$$

by a canonical recursion

Application of Uniform Substitution

substitution σ $U \subseteq V \cup \Omega$ tabooed variables and channels

$$\sigma^U(e_1 \sim e_2) \equiv \sigma^U(e_1) \sim \sigma^U(e_2)$$
$$\sigma^U(\mathsf{dx}(Y, e)) \equiv \{\cdot \mapsto \sigma^U(e \downarrow Y)\}^{\emptyset}(\sigma p(\cdot)) \quad \text{if } (\mathsf{FV}(\sigma p(\cdot)) \cup \mathsf{CN}(\sigma p(\cdot))) \cap U = \emptyset$$
$$\sigma^U(\neg\varphi) \equiv \neg\sigma^U(\varphi)$$
$$\sigma^U(\varphi \wedge \psi) \equiv \sigma^U(\varphi) \wedge \sigma^U(\psi)$$ homomorphic application
$$\sigma^U(\forall z\, \varphi) \equiv \forall z\, \sigma^{U \cup \{z\}}(\varphi)$$
$$\sigma^U([\alpha]_{\{\mathsf{A},\mathsf{C}\}}\psi) \equiv [\sigma_Z^{U,\emptyset}(\alpha)]_{\{\sigma^Z(\mathsf{A}),\sigma^Z(\mathsf{C})\}}\sigma^Z(\psi)$$

taboo

$$\sigma^U_{U \cup \mathsf{BV}(\sigma a) \cup \mathsf{CN}(\sigma a)}(a(\!|Y, \bar{z}|\!)) \equiv \sigma a \quad \text{if } \mathsf{BV}(\sigma a) \subseteq \bar{z} \text{ and } \mathsf{CN}(\sigma a) = Y$$
$$\sigma^U_{U \cup \{x\}}(x := \theta) \equiv x := \sigma^U(\theta)$$
$$\sigma^U_Z(\{x' = \theta\}) \equiv \{x' = \sigma^U(\theta)\} \quad \text{with } Z = U \cup \{x, x', \mu, \mu'\}$$
$$\sigma^U_{U \cup \{\mathsf{ch},h\}}(\mathsf{ch}(h)!\theta) \equiv \mathsf{ch}(h)!\sigma^U(\theta)$$
$$\sigma^U_{U \cup \{\mathsf{ch},h,x\}}(\mathsf{ch}(h)?x) \equiv \mathsf{ch}(h)?x$$
$$\sigma^U_{B \cup Z}(\alpha \cup \beta) \equiv \sigma^U_B(\alpha) \cup \sigma^U_Z(\beta)$$
$$\sigma^U_Z(\alpha; \beta) \equiv \sigma^U_B(\alpha); \sigma^B_Z(\beta)$$
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$$\sigma^U(\neg\varphi) \equiv \neg\sigma^U(\varphi)$$
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$$\sigma^U(\forall z\, \varphi) \equiv \forall z\, \sigma^{U \cup \{z\}}(\varphi)$$
$$\sigma^U([\alpha]_{\{\mathsf{A},\mathsf{C}\}}\psi) \equiv [\sigma_Z^{U,\emptyset}(\alpha)]_{\{\sigma^Z(\mathsf{A}),\sigma^Z(\mathsf{C})\}}\sigma^Z(\psi)$$

bound

$$\sigma^U_{U \cup \mathrm{BV}(\sigma a) \cup \mathrm{CN}(\sigma a)}(a(\!|Y,\bar{z}|\!)) \equiv \sigma a \quad \text{if } \mathrm{BV}(\sigma a) \subseteq \bar{z} \text{ and } \mathrm{CN}(\sigma a) = Y$$
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$$\sigma^U_{U \cup \{\mathsf{ch},h\}}(\mathsf{ch}(h)!\theta) \equiv \mathsf{ch}(h)!\sigma^U(\theta)$$
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$$\sigma^U(\neg\varphi) \equiv \neg\sigma^U(\varphi)$$

$$\sigma^U(\varphi \wedge \psi) \equiv$$

$$\sigma^U(\forall z\, \varphi) \equiv \forall z\, \sigma^{U \cup \{z\}}(\varphi)$$

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$$\sigma^U_Z(\{x' = \theta\}) \equiv \{x' = \sigma^U(\theta)\} \quad \text{with } Z = U \cup \{x, x', \mu, \mu'\}$$

$$\sigma^U_{U \cup \{\mathsf{ch},h\}}(\mathsf{ch}(h)!\theta) \equiv \mathsf{ch}(h)!\sigma^U(\theta)$$

$$\sigma^U_{U \cup \{\mathsf{ch},h,x\}}(\mathsf{ch}(h)?x) \equiv \mathsf{ch}(h)?x$$

$$\sigma^U_{B \cup Z}(\alpha \cup \beta) \equiv \sigma^U_B(\alpha) \cup \sigma^U_Z(\beta)$$

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$$\sigma^U_{U \cup \mathrm{BV}(\sigma a) \cup \mathrm{CN}(\sigma a)}(a(\!|Y,\bar{z}|\!)) \equiv \sigma a \quad \text{if } \mathrm{BV}(\sigma a) \subseteq \bar{z} \text{ and } \mathrm{CN}(\sigma a) = Y$$
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$$\sigma^U_Z(\{x' = \theta\}) \equiv \{x' = \sigma^U(\theta)\} \quad \text{with } Z = U \cup \{x, x', \mu, \mu'\}$$
$$\sigma^U_{U \cup \{\mathsf{ch},h\}}(\mathsf{ch}(h)!\theta) \equiv \mathsf{ch}(h)!\sigma^U(\theta)$$
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$$\sigma^U_{B \cup Z}(\alpha \parallel \beta) \equiv \sigma^U_B(\alpha) \parallel \sigma^U_Z(\beta)$$

input output

Implementation of

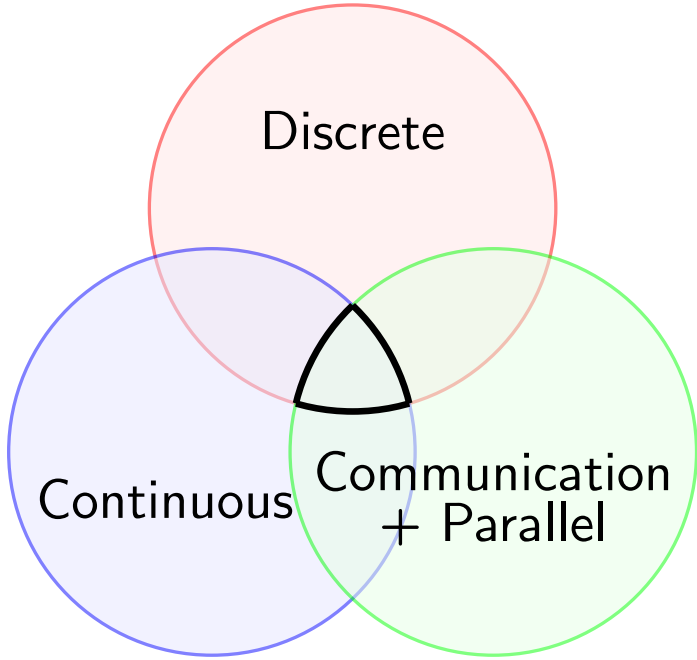
$$\frac{\phi}{\sigma\phi}\ \mathrm{US} \qquad \text{(B I) + (B II)}$$

by a canonical recursion

Uniform Substitution for $d\mathcal{L}_{\text{CHP}}$

Dynamic logic of CHPs

$d\mathcal{L}_{\text{CHP}} = d\mathcal{L}$ + CSP
+ ac-reasoning

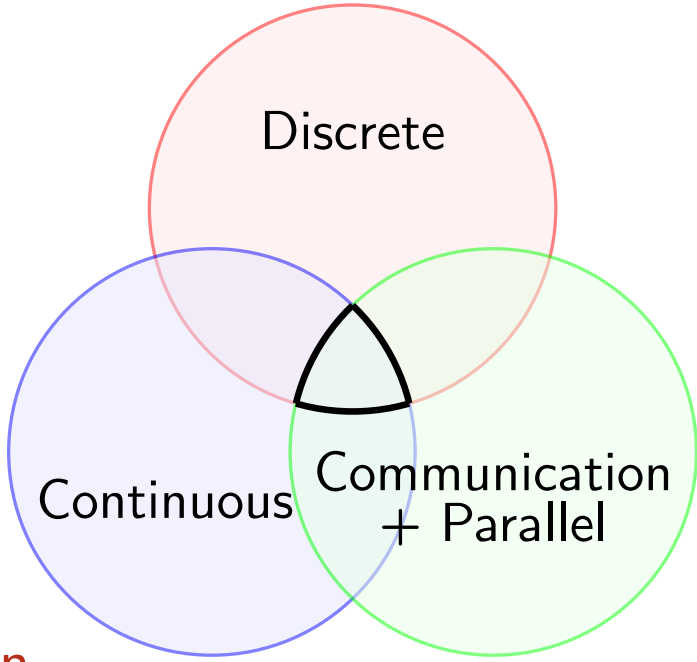


Uniform Substitution for $d\mathcal{L}_{\text{CHP}}$

Dynamic logic of CHPs

$d\mathcal{L}_{\text{CHP}} = d\mathcal{L}$ + CSP
+ ac-reasoning

- Uniform substitution for $d\mathcal{L}_{\text{CHP}}$ that operates linearly in the formulas
- Modular soundness argument
- Modular, thus smaller, prover implementation



Uniform Substitution for $d\mathcal{L}_{\text{CHP}}$

Dynamic logic of CHPs

$d\mathcal{L}_{\text{CHP}} = d\mathcal{L}$ + CSP
+ ac-reasoning

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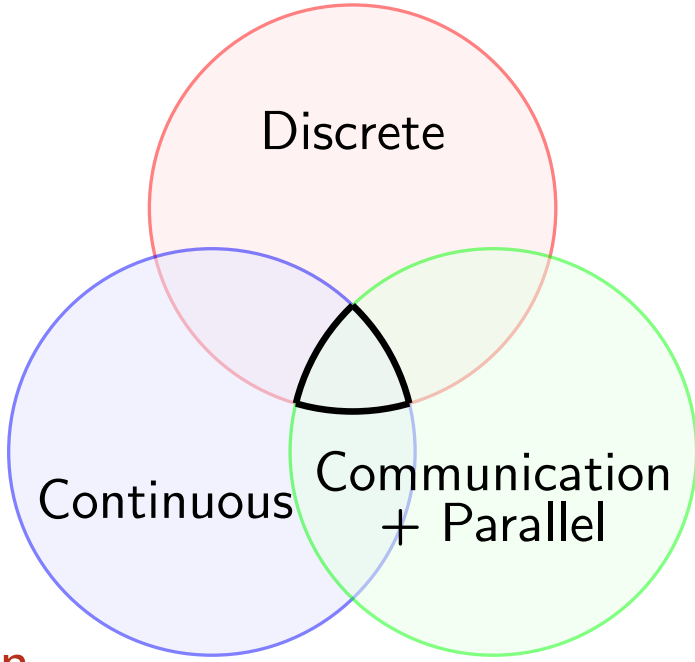

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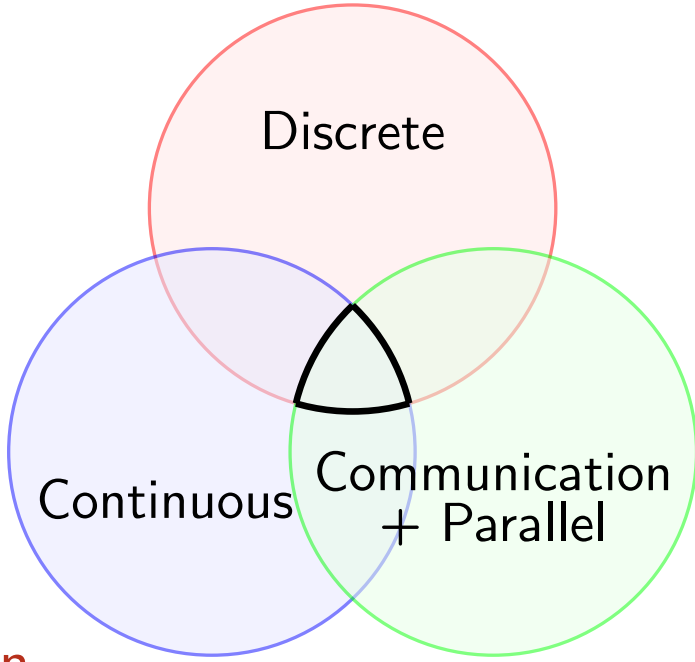

- Uniform substitution for $d\mathcal{L}_{\text{CHP}}$ that operates linearly in the formulas
- Modular soundness argument
- Modular, thus smaller, prover implementation
- Implementation in KeYmaera X: Ongoing effort shows promising progress
- All parallel reasoning reduces to multiple uses of the simple parallel injection axiom
- Discrete parallelism benefits as well

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Uniform Substitution for $d\mathcal{L}_{\text{CHP}}$

Theorem (substitution σ maps symbols to terms, formulas, or programs)

$$\frac{\phi}{\sigma\phi}\ \text{US}$$

provided for each operation $\otimes(e)$ and program constant $a(\!|Y,\bar{z}|\!)$ in ϕ:

(B I) $\mathsf{FV}(\sigma|_{\Sigma(e)}) \cap \mathsf{BV}(\otimes(\cdot)) = \emptyset$ *and* $\mathsf{CN}(\sigma|_{\Sigma(e)}) \cap \mathsf{CN}(\otimes(\cdot)) = \emptyset$

(B II) $\mathsf{BV}(\sigma a) \subseteq \mathsf{BV}(a(\!|Y,\bar{z}|\!))$ *and* $\mathsf{CN}(\sigma a) = \mathsf{CN}(a(\!|Y,\bar{z}|\!))$

Uniform substitution is sound if

(B I) it never introduces **free variables** or **channel access** into a **context** where the variable or channel is **written**

$$\frac{[t_0 := f]p(t_0) \leftrightarrow p(f)}{[t_0 := \mu][x' = \theta][\mathsf{ch}(h)!\theta]\varphi(t_0) \leftrightarrow [x' = \theta][\mathsf{ch}(h)!\theta]\varphi(\mu)}\ \text{clash}$$

free in a context where it is bound

$$\varphi(t) \equiv \mathtt{time}(h \downarrow \mathsf{ch}) = t$$

Uniform Substitution for $d\mathcal{L}_{\text{CHP}}$

Theorem (substitution σ maps symbols to terms, formulas, or programs)

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Uniform substitution is sound if

(B II) it never extends **bound variables** or **writes channels** beyond the **original** scope

$$[a(\!|\emptyset, V_{\mathbb{R}}|\!)]_{\{A,C\}} P \leftrightarrow C \wedge (A \rightarrow [a(\!|\emptyset, V_{\mathbb{R}}|\!)]P)$$

clash

$$[\mathsf{ch}(h)!\theta]_{\{\mathsf{true},|h\downarrow \mathsf{ch}|=0\}} x = 0 \leftrightarrow |h\downarrow \mathsf{ch}| = 0 \wedge (\mathsf{true} \rightarrow [\mathsf{ch}(h)!\theta]x = 0)$$

free in a context where it is bound

A, C, and P may mention channels

Program Semantics - Part I

Semantics $I[\![\alpha]\!] \subseteq \mathcal{S} \times \mathcal{T}_{\text{rec}} \times \mathcal{S}_\perp$ consists of state-trace-state triples

$I[\![a(\!|Y, \bar{z}|\!)]\!] = I(a(\!|Y, \bar{z}|\!))$

$I[\![x := \theta]\!] = \perp_{\mathcal{D}} \cup \{(v, \epsilon, w) \mid w = v_x^d \text{ where } d = Iv[\![\theta]\!]\}$

$I[\![x := *]\!] = \perp_{\mathcal{D}} \cup \{(v, \epsilon, w) \mid w = v_x^d \text{ where } d \in \mathbb{R}\}$

$I[\![?\chi]\!] = \perp_{\mathcal{D}} \cup \{(v, \epsilon, v) \mid Iv \vDash \chi\}$

$I[\![\{x' = \theta \,\&\, \chi\}]\!] = \perp_{\mathcal{D}} \cup \big\{(v, \epsilon, \varphi(s)) \mid v = \varphi(0) \text{ on } \{\mu', x'\}^{\complement},$
and $\varphi(\zeta) = \varphi(0)$ on $\{x, x', \mu, \mu'\}^{\complement}$, and $I\varphi(\zeta) \vDash \mu' = 1 \wedge x' = \theta \wedge \chi$
for all $\zeta \in [0, s]$ anda solution $\varphi : [0, s] \to \mathcal{S}$ with $\varphi(\zeta)(z') = \dfrac{d\varphi(t)(z)}{dt}(\zeta)$
for $z \in \{x, \mu\}\big\}$

where $\perp_{\mathcal{D}} = \mathcal{S} \times \{\epsilon\} \times \{\perp\}$ and $\mathcal{T}_{\text{rec}} = (V_{\mathcal{T}} \times \Omega \times \mathbb{R} \times \mathbb{R})^*$

Program Semantics - Part II

$$I[\![\mathsf{ch}(h)!\theta]\!] = \{(v,\tau,w) \mid (\tau,w) \preceq (\langle h,\mathsf{ch},d,v(\mu)\rangle, v) \text{ where } d = Iv[\![\theta]\!]\}$$

$$I[\![\mathsf{ch}(h)?x]\!] = \{(v,\tau,w) \mid (\tau,w) \preceq (\langle h,\mathsf{ch},d,v(\mu)\rangle, v^d_x) \text{ where } d \in \mathbb{R}\}$$

$$I[\![\alpha \cup \beta]\!] = I[\![\alpha]\!] \cup I[\![\beta]\!]$$

$$I[\![\alpha;\beta]\!] = I[\![\alpha]\!] \mathbin{\hat{\circ}} I[\![\beta]\!] \overset{\mathsf{def}}{=} (I[\![\alpha]\!])_\perp \cup (I[\![\alpha]\!] \triangleright I[\![\beta]\!])$$

$$I[\![\alpha^*]\!] = \bigcup_{n\in\mathbb{N}} (I[\![\alpha]\!])^n = \bigcup_{n\in\mathbb{N}} I[\![\alpha^n]\!] \quad \text{where } \alpha^0 \equiv ?\mathsf{T} \text{ and } \alpha^{n+1} = \alpha;\alpha^n$$

$$I[\![\alpha_1 \parallel \alpha_2]\!] = \left\{ (v,\tau,w_{\alpha_1} \oplus w_{\alpha_2}) \;\middle|\; \begin{array}{l} (v,\tau\downarrow\alpha_j,w_{\alpha_j}) \in I[\![\alpha_j]\!] \text{ for } j=1,2, \text{ and} \\ w_{\alpha_1}(\mu) = w_{\alpha_2}(\mu), \text{ and } \tau = \tau\downarrow(\alpha_1 \parallel \alpha_2) \end{array} \right\}$$

Static Semantics

Definition (Static semantics)

For term or formula e, and program α, free variables $\mathsf{FV}(e)$ and $\mathsf{FV}(\alpha)$, bound variables $\mathsf{BV}(\alpha)$, accessed channels $\mathsf{CN}(e)$, and written channels $\mathsf{CN}(\alpha)$ form the static semantics.

$$\mathsf{FV}(e) = \{z \in V \mid \exists I, v, \tilde{v} : v = \tilde{v} \text{ on } \{z\}^{\mathsf{C}} \text{ and } Iv[\![e]\!] \neq I\tilde{v}[\![e]\!]\}$$

$$\mathsf{CN}(e) = \{\mathsf{ch} \in \Omega \mid \exists I, v, \tilde{v} : v \downarrow \{\mathsf{ch}\}^{\mathsf{C}} = \tilde{v} \downarrow \{\mathsf{ch}\}^{\mathsf{C}} \text{ and } Iv[\![e]\!] \neq I\tilde{v}[\![e]\!]\}$$

$$\mathsf{FV}(\alpha) = \{z \in V \mid \exists I, v, \tilde{v}, \tau, w : v = \tilde{v} \text{ on } \{z\}^{\mathsf{C}} \text{ and } (v, \tau, w) \in I[\![\alpha]\!], \text{ and there is no } (\tilde{v}, \tilde{\tau}, \tilde{w}) \in I[\![\alpha]\!] : \tilde{\tau} = \tau \text{ and } w = \tilde{w} \text{ on } \{z\}^{\mathsf{C}}\}$$

$$\mathsf{BV}(\alpha) = \{z \in V \mid \exists I, (v, \tau, w) \in I[\![\alpha]\!] : w \neq \bot \text{ and } (w \cdot \tau)(z) \neq v(z)\}$$

$$\mathsf{CN}(\alpha) = \{\mathsf{ch} \in \Omega \mid \exists I, (v, \tau, w) \in I[\![\alpha]\!] : \tau \downarrow \{\mathsf{ch}\} \neq \epsilon\}$$

Communication-aware Coincidence

Lemma (Bound effect property)

$\mathsf{BV}(\alpha)$ *and* $\mathsf{CN}(\alpha)$ *are the smallest sets with the bound effect property for program* α*. That is,* $v = w$ *on* $V_{\mathcal{T}}$ *and* $v = w \cdot \tau$ *on* $\mathsf{BV}(\alpha)^{\complement}$ *if* $w \neq \bot$*, and* $\tau \downarrow \mathsf{CN}(\alpha)^{\complement} = \epsilon$ *for all* $(v, \tau, w) \in I[\![\alpha]\!]$*.*

Lemma (Coincidence for terms and formulas)

$\mathsf{FV}(e)$ *and* $\mathsf{CN}(e)$ *are the smallest sets with the communication-aware coincidence property for term or formula* e*: If* $v \downarrow \mathsf{CN}(e) = \tilde{v} \downarrow \mathsf{CN}(e)$ *on* $\mathsf{FV}(e)$ *and* $I = J$ *on* $\Sigma(e)$*, then* $Iv[\![e]\!] = J\tilde{v}[\![e]\!]$*.*

Lemma (Coincidence for programs)

$\mathsf{FV}(\alpha)$ *is the smallest set with the coincidence property for program* α*: If* $v = \tilde{v}$ *on* $X \supseteq \mathsf{FV}(\alpha)$*, and* $I = J$ *on* $\Sigma(\alpha)$*, and* $(v, \tau, w) \in I[\![\alpha]\!]$*, then* $\exists (\tilde{v}, \tilde{\tau}, \tilde{w}) \in J[\![\alpha]\!] : w = \tilde{w}$ *on* X*, and* $\tau = \tilde{\tau}$*, and* ($w = \bot$ *iff* $\tilde{w} = \bot$)*.*

Soundness argument

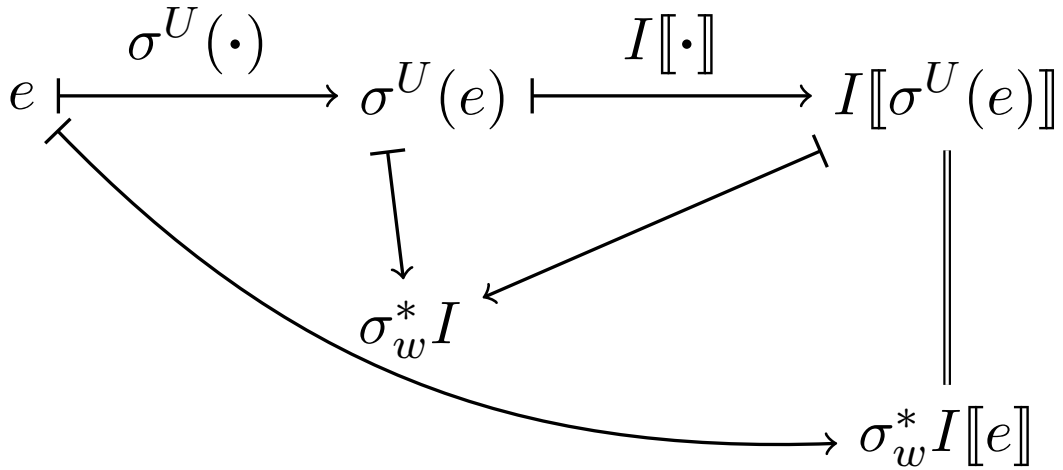
Lemma (Semantic uniform substitution)

Terms e, and formulas ϕ, and programs α evaluate equally under substitution σ^U and adjoint interpretation $\sigma_w^* I$ for all U-variations v of w:

$$I[\![\sigma^U(e)]\!] = \sigma_w^* I[\![e]\!]$$

$$Iv \vDash \sigma^U(\phi) \text{ iff } \sigma_w^* Iv \vDash \phi$$

$$(v, \tau, o) \in I[\![\sigma_Z^U(\alpha)]\!] \text{ iff } (v, \tau, o) \in \alpha[\![\phi]\!]$$



Axiomatization - Part I

$[:=]$ $[x := g^{\mathbb{R}}]p(x) \leftrightarrow p(g^{\mathbb{R}})$

$[:*]$ $[x := *]p(x) \leftrightarrow \forall x\, p(x)$

$[?]$ $[?q_{\mathbb{R}}]p \leftrightarrow (q_{\mathbb{R}} \rightarrow p)$

$[\mu]$ $[\{\bar{x}' = g^{\mathbb{R}}(\bar{x},\mu)\ \&\ q_{\mathbb{R}}(\bar{x},\mu)\}]p(\bar{x},\mu) \leftrightarrow [\{\mu' = 1, \bar{x}' = g^{\mathbb{R}}(\bar{x},\mu)\ \&\ q_{\mathbb{R}}(\bar{x},\mu)\}]p(\bar{x},\mu)$

$[\mathsf{ch}!]$ $[\mathsf{ch}(h)!g^{\mathbb{R}}]p(\mathsf{ch},h) \leftrightarrow \forall h_0 \left(h_0 = h \cdot \langle \mathsf{ch}, g^{\mathbb{R}}, \mu\rangle \rightarrow p(\mathsf{ch},h_0)\right)$

$[\mathsf{ch}!]_{\mathsf{AC}}$ $[\mathsf{ch}(h)!g^{\mathbb{R}}]_{\{\widehat{r},\widehat{q}\}}\widehat{p} \leftrightarrow \widehat{q} \wedge \left(\widehat{r} \rightarrow [\mathsf{ch}(h)!g^{\mathbb{R}}]\left(\widehat{q} \wedge (\widehat{r} \rightarrow \widehat{p})\right)\right)$

$[\mathsf{ch}?]_{\mathsf{AC}}$ $[\mathsf{ch}(h)?x]_{\{\widehat{r},\widehat{q}\}}p(\mathsf{ch},h,x) \leftrightarrow [x := *][\mathsf{ch}(h)!x]_{\{\widehat{r},\widehat{q}\}}p(\mathsf{ch},h,x)$

$\mathsf{P}_j \equiv p_j(Y,\bar{z})$, and $\mathsf{R}_j \equiv r_j(Y,\bar{h})$, and $\mathsf{Q}_j \equiv q_j(Y,\bar{h})$, and $\widehat{\chi} \equiv \chi(\mathsf{ch},h)$, where j may be blank, and $Y \subseteq \Omega$, $\bar{z} \subseteq V_{\mathbb{R}} \cup V_{\mathcal{T}}$, and $\bar{h} \subseteq V_{\mathcal{T}}$ are (co)finite.

Axiomatization - Part II

$[;]_{\mathsf{AC}}$ $[a;b]_{\{\mathsf{R},\mathsf{Q}\}}\mathsf{P} \leftrightarrow [a]_{\{\mathsf{R},\mathsf{Q}\}}[b]_{\{\mathsf{R},\mathsf{Q}\}}\mathsf{P}$

$[\cup]_{\mathsf{AC}}$ $[a \cup b]_{\{\mathsf{R},\mathsf{Q}\}}\mathsf{P} \leftrightarrow [a]_{\{\mathsf{R},\mathsf{Q}\}}\mathsf{P} \wedge [b]_{\{\mathsf{R},\mathsf{Q}\}}\mathsf{P}$

$[^*]_{\mathsf{AC}}$ $[a^*]_{\{\mathsf{R},\mathsf{Q}\}}\mathsf{P} \leftrightarrow [a^0]_{\{\mathsf{R},\mathsf{Q}\}}\mathsf{P} \wedge [a]_{\{\mathsf{R},\mathsf{Q}\}}[a^*]_{\{\mathsf{R},\mathsf{Q}\}}\mathsf{P}$

I_{AC} $[a^*]_{\{\mathsf{R},\mathsf{Q}\}}\mathsf{P} \leftrightarrow [a^0]_{\{\mathsf{R},\mathsf{Q}\}}\mathsf{P} \wedge [a^*]_{\{\mathsf{R},\mathsf{true}\}}(\mathsf{P} \rightarrow [a]_{\{\mathsf{R},\mathsf{Q}\}}\mathsf{P})$

$\mathsf{P}_j \equiv p_j(Y,\bar{z})$, and $\mathsf{R}_j \equiv r_j(Y,\bar{h})$, and $\mathsf{Q}_j \equiv q_j(Y,\bar{h})$, and $\widehat{\chi} \equiv \chi(\mathsf{ch},h)$, where j may be blank, and $Y \subseteq \Omega$, $\bar{z} \subseteq V_{\mathbb{R}} \cup V_{\mathcal{T}}$, and $\bar{h} \subseteq V_{\mathcal{T}}$ are (co)finite.

Axiomatization - Part III

$[]_{\top,\top}$ $\quad [a]\mathsf{P} \leftrightarrow [a]_{\{\mathsf{true},\mathsf{true}\}}\mathsf{P}$

$[\epsilon]_{\mathsf{AC}}$ $\quad [a(\!|\emptyset, V_{\mathbb{R}}|\!)]_{\{\mathsf{R},\mathsf{Q}\}}\mathsf{P} \leftrightarrow \mathsf{Q} \wedge (\mathsf{R} \rightarrow [a(\!|\emptyset, V_{\mathbb{R}}|\!)]\mathsf{P})$

$[]_{\mathsf{WA}}$ $\quad [a]_{\{\mathsf{true},\mathsf{W}_\mathsf{A}\}}\mathsf{true} \wedge [a]_{\{\mathsf{R}_1\wedge\mathsf{R}_2,\mathsf{Q}_1\wedge\mathsf{Q}_2\}}\mathsf{P} \rightarrow [a]_{\{\mathsf{R},\mathsf{Q}_1\wedge\mathsf{Q}_2\}}\mathsf{P}$

$\mathsf{W}[]_{\mathsf{AC}}$ $\quad [a]_{\{\mathsf{R},\mathsf{Q}\}}\mathsf{P} \leftrightarrow \mathsf{Q} \wedge [a]_{\{\mathsf{R},\mathsf{Q}\}}(\mathsf{Q} \wedge (\mathsf{R} \rightarrow \mathsf{P}))$

K_{AC} $\quad [a]_{\{\mathsf{R},\mathsf{Q}_1\rightarrow\mathsf{Q}_2\}}(\mathsf{P}_1 \rightarrow \mathsf{P}_2) \rightarrow ([a]_{\{\mathsf{R},\mathsf{Q}_1\}}\mathsf{P}_1 \rightarrow [a]_{\{\mathsf{R},\mathsf{Q}_2\}}\mathsf{P}_2)$

$\mathsf{P}_j \equiv p_j(Y, \bar{z})$, and $\mathsf{R}_j \equiv r_j(Y, \bar{h})$, and $\mathsf{Q}_j \equiv q_j(Y, \bar{h})$, and $\widehat{\chi} \equiv \chi(\mathsf{ch}, h)$, where j may be blank, and $Y \subseteq \Omega$, $\bar{z} \subseteq V_{\mathbb{R}} \cup V_{\mathcal{T}}$, and $\bar{h} \subseteq V_{\mathcal{T}}$ are (co)finite.

Axiomatization - Part IV

$$\text{MP}\quad \frac{p \to q \quad p}{q}$$

$$\text{G}_{\text{AC}}\quad \frac{\mathsf{Q} \wedge \mathsf{P}}{[a]_{\{\mathsf{R},\mathsf{Q}\}}\mathsf{P}}$$

$$\forall\quad \frac{p(x)}{\forall x\, p(x)}$$

$$\text{CE}\quad \frac{\mathsf{P}_1 \leftrightarrow \mathsf{P}_2}{C(\mathsf{P}_1) \leftrightarrow C(\mathsf{P}_2)}$$

$\mathsf{P}_j \equiv p_j(Y, \bar{z})$, and $\mathsf{R}_j \equiv r_j(Y, \bar{h})$, and $\mathsf{Q}_j \equiv q_j(Y, \bar{h})$, and $\widehat{\chi} \equiv \chi(\mathsf{ch}, h)$, where j may be blank, and $Y \subseteq \Omega$, $\bar{z} \subseteq V_{\mathbb{R}} \cup V_{\mathcal{T}}$, and $\bar{h} \subseteq V_{\mathcal{T}}$ are (co)finite.

Don't Release the Synchronization Dragon!

This instantiation is unsound as it realeses dh from synchronization!

$$\frac{[a(\langle\mathsf{ch}\rangle) \cup (?\mathsf{false}; c(\langle\mathsf{dh}\rangle))]p(\mathsf{dh}) \rightarrow [a(\langle\mathsf{ch}\rangle) \cup (?\mathsf{false}; c(\langle\mathsf{dh}\rangle)) \parallel \mathsf{dh}?x]p(\mathsf{dh})}{[\mathsf{ch}(h)!\theta \cup (?\mathsf{false}; ?\mathsf{true})]\varphi(\mathsf{dh}) \rightarrow [\mathsf{ch}(h)!\theta \cup (?\mathsf{false}; ?\mathsf{true}) \parallel \mathsf{dh}?x]\varphi(\mathsf{dh})}$$

free in a context where it is bound

Luckily uniform substitution sorts it out by a ↯ clash.

Parallel Decomposition

$$\mathsf{G}_{\mathsf{AC}}\ \dfrac{\dfrac{\vdots}{\dfrac{\dfrac{\dfrac{\Gamma \vdash [\alpha]_{\{\mathsf{A}_1,\mathsf{C}_1\}}\psi_1}{\Gamma \vdash [\alpha \parallel \beta]_{\{\mathsf{A}_1,\mathsf{C}_1\}}\psi_1}\,[\|\_]_{\mathsf{AC}}}{\Gamma \vdash [\alpha \parallel \beta]_{\{\mathsf{A}_1\wedge\mathsf{A}_2,\mathsf{C}_1\}}\psi_1}\,\mathsf{M}[\cdot]_{\mathsf{AC}} \qquad \vdots}{\Gamma \vdash [\alpha \parallel \beta]_{\{\mathsf{A}_1\wedge\mathsf{A}_2,\mathsf{C}_1\wedge\mathsf{C}_2\}}(\psi_1 \wedge \psi_2)}\,[]_{\mathsf{AC}}\wedge \qquad \vdash \mathsf{W}_\mathsf{A} \wedge \mathsf{true}}{\Gamma \vdash [\alpha \parallel \beta]_{\{\mathsf{true},\mathsf{W}_\mathsf{A}\}}\mathsf{true} \wedge [\alpha \parallel \beta]_{\{\mathsf{A}_1\wedge\mathsf{A}_2,\mathsf{C}_1\wedge\mathsf{C}_2\}}(\psi_1 \wedge \psi_2)}\,\wedge\mathsf{R}}{\Gamma \vdash [\alpha \parallel \beta]_{\{\mathsf{true},\mathsf{C}_1\wedge\mathsf{C}_2\}}(\psi_1 \wedge \psi_2)}\,[]_{\mathsf{WA}}}{\Gamma \vdash [\alpha \parallel \beta]_{\{\mathsf{true},\mathsf{true}\}}\psi}\,\mathsf{M}[\cdot]_{\mathsf{AC}}}{\Gamma \vdash [\alpha \parallel \beta]\psi}\,[]_{\top,\top}$$

$$\mathsf{W}_\mathsf{A} \equiv (\mathsf{A}_2 \to \mathsf{C}_1) \wedge (\mathsf{A}_1 \to \mathsf{C}_2)$$