Implicit Definitions with Differential Equations for KeYmaera X (System Description)

James Gallicchio **Yong Kiam Tan** Stefan Mitsch André Platzer

Computer Science Department, Carnegie Mellon University

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Outline

1. Hybrid System Verification
2. Implicit Definitions in Differential Dynamic Logic
3. Implementation in KeYmaera X
4. Conclusion

Outline

Motivation: Cyber-Physical Systems (CPSs)



Challenge: How can we formally ensure correctness for cyber-physical systems that feature interacting discrete and continuous dynamics?

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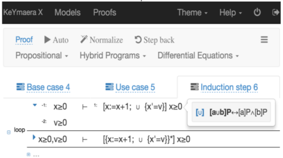
Model as hybrid system & specify correctness



Hybrid system verification tool

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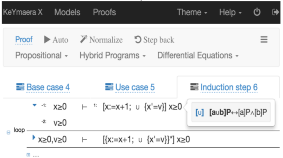
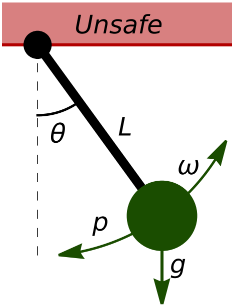


Hybrid system verification tool

Toy Example: safely pushed swing

Challenge: How can we formally ensure correctness for cyber-physical systems that feature interacting discrete and continuous dynamics?

Safely Pushed Swing



Discrete controlled pushes p
Continuous ODEs:

$$\theta' = \omega, \omega' = -\frac{g}{L}\sin(\theta) - k\omega$$

Safety:
Swing stays below horizontal

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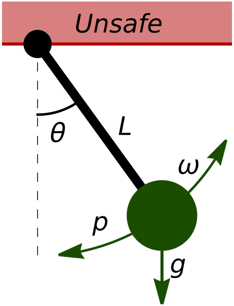

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Challenges:

- Hybrid system model + specification

Need adequate modeling of interacting discrete & continuous dynamics

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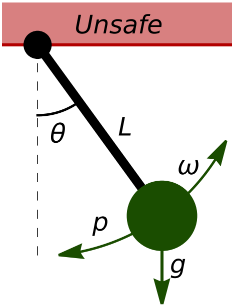

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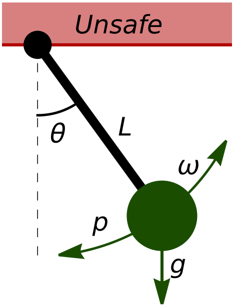

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- Proving safety & correctness

Need sound + (semi-)automated reasoning for hybrid dynamics

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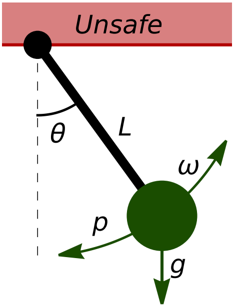

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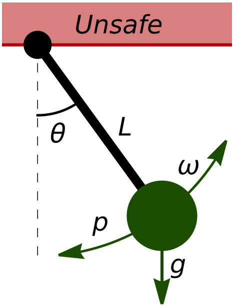

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Need extensible support for new defs.

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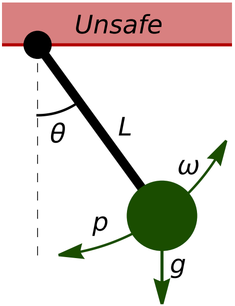

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$$\sin(\theta) = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \dots$$

Series defs. in foundational provers
×Lose hybrid system support & autom.

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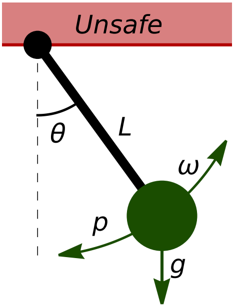

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✓**This Work:**
Definitions package for user-defined functions in dL and KeYmaera X

Domain-specific support for hybrid systems
e.g., $\sin(\theta)$ solves $s' = c, c' = -s$

KeYmaera X Package for Implicit Definitions

Modeling Interface:

```
Definitions
  implicit Real sin(Real t), cos(Real t) =
    {{sin:=0;cos:=1;}; {sin'=cos,cos'=sin}};
  Real g; /* Gravity */
  Real L; /* Length of rod */
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End.
...
Problem
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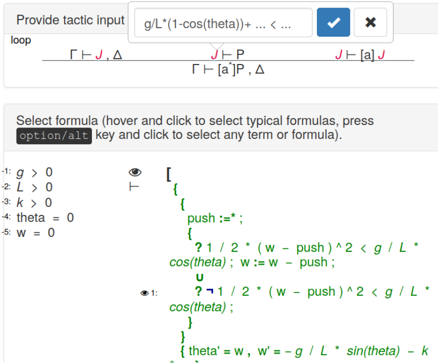

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Users define their desired functions using sugared syntax in KeYmaera X.

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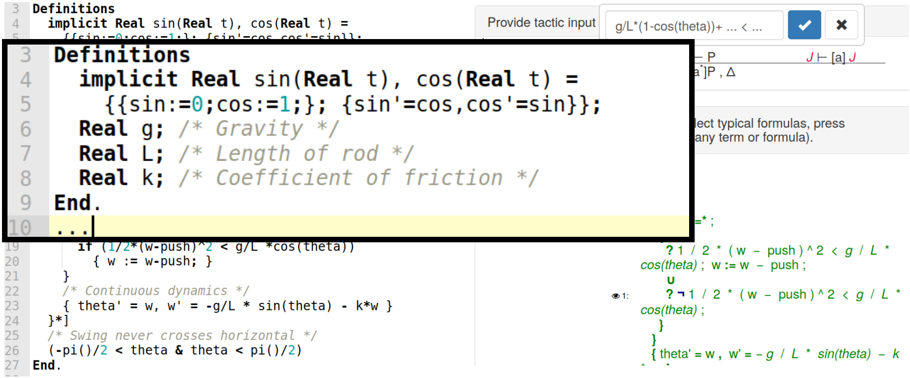

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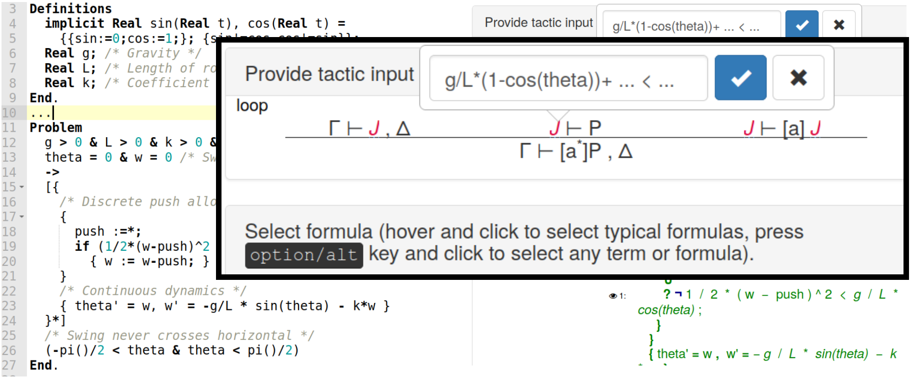
Seamlessly use functions throughout existing specifications and proof methods.



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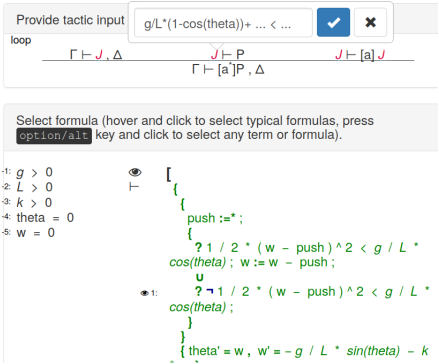

Proof: ✔ All goals closed

Export proof | Browse proof | Redo proof

```
Provable(  ==>  g()>0&L()>0&k()>0&theta=0&w=0->... proved)
```

Outline

Background: Differential Dynamic Logic (dL)

Hybrid programs model hybrid systems; terms in red (polynomials, etc.)

$$\alpha, \beta \;::=\; x' = f(x) \,\&\, Q \mid x := e \mid ?Q \mid \alpha ; \beta \mid \alpha \cup \beta \mid \alpha^*$$

ODE | Assign | Test | Compose | Choice | Loop

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Properties of hybrid program α are specified in dL's **formula** language.

$$\phi, \psi ::= e \sim \tilde{e} \mid \phi \wedge \psi \mid \cdots \mid \forall x\, \phi \mid \exists x\, \phi \mid [\alpha]\phi \mid \langle \alpha \rangle \phi$$

Compare $\geq, >, =$ | And, Or, etc. | For all, Exists | ϕ true after all α runs | ϕ true after some α run

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This Work: Expand term language with **implicitly defined functions**

$$f_{\ll\phi\gg}(t) = x \leftrightarrow \phi(x, t)$$

Function $f_{\ll\phi\gg}$ is interpreted using its **graph** characterized by ϕ.

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This Work: Expand term language w

$$f_{\ll \phi \gg}(t) =$$

n.b. Not all dL formulas characterize graphs of (suitable) functions (see paper for restrictions).

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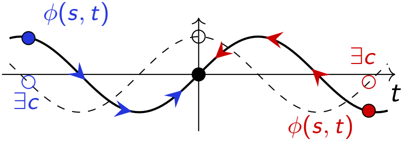
Differentially-Defined Functions

Example: Implicitly defined trigonometric sine function $\sin(t) = s$

$$\phi(s,t) \equiv \underbrace{\exists c \left\langle \begin{array}{l} \color{red}{s' = -c, c' = s, t' = -1 \cup} \\ \color{blue}{s' = c, c' = -s, t' = 1} \end{array} \right\rangle}_{ODE} \underbrace{\begin{pmatrix} s=0 \land \\ c=1 \land \\ t=0 \end{pmatrix}}_{Init.}$$



Intuition: Initial point is reachable by following ODE forward or backward.
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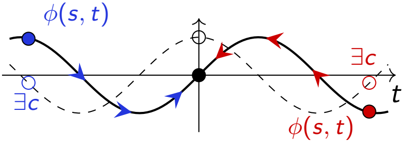

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General Case: Any projection of an ODE system solution is implicitly characterizable in dL (soundness proof in paper).

Thm. [JACM'20]: dL extended with Noetherian functions (incl. solutions of polynomial ODEs) has sound and complete ODE invariance reasoning.

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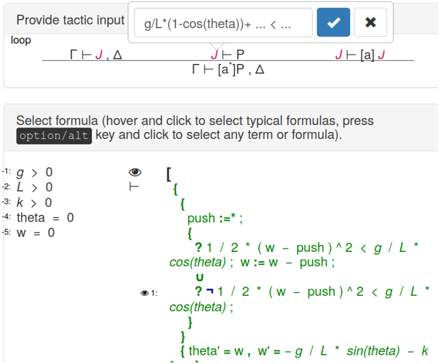

Implementation Details

Soundness-critical changes: Syntax & axiom schema for implicit defs.
Follows KeYmaera X's small trusted kernel design, $\approx$170 lines extension

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Non-critical (core-adjacent): Syntactic sugar for parsing and UI pretty-printing of user-defined functions

Non-critical (user automation):

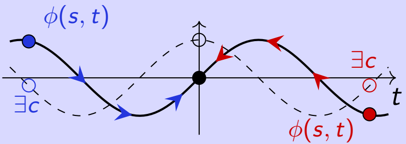
- Auto. derive base properties of functions from underlying ODEs:

Initial value: $\sin(0) = 0$

```
Provable(  ==>  sin<< ... >>(0)=0 proved)
```

Derivative: $\sin(e)' = \cos(e)(e)'$

```
Provable(  ==>
  (sin<< ... >>(x))'=cos<< ... >>(x)*(x)' proved)
```



- Prove additional arithmetic properties with ODE analysis (next slide)

Specialized Arithmetic Support

Adapt existing KeYmaera X sound abstraction & ODE analysis
$+$ arithmetic export to external real arithmetic solvers

$$x(\tanh(\lambda x) - \tanh(\lambda y)) + y(\tanh(\lambda x) + \tanh(\lambda y)) \leq 2\sqrt{x^2 + y^2}$$

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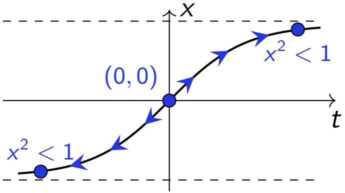
ODE analysis

$$\tanh(\lambda x)^2 < 1 \land \tanh(\lambda y)^2 < 1 \rightarrow$$
$$x(\tanh(\lambda x) - \tanh(\lambda y)) + y(\tanh(\lambda x) + \tanh(\lambda y)) \leq 2\sqrt{x^2 + y^2}$$

Claim: $\tanh(t)^2 < 1$ for all t.

Intuition: Property is always preserved along ODE, forward and backward from initial point.



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$$\Downarrow$$

Abstraction (replace tanh with fresh variables):

$${t_x}^2 < 1 \land {t_y}^2 < 1 \rightarrow x(t_x - t_y) + y(t_x + t_y) \leq 2\sqrt{x^2 + y^2}$$

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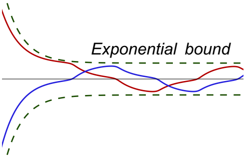
$$\Downarrow$$

Provable by solvers without native support for tanh

Proof: ✔ All goals closed

```
Provable(  ==>
  x*(tanh<< ... >>(lambda()*x)-tanh<< ... >>(lambda()*y)) +
  y*(tanh<< ... >>(lambda()*x)+tanh<< ... >>(lambda()*y)) <=
  2*(x^2+y^2)^(1/2) proved)
```

Examples (see paper [IJCAR'22])



2 neuron interaction, asymptotic norm bound

```
Problem
  tau > 0 -> \forall eps( eps > 0 ->
    <{
      x' = -x/tau + tanh(lambda*x)
         - tanh(lambda*y),
      y' = -y/tau + tanh(lambda*x)
         + tanh(lambda*y)
    }>
    [{
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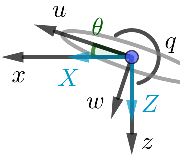
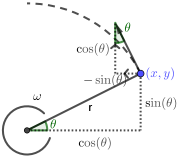
    (x^2 + y^2)^(1/2) <= (2*tau + eps)
  )
End.
```

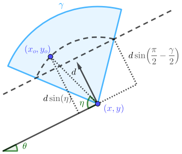


Invariants of longitudinal flight dynamics

```
Definitions
  ...
  Real inv1(Real z, Real u, Real w,
            Real theta, Real q) =
    M*z/Iyy + g*theta
    + (X/m-q*w)*cos(theta)
    + (Z/m+q*u)*sin(theta);
  ...
End.
...
Problem
  assmpts() & inv(x,z,u,w,theta,q)
  -> [motion;]inv(x,z,u,w,theta,q)
End.
```



Robot collision avoid., trajectory & vision limits



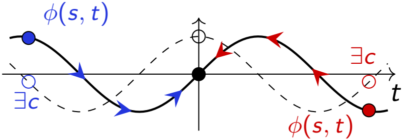
Takeaway: Package enables succinct models and powerful reasoning support for user-defined functions in KeYmaera X.

Outline

Summary

Theory: Implicit defs. in dL

$$f_{\ll\phi\gg}(t) = x \leftrightarrow \phi(x, t)$$



Practice: KeYmaera X package

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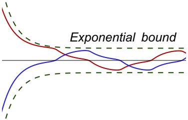
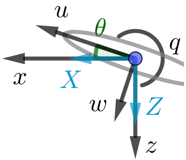
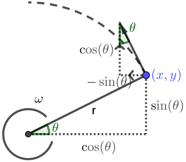
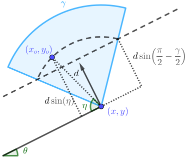




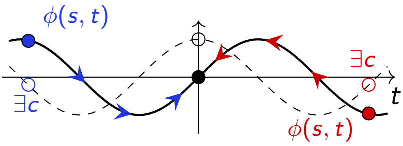




Summary

Theory: Implicit defs. in dL

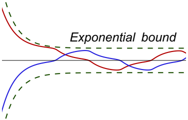
$$f_{\ll\phi\gg}(t) = x \leftrightarrow \phi(x,t)$$

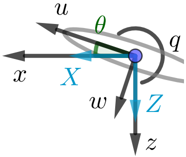


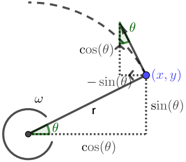
Future Work: Defining and reasoning about multivariate & non-smooth functions in dL

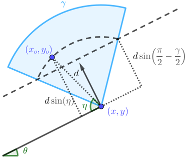
Practice: KeYmaera X package

```
Definitions
  implicit Real sin(Real t), cos(Real t) =
    {{sin:=0;cos:=1;}; {sin'=cos,cos'=sin}};
  Real g; /* Gravity */
  Real L; /* Length of rod */
  Real k; /* Coefficient of friction */
End.
...
```









Check it out: `http://keymaerax.org/keymaeraXfunc/`

References I

[1] Gallicchio, J., Tan, Y. K., Mitsch, S., and Platzer, A. (2022). Implicit definitions with differential equations for KeYmaera X (system description). In Blanchette, J., Kovacs, L., and Pattinson, D., editors, IJCAR, volume 13385 of LNCS, pages 723–733. Springer.

[2] Platzer, A. and Tan, Y. K. (2020). Differential equation invariance axiomatization. J. ACM, 67(1):6:1–6:66.