# An Axiomatic Approach to Liveness for Differential Equations

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# Outline

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# Motivation : Cyber-Physical Systems (CPSs)

Hybrid system models enable formal analysis of safety-critical CPSs:

**Discrete control:**

```
if (v > speed_limit)
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else
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$$\underbrace{x' = v, v' = a}_{\textbf{Ordinary Differential Equations (ODEs)}}$$

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$$\underbrace{x' = v, v' = a}_{\textbf{ODEs need proofs too!}}$$

# Correctness Specifications for CPSs


✓Safely under speed limit

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# Correctness Specifications for CPSs


✓Safely under speed limit
✓Gets to destination

System is safe and **live**

✓Safely under speed limit
✗Not moving at all!

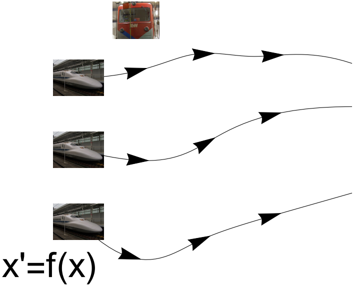
System is safe but not **live**

# ODEs and Domain Constraints

Ordinary Differential Equation (ODE)

$$\overbrace{x' = f(x)}$$

**ODE**: Models continuous physics of the system

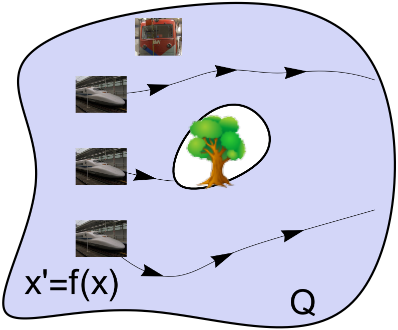



Trains drive on tracks prescribed by the ODEs.

# ODEs and Domain Constraints

$$\overbrace{x' = f(x) \,\&\, Q}^{\text{ODE with domain } Q}$$

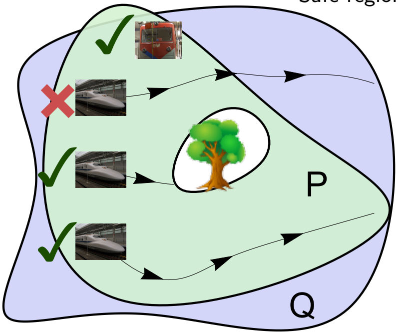
**Domain**: Specifies the domain of definition for ODEs




There are no train tracks across the national park!

# Safety & Liveness for ODEs

$$\textbf{Safety:}\ [\overbrace{x' = f(x)\,\&\,Q}^{\text{ODE with domain } Q}]\underbrace{P}_{\text{Safe region}}$$

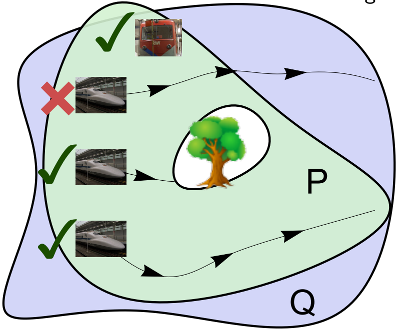


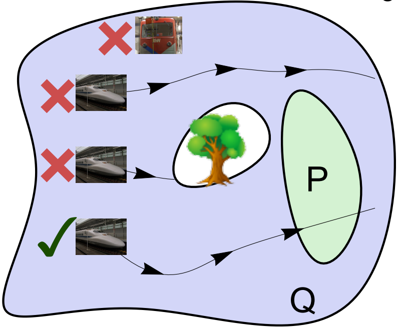

✓Trains stay in Porto ($P$) while driving on tracks.

# Safety & Liveness for ODEs

**Safety:** $\overbrace{[x' = f(x) \,\&\, Q]}^{\text{ODE with domain } Q}\ \underbrace{P}_{\text{Safe region}}$

**Liveness:** $\overbrace{\langle x' = f(x) \,\&\, Q \rangle}^{\text{ODE with domain } Q}\ \underbrace{P}_{\text{Goal region}}$




✓Trains stay in Porto ($P$) while driving on tracks.

✓Trains reach Porto ($P$) by driving on tracks.

# Safety & Liveness for ODEs

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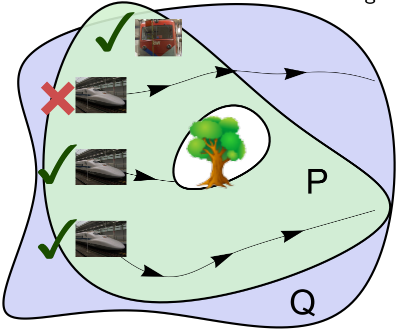


**Prior work:** complete invariance proofs for ODE safety [LICS'18]

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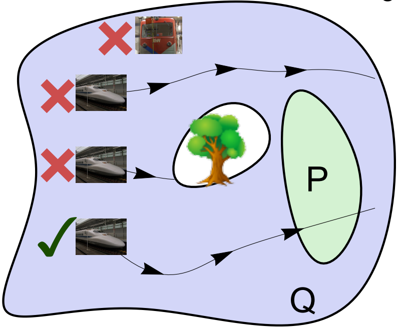


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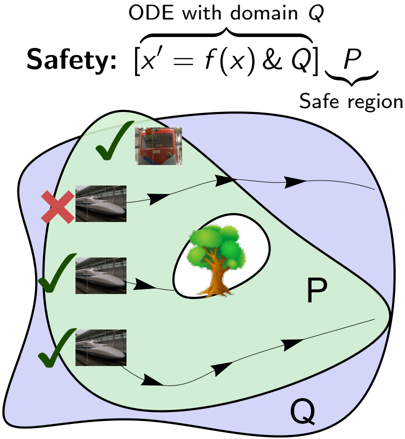
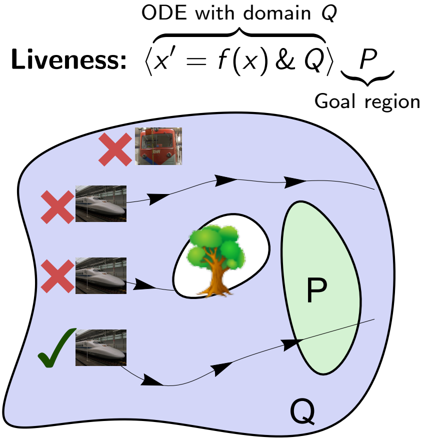
# Safety & Liveness for ODEs




**Prior work:** complete invariance proofs for ODE safety [LICS'18]

**This talk:** proving ODE liveness in differential dynamic logic (dL)

# An Axiomatic Approach to Liveness for ODEs

Why take a **logical** approach?

| Surveyed Liveness Arguments | Goals of surveyed paper |
|---|---|
| Differential Variants [1] | Liveness proofs for inequalities |
| Bounded/Compact Eventuality [3, 4] | Automatic SOS liveness proofs |
| Set Lyapunov Functions [5] | Finding basin of attraction |
| Staging Sets + Progress [6] | Indirect liveness proofs for $P$ |
| Eq. Differential Variants [7] | Synthesizing switching logic |

Liveness arguments in the literature are used for a wide variety of purposes.

# An Axiomatic Approach to Liveness for ODEs

Why take a **logical** approach?

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| Differential Variants [1] | | × |
| Bounded/Compact Eventuality [3, 4] | × | × |
| Set Lyapunov Functions [5] | × | × |
| Staging Sets + Progress [6] | | |
| Eq. Differential Variants [7] | × | × |

Several arguments have technical glitches, making them unsound (×).

# An Axiomatic Approach to Liveness for ODEs

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| Set Lyapunov Functions [5] | ✗ ⇝ ✓ | ✗ ⇝ ✓ |
| Staging Sets + Progress [6] | ✓ | ✓ |
| Eq. Differential Variants [7] | ✗ ⇝ ✓ | ✗ ⇝ ✓ |

Our approach formalizes the underlying liveness arguments in a sound (✓), foundational, and uniform framework. It also corrects (✗ ⇝ ✓) the technical glitches.

# An Axiomatic Approach to Liveness for ODEs

Why take a **logical** approach?

- Understand the core principles behind ODE liveness proofs.

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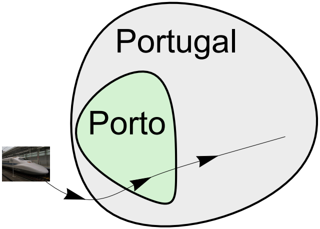
| Staging Sets + Progress [6] | ✓ | ✓ |
| Eq. Differential Variants [7] | ✗ ⇝ ✓ | ✗ ⇝ ✓ |

- Yields generalizations of existing liveness arguments "for free".

| New Liveness Arguments | Without Domains | With Domains |
|---|---|---|
| Higher Differential Variants | ✓ | - |
| [1] + [3, 4] + [6] | ✓ | - |
| [1] + [3, 4] + [6] + Higher Diff. Var. | - | ✓ |

# Outline

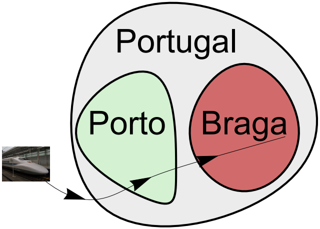
# A Simple Liveness Refinement




Trains that reach Porto also reach Portugal since Porto is part of Portugal.

✓ $\langle x' = f(x)\rangle \texttt{Porto} \rightarrow \langle x' = f(x)\rangle \texttt{Portugal}$

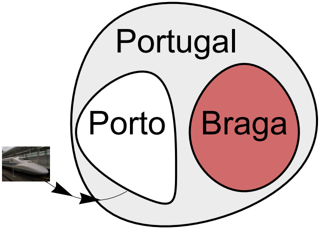
# A Simple Liveness Refinement




Can train reach Porto if it reaches Braga? Not true for **all** trains.

$$? \quad \langle x' = f(x) \rangle \texttt{Braga} \rightarrow \langle x' = f(x) \rangle \texttt{Porto}$$

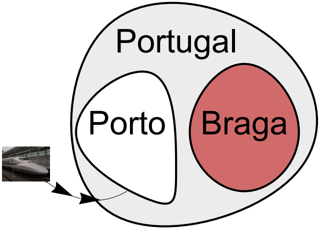
# A Simple Liveness Refinement




Must use specific properties of the ODE / train track.

$$[x' = f(x) \,\&\, \neg\texttt{Porto}]\neg\texttt{Braga} \rightarrow \big(\langle x' = f(x)\rangle\texttt{Braga} \rightarrow \langle x' = f(x)\rangle\texttt{Porto}\big)$$

# A Simple Liveness Refinement

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$$[x' = f(x) \,\&\, \neg\texttt{Porto}]\neg\texttt{Braga} \rightarrow \big(\underbrace{\langle x' = f(x)\rangle \texttt{Braga}}_{\text{Known liveness property}} \rightarrow \underbrace{\langle x' = f(x)\rangle \texttt{Porto}}_{\text{Desired liveness property}}\big)$$

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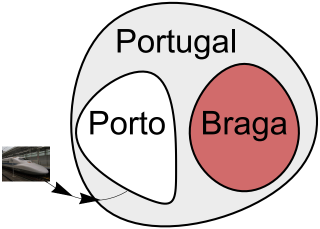


Must use specific properties of the ODE / train track.

$$\underbrace{[x' = f(x) \,\&\, \neg\mathtt{Porto}]\neg\mathtt{Braga}}_{\text{Need to show}} \rightarrow \big(\underbrace{\langle x' = f(x)\rangle\mathtt{Braga}}_{\text{Known liveness property}} \rightarrow \underbrace{\langle x' = f(x)\rangle\mathtt{Porto}}_{\text{Desired liveness property}}\big)$$

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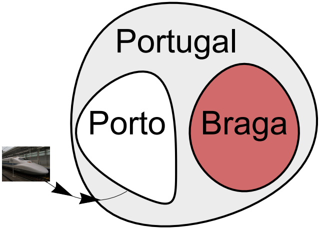


**Key Idea:** Liveness arguments can and should be understood using liveness refinement steps.

$$\underbrace{[x' = f(x) \,\&\, \neg\mathtt{Porto}]\neg\mathtt{Braga}}_{\text{Need to show}} \rightarrow \underbrace{(\langle x' = f(x)\rangle\mathtt{Braga}}_{\text{Known liveness property}} \rightarrow \underbrace{\langle x' = f(x)\rangle\mathtt{Porto})}_{\text{Desired liveness property}}$$

# Diamond Refinement Axioms

$$\underbrace{[x' = f(x) \,\&\, \neg\texttt{Porto}]\neg\texttt{Braga}}_{\text{Need to show}} \rightarrow \big(\underbrace{\langle x' = f(x)\rangle\texttt{Braga}}_{\text{Known liveness property}} \rightarrow \underbrace{\langle x' = f(x)\rangle\texttt{Porto}}_{\text{Desired liveness property}}\big)$$

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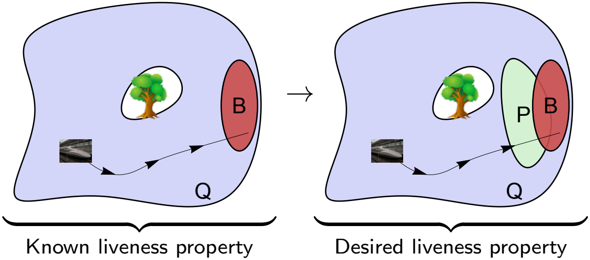

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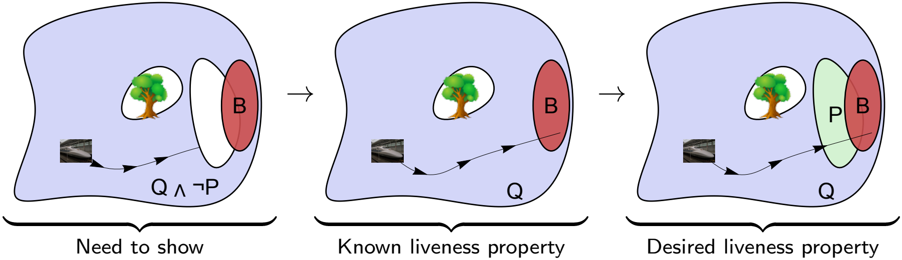

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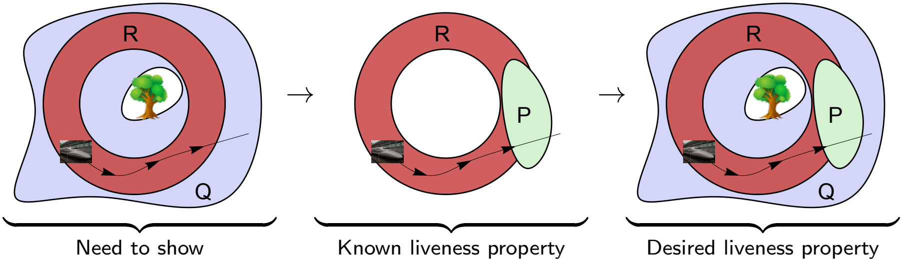

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**Idea 1:** ODE safety has effective reasoning principles [LICS'18], so use ODE safety to justify refinement steps.

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$$\langle x' = f(x) \,\&\, Q\rangle B \overbrace{\rightarrow}^{\substack{\mathrm{K}\langle\&\rangle \\ [x'=f(x)\,\&\,\neg P]\neg B}} \langle x' = f(x) \,\&\, Q\rangle P$$

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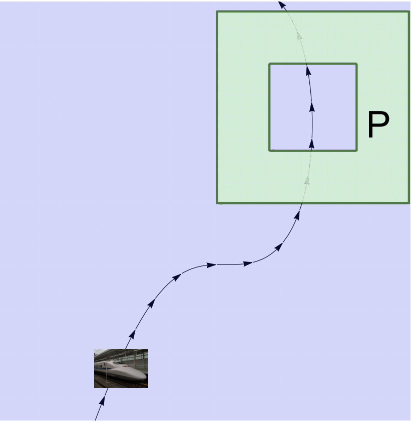
**Idea 2:** Implication chains build complicated liveness arguments from simple building blocks.

# Outline

# ODE Liveness Example




**Example:** Train reaches Porto suburbs ($P$).
For simplicity, no domain constraint.

**Model ODE:**

$$x' = -y, y' = 4x^2$$

# Equational Differential Variants

| Surveyed Liveness Arguments | Goals of surveyed paper |
|---|---|
| Eq. Differential Variants [7] | Synthesizing switching logic |

Derived proof rule:

$$\mathrm{dV}^{M}_{=}\ \frac{p = 0 \vdash P \quad p < 0 \vdash p' \geq \varepsilon()}{\Gamma, \varepsilon() > 0, p \leq 0 \vdash \langle x' = f(x) \rangle P}$$

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**Additional condition for soundness** ✓:
Either solution exists for sufficient duration or $x' = f(x)$ is globally Lipschitz continuous.

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Derived proof rule:

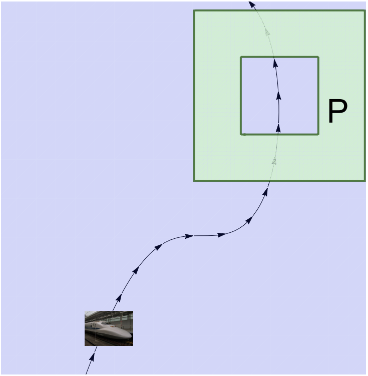
$$\mathrm{dV}^{M}_{=}\ \frac{\overbrace{p = 0 \vdash P}^{\text{Step 3}} \quad \overbrace{p < 0 \vdash p' \geq \varepsilon()}^{\text{Step 1}}}{\Gamma, \varepsilon() > 0, \underbrace{p \leq 0}_{\text{Step 2}} \vdash \langle x' = f(x) \rangle P}$$

Underlying refinement chain:

$$\langle x' = f(x), t' = 1 \rangle t > c() \overset{\substack{\text{Step 1}\\ \mathrm{K}\langle\&\rangle}}{\overbrace{\rightarrow}} \langle x' = f(x) \rangle p \geq 0 \overset{\substack{\text{Step 2}\\ \mathrm{K}\langle\&\rangle}}{\overbrace{\rightarrow}} \langle x' = f(x) \rangle p = 0 \overset{\substack{\text{Step 3}\\ \mathrm{K}\langle\&\rangle}}{\overbrace{\rightarrow}} \langle x' = f(x) \rangle P$$

# Proving Liveness for Train

$$\langle x' = f(x), t' = 1\rangle t > 1.4 \overbrace{\rightarrow}^{\substack{\text{Step 1}\\ K\langle\&\rangle}} \langle x' = f(x)\rangle r \leq 1 \overbrace{\rightarrow}^{\substack{\text{Step 2}\\ K\langle\&\rangle}} \langle x' = f(x)\rangle r = 1 \overbrace{\rightarrow}^{\substack{\text{Step 3}\\ K\langle\&\rangle}} \langle x' = f(x)\rangle P$$




**Intuition:** Reduce liveness for (complicated) region $P$ to (simple) circle.

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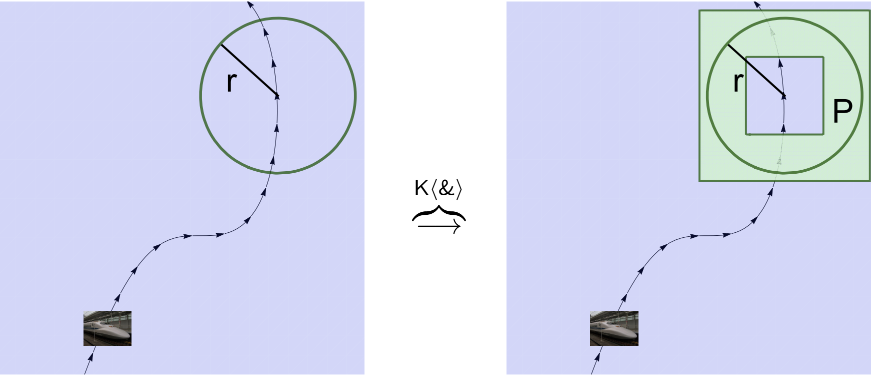


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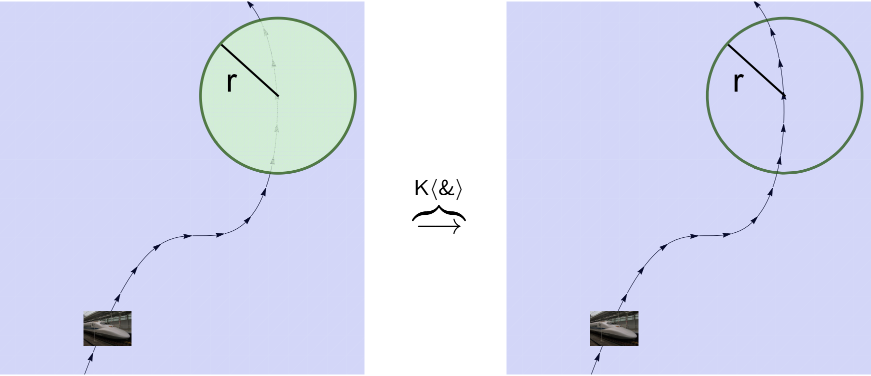


**Intuition:** Since train starts outside circle, reduce further to liveness for disk.

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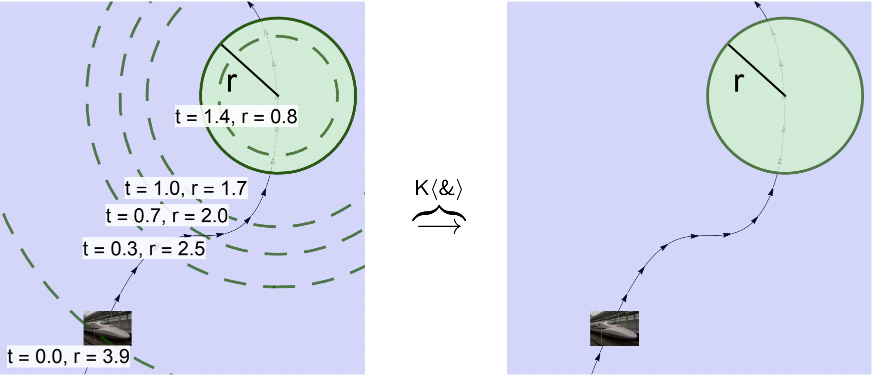


**Intuition:** Symbolically analyze derivatives to lower bound time required to reach disk for the train.

# Proving Liveness for Train

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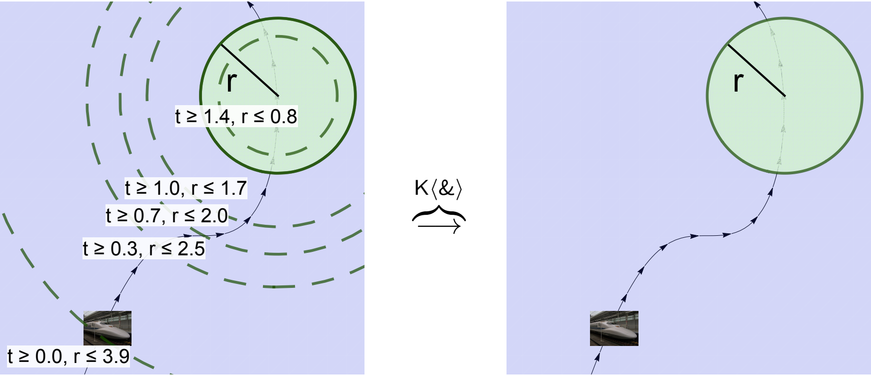


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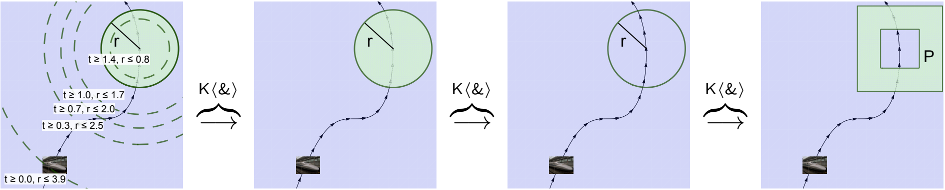


The train reaches Porto ($P$) if it is driven for $> 1.4$ hours:

$$\langle x' = f(x), t' = 1\rangle t > 1.4 \to \langle x' = f(x)\rangle P$$

# Existence Properties

**Idea 3:** Basic liveness properties of ODEs can be justified by a small number of simple axioms.

GEx $\langle x' = f(x), t' = 1 \rangle t > c()$ (if $x' = f(x)$ globally Lipschitz)

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$$\text{GEx}\ \langle x' = f(x), t' = 1\rangle t > c() \quad \text{(if } x' = f(x) \text{ globally Lipschitz)}$$

Apply to ODE example:

$$\dfrac{\overbrace{\langle x' = f(x), t' = 1\rangle t > 1.4}^{\text{GEx}} \qquad \overbrace{\langle x' = f(x), t' = 1\rangle t > 1.4 \rightarrow \langle x' = f(x)\rangle P}^{\text{Train reaches Porto } (P) \text{ if driven for } > 1.4 \text{ hours}}}{\langle x' = f(x)\rangle P}$$

# Existence Properties

**Idea 3:** Basic liveness properties of ODEs can be justified by a small number of simple axioms.

$$\text{GEx } \langle x' = f(x), t' = 1\rangle t > c() \quad \text{(if } x' = f(x) \text{ globally Lipschitz)}$$

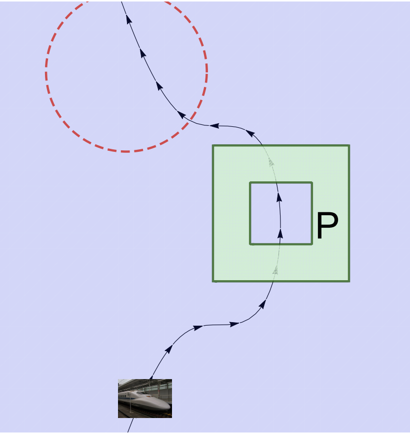
Apply to ODE example:

$$\frac{\overbrace{\cancel{\langle x' = f(x), t' = 1\rangle t > 1.4}}^{\text{Not for } x' = -y,\, y' = 4x^2} \qquad \overbrace{\langle x' = f(x), t' = 1\rangle t > 1.4 \rightarrow \langle x' = f(x)\rangle P}^{\text{Train reaches Porto } (P) \text{ if driven for } > 1.4 \text{ hours}}}{\cancel{\langle x' = f(x)\rangle P}}$$

# Existence Properties

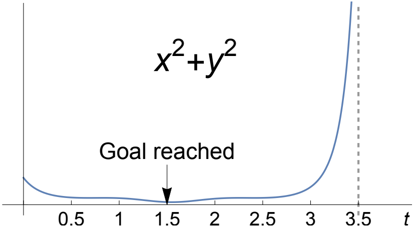
**Idea 3:** Basic liveness properties of ODEs can be justified by a small number of simple axioms.

GEx $\langle x' = f(x), t' = 1 \rangle t > c()$ (if $x' = f(x)$ globally Lipschitz)




**Problem:** Finite time blowup may prevent solutions from reaching goal.

$$\underbrace{x' = -y, y' = 4x^2}_{\text{This non-linear ODE is not globally Lipschitz!}}$$



# Equational Differential Variants

| Surveyed Liveness Arguments | Goals of surveyed paper |
|---|---|
| Eq. Differential Variants [7] | Synthesizing switching logic |

Derived proof rule:

$$\mathrm{dV}_{=}^{M}\ \frac{p = 0 \vdash P \quad p < 0 \vdash p' \geq \varepsilon()}{\Gamma, \varepsilon() > 0, p \leq 0 \vdash \langle x' = f(x) \rangle P}$$

**Additional condition for soundness** ✓:
Either solution exists for sufficient duration or $x' = f(x)$ is globally Lipschitz continuous.

# A Common Technical Glitch

Several errors ($\times$) due to insufficient technical assumptions about existence of solutions.

| Surveyed Liveness Arguments | Without Domains | With Domains |
|---|---|---|
| Differential Variants [1] | | |
| Bounded/Compact Eventuality [3, 4] | $\times$ | |
| Set Lyapunov Functions [5] | $\times$ | $\times$ |
| Staging Sets + Progress [6] | | |
| Eq. Differential Variants [7] | $\times$ | $\times$ |

# A Common Technical Glitch

Other errors ($\times$) were due to more subtle issues but they were also caught by our approach.

| Surveyed Liveness Arguments | Without Domains | With Domains |
|---|---|---|
| Differential Variants [1] | | $\times$ |
| Bounded/Compact Eventuality [3, 4] | | $\times$ |
| Set Lyapunov Functions [5] | | |
| Staging Sets + Progress [6] | | |
| Eq. Differential Variants [7] | | |

# Outline

# More Diamond Refinement Axioms

K⟨&⟩ $[x' = f(x) \,\&\, Q \land \neg P]\neg B \rightarrow (\langle x' = f(x) \,\&\, Q\rangle B \rightarrow \langle x' = f(x) \,\&\, Q\rangle P)$

DR⟨·⟩ $[x' = f(x) \,\&\, R]Q \rightarrow (\langle x' = f(x) \,\&\, R\rangle P \rightarrow \langle x' = f(x) \,\&\, Q\rangle P)$

COR $\neg P \land [x' = f(x) \,\&\, R \land \neg P]Q \rightarrow (\langle x' = f(x) \,\&\, R\rangle P \rightarrow \langle x' = f(x) \,\&\, Q\rangle P)$

SAR $[x' = f(x) \,\&\, R \land \neg(P \land Q)]Q \rightarrow (\langle x' = f(x) \,\&\, R\rangle P \rightarrow \langle x' = f(x) \,\&\, Q\rangle P)$

GEx $\langle x' = f(x), t' = 1\rangle t > c()$ (if $x' = f(x)$ globally Lipschitz)

BEx $\langle x' = f(x), t' = 1\rangle(\neg B(x) \lor t > c())$

# More Diamond Refinement Axioms

**Idea 1:** ODE safety has effective reasoning principles [LICS'18], so use ODE safety to justify refinement steps.

COR $\neg P \wedge [x' = f(x) \,\&\, R \wedge \neg P]Q \rightarrow \big(\langle x' = f(x) \,\&\, R\rangle P \rightarrow \langle x' = f(x) \,\&\, Q\rangle P\big)$

SAR $[x' = f(x) \,\&\, R \wedge \neg(P \wedge Q)]Q \rightarrow \big(\langle x' = f(x) \,\&\, R\rangle P \rightarrow \langle x' = f(x) \,\&\, Q\rangle P\big)$

GEx $\langle x' = f(x), t' = 1\rangle t > c()$ (if $x' = f(x)$ globally Lipschitz)

BEx $\langle x' = f(x), t' = 1\rangle(\neg B(x) \vee t > c())$

# More Diamond Refinement Axioms

**Idea 1:** ODE safety has effective reasoning principles [LICS'18], so use ODE safety to justify refinement steps.

**Idea 2:** Implication chains build complicated liveness arguments from simple building blocks.

GEx $\langle x' = f(x), t' = 1 \rangle t > c()$ (if $x' = f(x)$ globally Lipschitz)

BEx $\langle x' = f(x), t' = 1 \rangle (\neg B(x) \vee t > c())$

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**Idea 1:** ODE safety has effective reasoning principles [LICS'18], so use ODE safety to justify refinement steps.

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**Idea 4:** Reducing ODE liveness arguments to basic liveness refinements isolates and minimizes the possibility of soundness errors.

# More Diamond Refinement Axioms

**Idea 1:** ODE safety has effective reasoning principles [LICS'18], so use ODE safety to justify refinement steps.

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**Idea 3:** Basic liveness properties of ODEs can be justified by a small number of simple axioms.

**Idea 4:** Reducing ODE liveness arguments to basic liveness refinements isolates and minimizes the possibility of soundness errors.

**Key Idea:** Liveness arguments can and should be understood using liveness refinement steps.

# An Axiomatic Approach to Liveness for ODEs

Why take a **logical** approach?

- Understand the core principles behind ODE liveness proofs.

| Surveyed Liveness Arguments | Without Domains | With Domains |
|---|---|---|
| Differential Variants [1] | ✓ | ✗ ⇝ ✓ |
| Bounded/Compact Eventuality [3, 4] | ✗ ⇝ ✓ | ✗ ⇝ ✓ |
| Set Lyapunov Functions [5] | ✗ ⇝ ✓ | ✗ ⇝ ✓ |
| Staging Sets + Progress [6] | ✓ | ✓ |
| Eq. Differential Variants [7] | ✗ ⇝ ✓ | ✗ ⇝ ✓ |

- Yields generalizations of existing liveness arguments "for free".

| New Liveness Arguments | Without Domains | With Domains |
|---|---|---|
| Higher Differential Variants | ✓ | - |
| [1] + [3, 4] + [6] | ✓ | - |
| [1] + [3, 4] + [6] + Higher Diff. Var. | - | ✓ |

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