Deductive Stability Proofs for Ordinary Differential Equations

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Outline

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Motivation : Cyber-Physical Systems (CPSs)

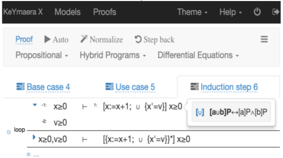


Challenge: How can we formally ensure correctness for cyber-physical systems that feature interacting discrete and continuous dynamics?

Motivation : Cyber-Physical Systems (CPSs)



Model as hybrid system & specify correctness



Hybrid system model:

$$\texttt{if}(v > \mathit{limit}) \quad x' = v, v' = \mathit{brake}$$

$$\texttt{else} \quad x' = v, v' = \mathit{accel}$$

Discrete + Continuous

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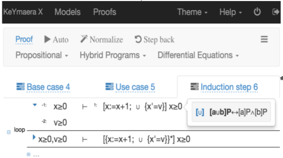

✓Deductive hyb. sys. safety proofs

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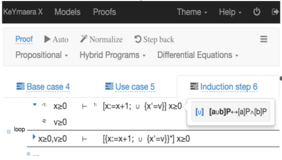

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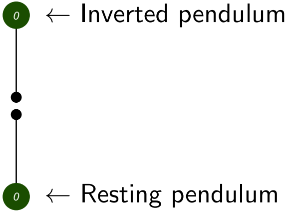
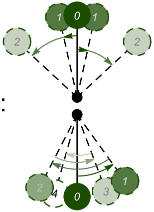
✓**Deductive ODE stability proofs**

This work: Deductive proofs of stability for continuous dynamics described by ordinary differential equations (ODEs).

Why Stability?

Theoretically: 

In Practice: 

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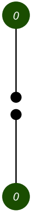
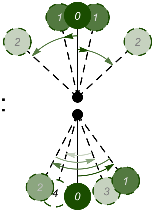
Theoretically: 

← Inverted pendulum

← Resting pendulum

In Practice: 

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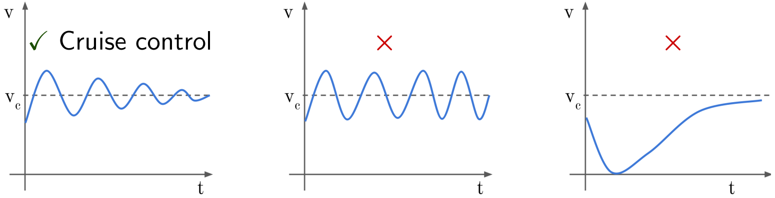

Stability is often **the** correctness criterion for control system designs.

Which Stability?

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- **Region stability**: stability-like notion for hybrid systems, Podelski and Wagner (2007).
- ε**-stability**: relaxed notion of Lyapunov stability, suitable for applying numerically-driven decision procedures, Gao et al. (2019).

Contributions

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- **Asymptotic stability**: Lyapunov stability & asymptotic convergence to an equilibrium point from nearby states, Lyapunov (1892).
- **Exponential stability**: exponential convergence to an equilibrium

This work: Formal specification of various stability properties in differential dynamic logic (dL), enabling:

- Rigorous proofs of ODE stability from sound dL foundations.
- Formalization of logical relationships between stability notions.
- Practical deployment of stability proofs in KeYmaera X, a hybrid systems prover based on dL.

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Outline

Background: Differential Dynamic Logic

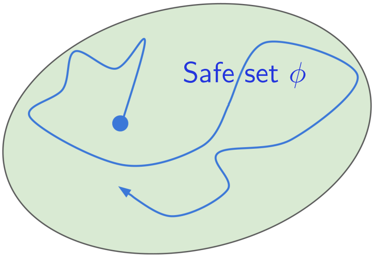
Specifications: $\phi, \psi ::= e \sim \tilde{e} \mid \phi \wedge \psi \mid \phi \vee \psi \mid \neg\phi \mid \cdots \mid \forall x\, \phi \mid \exists x\, \phi$

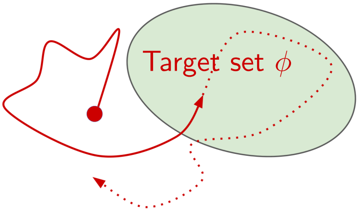
$\mid \underbrace{[\alpha]\phi \mid \langle\alpha\rangle\phi}$

α is a hybrid system modeled using dL's language of hybrid programs

Semantics: $[x' = f(x)]\phi$ iff $x' = f(x)$ solution always stays in ϕ.

$\langle x' = f(x)\rangle\phi$ iff $x' = f(x)$ solution eventually reaches ϕ.





Background: Differential Dynamic Logic

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Axiomatics:

[JACM'20] provides sound and complete dL ODE invariance reasoning for proving ODE safety properties $[x' = f(x)]\phi$.

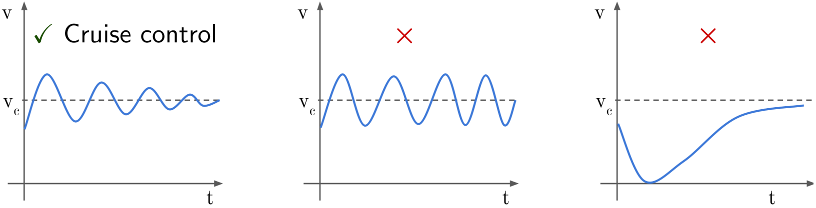
[FAC'21] provides a general, dL refinement-based approach for proving ODE liveness properties $\langle x' = f(x)\rangle\phi$.

Asymptotic Stability for ODEs

Stability: Stable systems stay close to their desired operating state(s) when slightly perturbed from those state(s).

Attractivity: Attractive systems dissipate small initial perturbations from their desired operating state(s).

Example: Cruise controllers stabilize car's velocity v at targeted value v_c.



✓ Stable + Attractive → **Asymptotic Stability**

✓ Stable
× Not attractive

✓ Attractive
× Not stable

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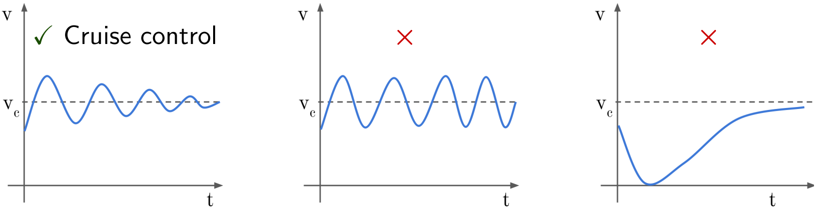


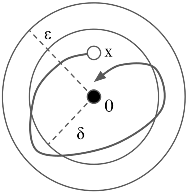
Key Idea: Specify ODE stability in dL, then derive stability proof rules by combining dL's ODE safety and liveness reasoning.

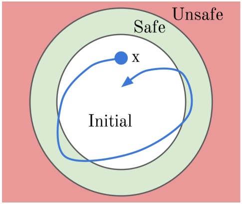
Formal Specification & Deduction (Stability)

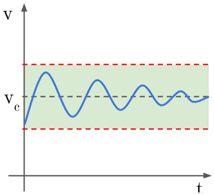
Definition (Stability)

The origin $0 \in \mathbb{R}^n$ of the n-dimensional ODE $x' = f(x)$ is:
stable if for all $\varepsilon{>}0$, there exists $\delta{>}0$ s.t. for all $x{=}x(0)$ with $\|x\|{<}\delta$, the ODE solution $x(t) : [0, T) \to \mathbb{R}^n$ always satisfies $\|x(t)\|{<}\varepsilon$.

$$\text{Stab}(x' = f(x)) \equiv \forall \varepsilon{>}0\, \exists \delta{>}0\, \forall x \left(\|x\|^2 < \delta^2 \to [x' = f(x)]\, \|x\|^2 < \varepsilon^2\right)$$





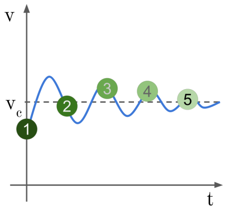


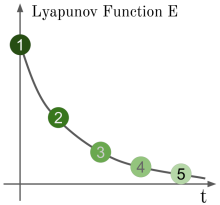
Formal Specification & Deduction (Stability)

Lemma (Lyapunov Function Proof Rule)

The following Lyapunov function proof rule is derivable in dL.

$$\text{Lyap}_{\geq}\ \frac{\vdash f(0)=0 \land E(0)=0 \qquad 0<\|x\|^2 \vdash E>0 \land E' \leq 0}{\vdash \underbrace{\forall \varepsilon{>}0\, \exists \delta{>}0\, \forall x\, \left(\|x\|^2 < \delta^2 \rightarrow [x'=f(x)]\, \|x\|^2 < \varepsilon^2\right)}_{\text{Stab}(x'=f(x))}}$$





Lyapunov functions are an energy-like auxiliary measure used to certify (asymptotic) stability for a given system.

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Deductive reasoning over $\forall, \exists$
& ODE safety reasoning [JACM'20]

Formal Specification & Deduction (Attractivity)

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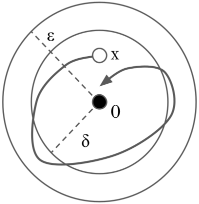
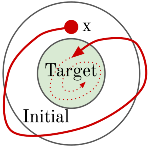
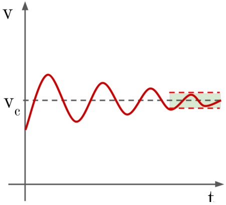
The origin $0 \in \mathbb{R}^n$ of the n-dimensional ODE $x' = f(x)$ is:
attractive if there exists $\delta > 0$ s.t. for all $x = x(0)$ with $\|x\| < \delta$, the ODE solution $x(t) : [0, T) \to \mathbb{R}^n$ satisfies $\lim_{t \to T} x(t) = 0$.

$$\mathrm{Attr}(x' = f(x)) \equiv \exists \delta{>}0\, \forall x \left(\|x\|^2 < \delta^2 \to \underbrace{\mathrm{Asym}(x' = f(x), x = 0)}_{\forall \varepsilon > 0\, \langle x' = f(x) \rangle [x' = f(x)]\, \|x\|^2 < \varepsilon^2}\right)$$







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The origin $0 \in \mathbb{R}^n$ of the n-dimensional ODE $x' = f(x)$ is:
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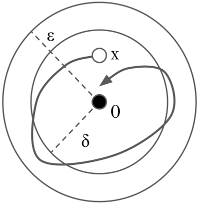

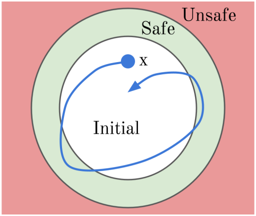

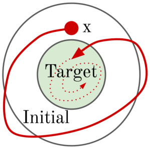

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Deductive reasoning over $\forall, \exists$
& ODE safety reasoning [JACM'20]
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In paper: Proofs of logical relationships between stability notions in dL and various verified stability examples in KeYmaera X:

- Asymptotic stability of a PD inverted pendulum controller.
- Set stability for the axes of a 3D rigid body.
- ϵ-stability and other stability examples from verification literature (nonlinear ODEs, up to 6 dimensions) [CAV'19, TACAS'20].

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Conclusion and Future Work

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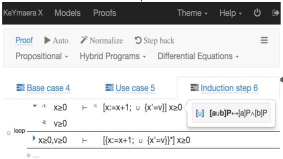

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✓**Deductive ODE stability proofs**

Future work: stability with respect to continuous ODE disturbances, hybrid systems stability, and KeYmaera X automation for stability proofs.

References

[1] Ahmed, D., Peruffo, A., and Abate, A. (2020). Automated and sound synthesis of Lyapunov functions with SMT solvers. In Biere, A. and Parker, D., editors, TACAS, volume 12078 of LNCS, pages 97–114. Springer.

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