KeYmaera: A Hybrid Theorem Prover for Hybrid Systems

André Platzer **Jan-David Quesel**

University of Oldenburg, Department of Computing Science, Germany

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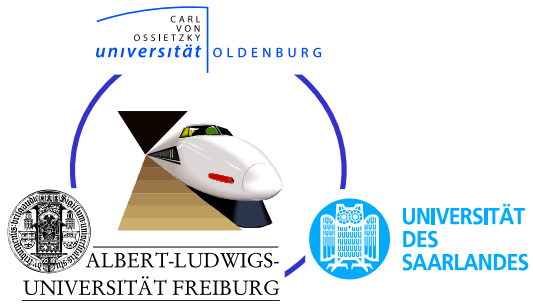
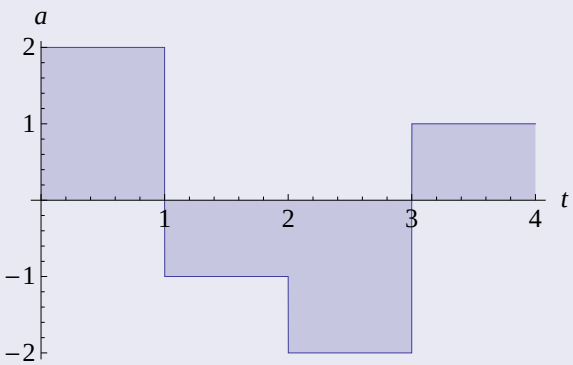


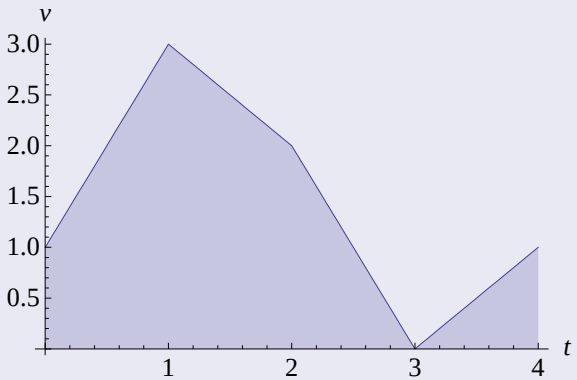
Motivation

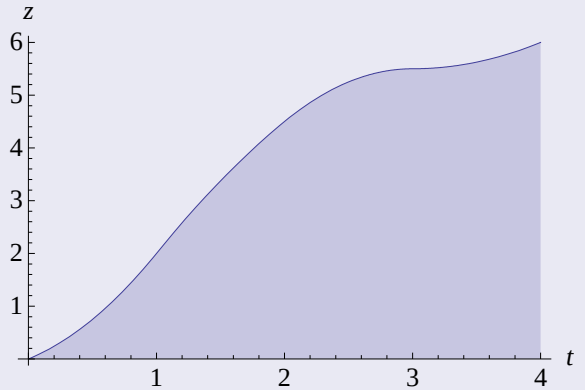
KeYmaera: Verification tool for hybrid systems

Hybrid System

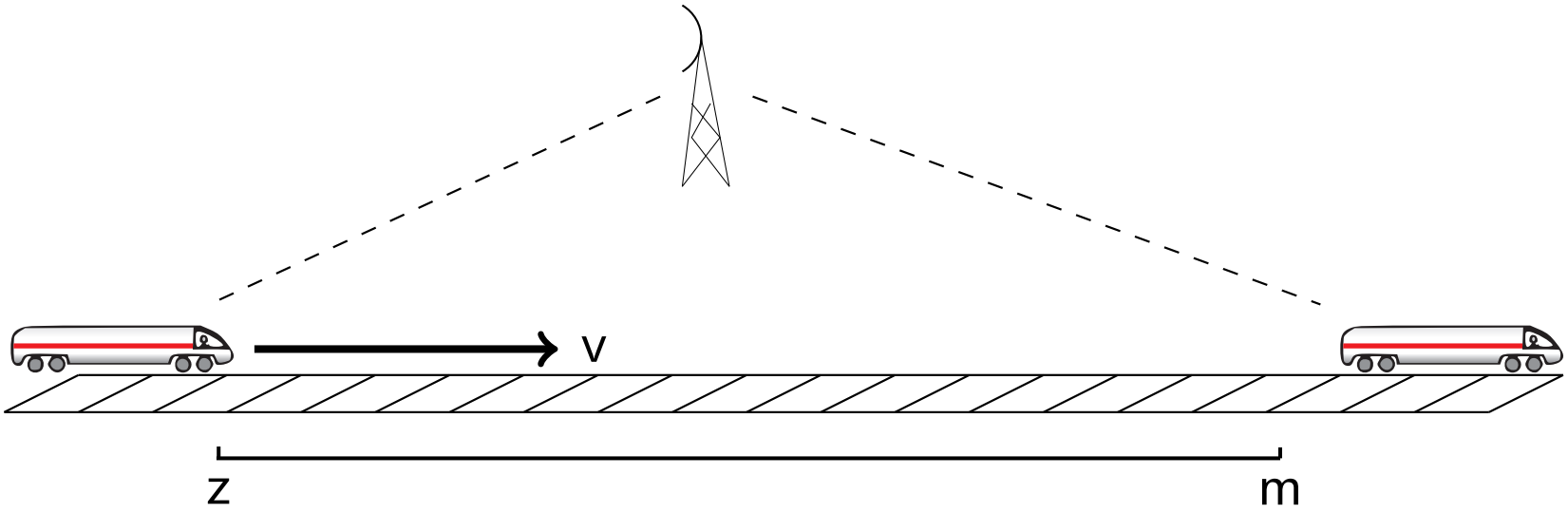
- Continuous evolutions (differential equations)
- Discrete jumps (control decisions)







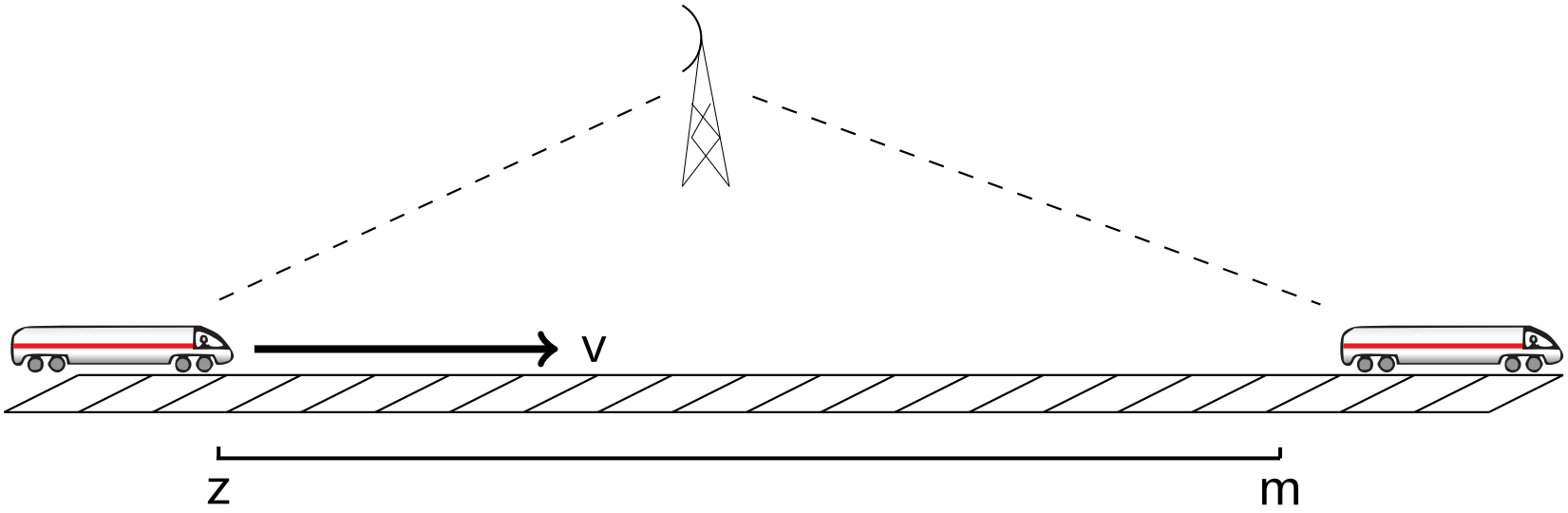
Differential Dynamic Logic (d$\mathcal{L}$)



Example

$$\underbrace{\qquad}_{\text{Precondition}} \longrightarrow [\underbrace{\qquad\qquad\qquad\qquad}_{\text{Operational model}}](\underbrace{\quad}_{\text{Property}})$$

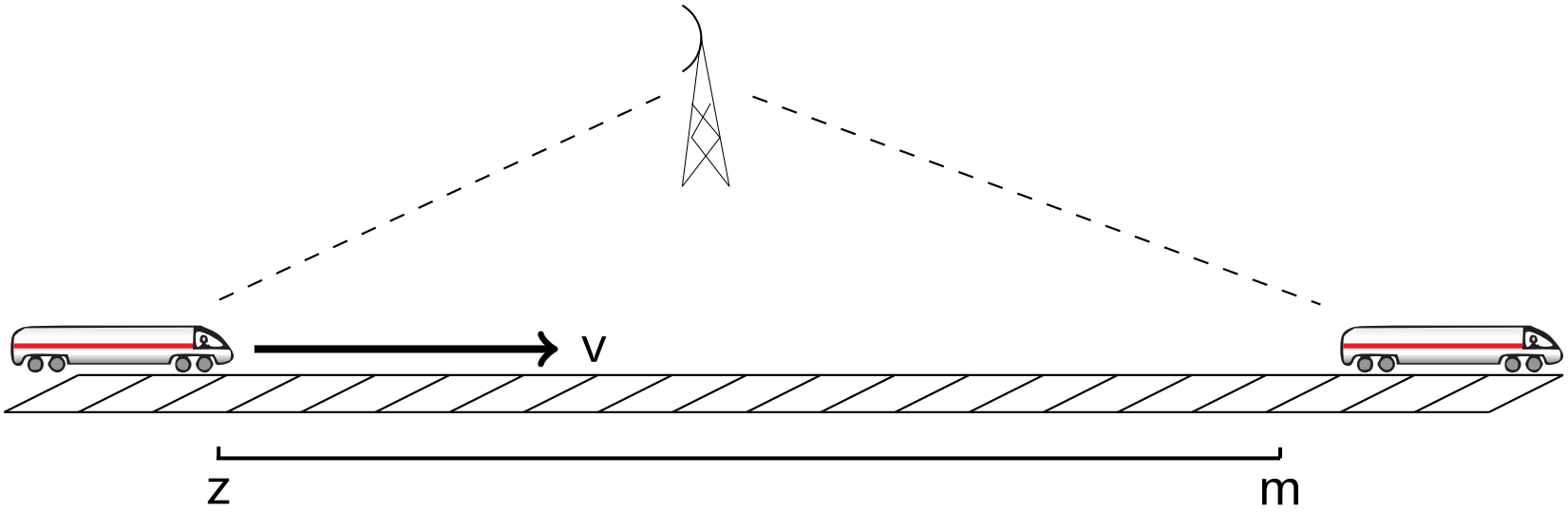
Differential Dynamic Logic (d$\mathcal{L}$)



Example

$$\underbrace{v^2 \leq 2b(m-z)}_{\text{Precondition}} \longrightarrow [\underbrace{\qquad\qquad}_{\text{Operational model}}](\underbrace{z \leq m}_{\text{Property}})$$

Differential Dynamic Logic (dℒ)

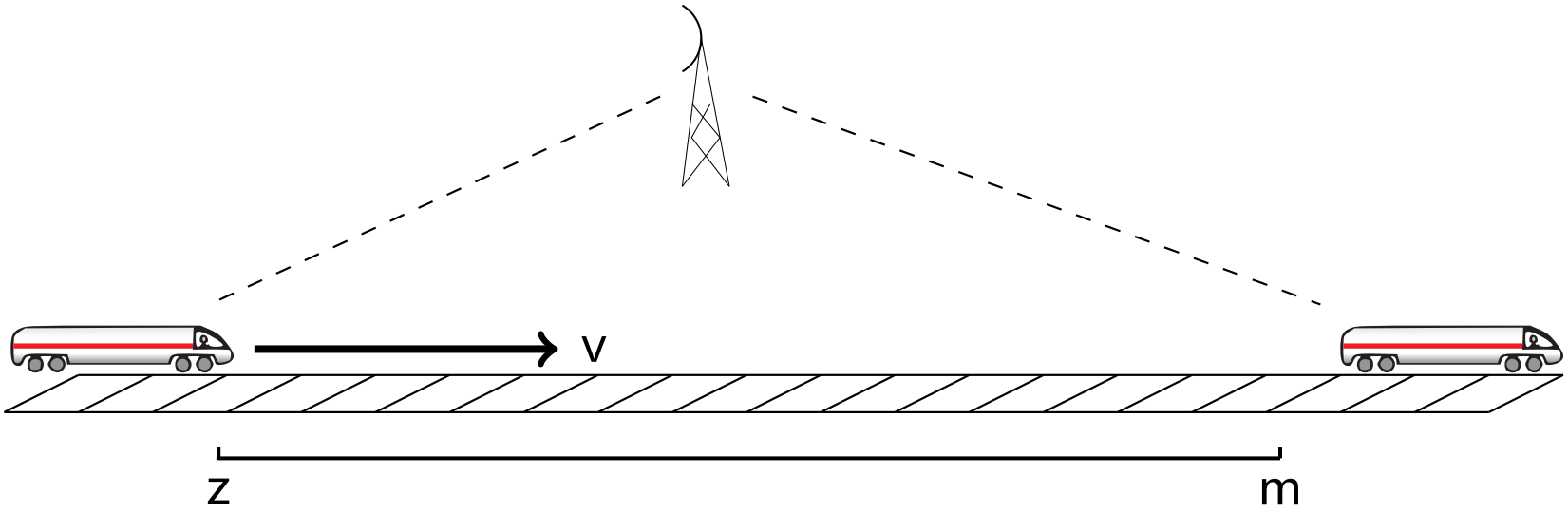


Example

$$\underbrace{v^2 \leq 2b(m-z)}_{\text{Precondition}} \longrightarrow \underbrace{[\underbrace{z' = v, v' = a}_{\text{Operational model}}]\underbrace{(z \leq m)}_{\text{Property}}}$$

Continuous evolution: differential equation

Differential Dynamic Logic (d$\mathcal{L}$)

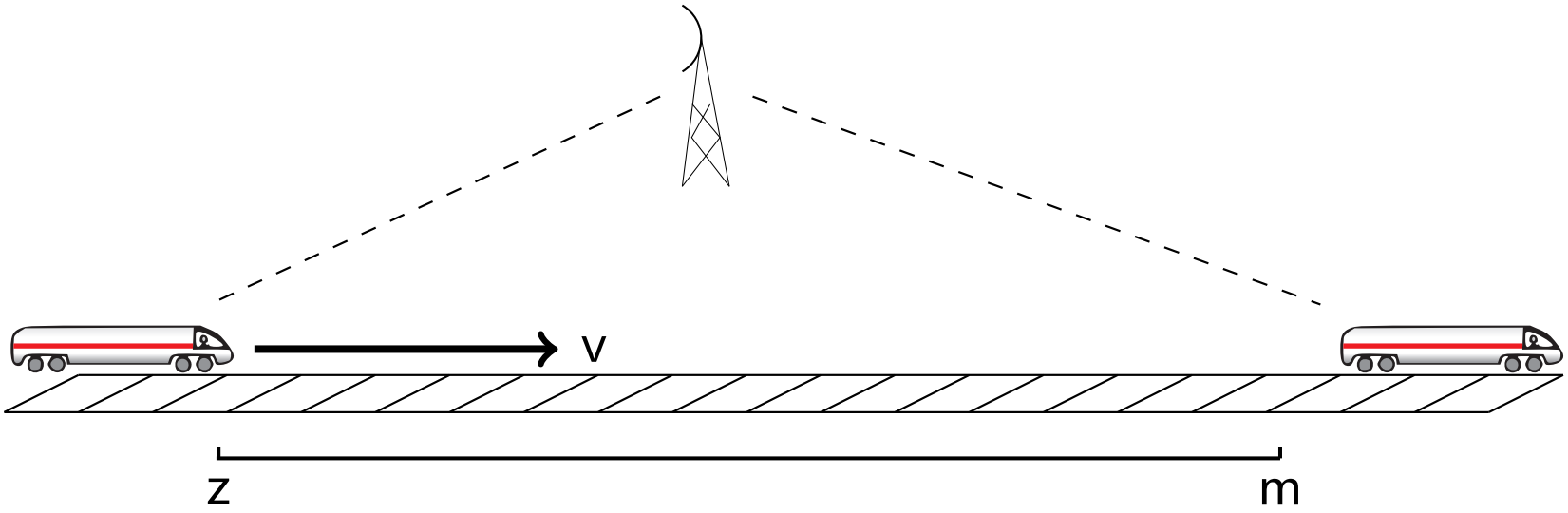


Example

$$\underbrace{v^2 \leq 2b(m-z)}_{\text{Precondition}} \longrightarrow \underbrace{[a := *; \quad z' = v, v' = a]}_{\text{Operational model}}\underbrace{(z \leq m)}_{\text{Property}}$$

Random assignment

Differential Dynamic Logic (dℒ)



Example

$$\underbrace{v^2 \leq 2b(m-z)}_{\text{Precondition}} \longrightarrow \underbrace{[a := *;\ ?a \leq -b;\ z' = v, v' = a]}_{\text{Operational model}}\underbrace{(z \leq m)}_{\text{Property}}$$

Test

Syntax of Differential Dynamic Logic

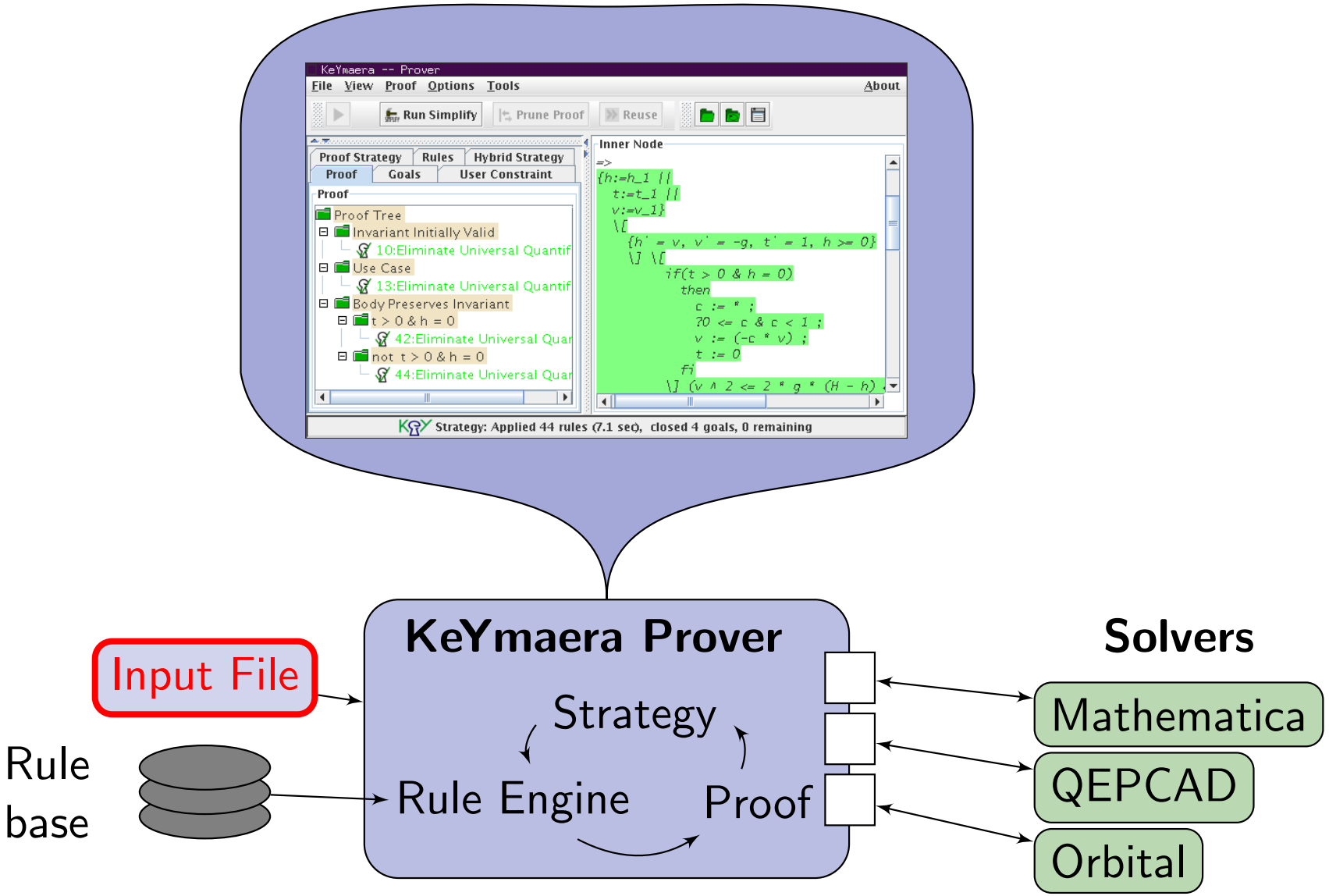
dℒ Formulas

$$\phi ::= \theta_1 \sim \theta_2 \mid \neg\phi \mid \phi \wedge \psi \mid \forall x \phi \mid \exists x \phi \mid [\alpha]\phi \mid \langle\alpha\rangle\phi$$

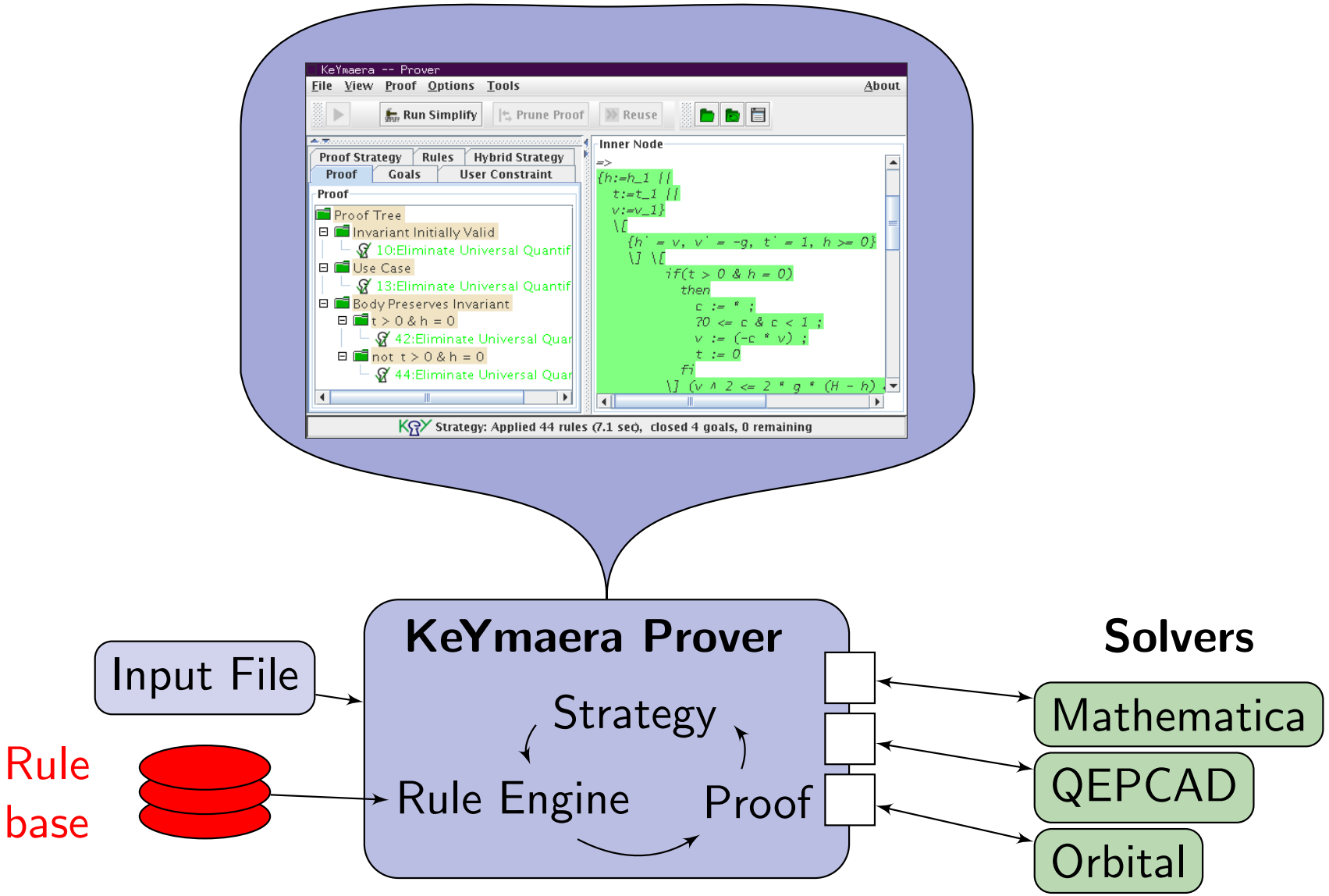
Hybrid Program	Effect
$\alpha;\ \beta$	sequential composition
$\alpha\ \cup\ \beta$	nondeterministic choice
α^*	nondeterministic repetition
$x := \theta$	discrete assignment (jump)
$x := *$	nondeterministic assignment
$(x_1' = \theta_1, \ldots, x_n' = \theta_n, F)$	continuous evolution of x_i
$?F$	check if formula F holds

A. Platzer.
Differential Dynamic Logic for Hybrid Systems.
Journal of Automated Reasoning, 41(2), 2008, to appear.

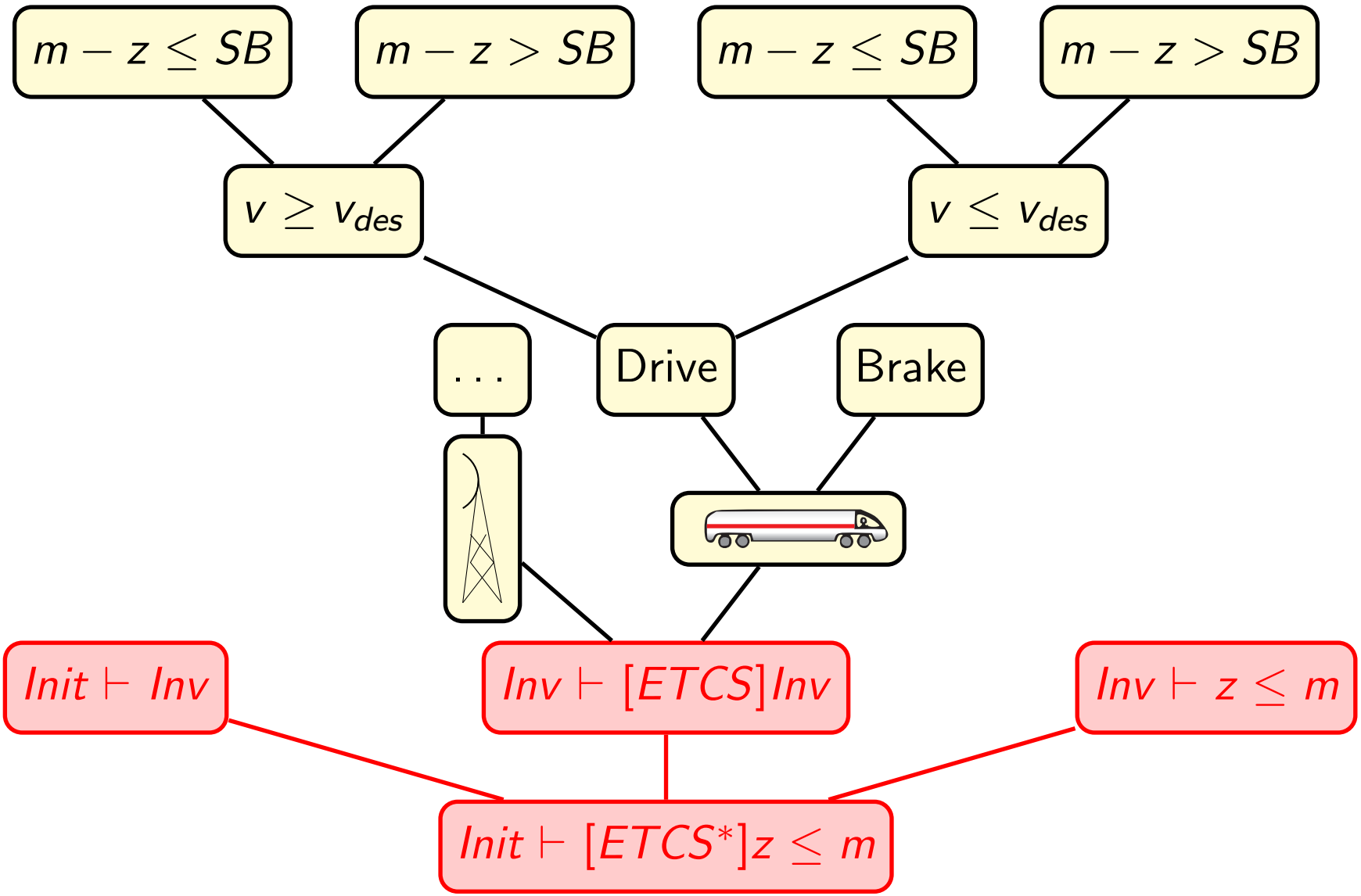
KeYmaera Architecture



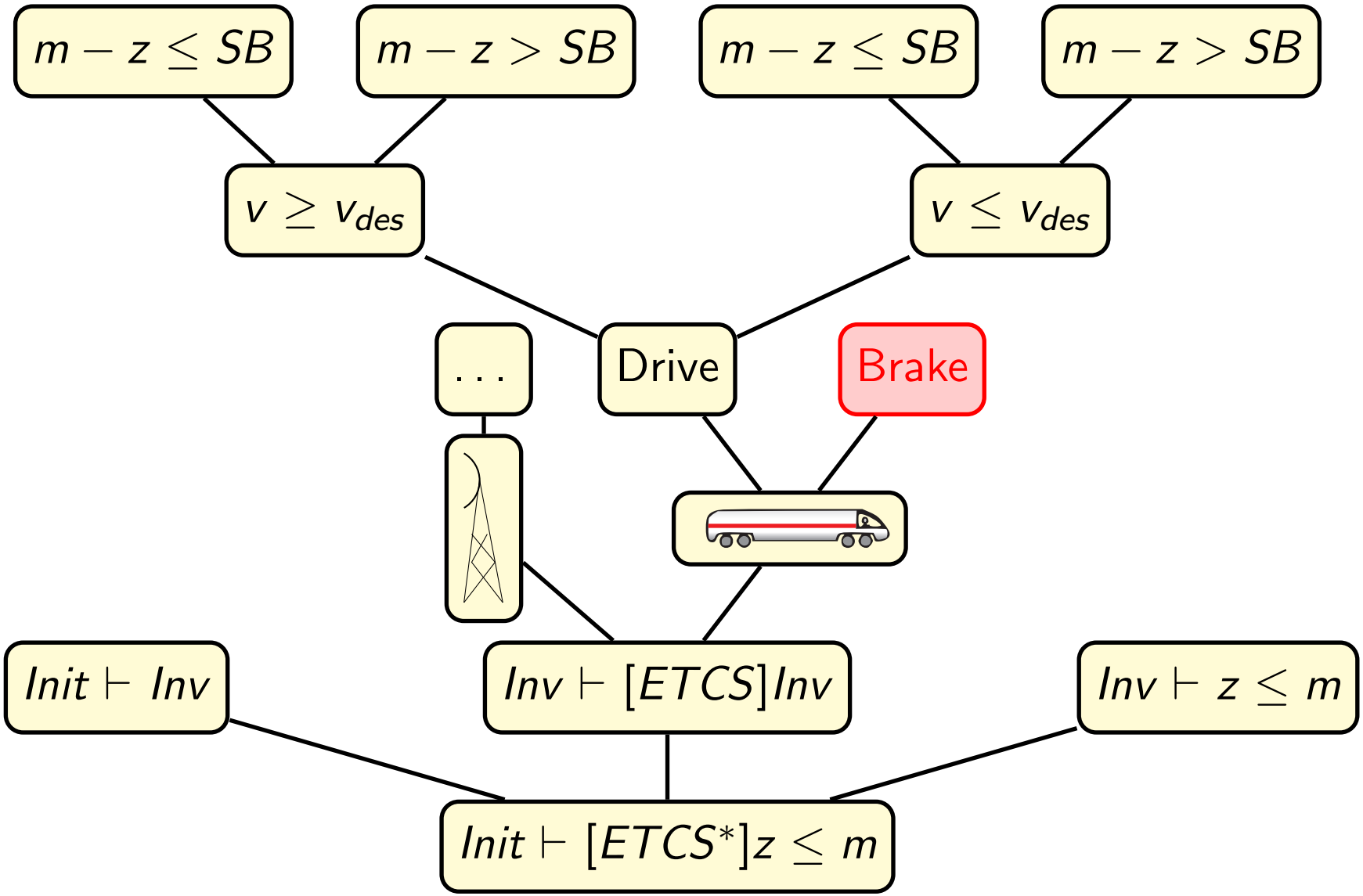
KeYmaera Architecture



Proof Sketch



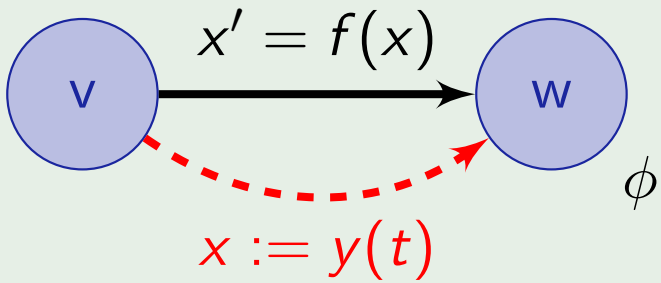
Proof Sketch



Handling Differential Equations

Example

$$\frac{\forall t \geq 0\ [x := y(t)]\,\phi}{[x' = f(x)]\,\phi}$$



$$\ldots \vdash [z' = v, v' = -b]z \leq m$$

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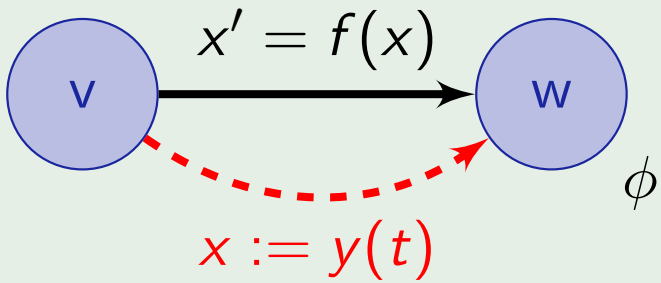

$$\frac{}{\ldots \vdash \forall t \geq 0\,[z := -\tfrac{1}{2}bt^2 + tv + z]z \leq m}$$
$$\ldots \vdash [z' = v, v' = -b]z \leq m$$

Handling Differential Equations

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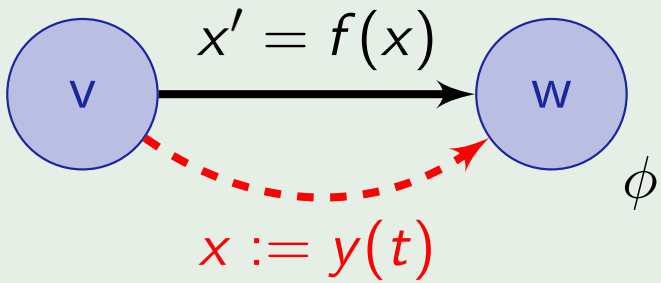
$$\frac{\forall t \geq 0\ [x := y(t)]\,\phi}{[x' = f(x)]\,\phi}$$



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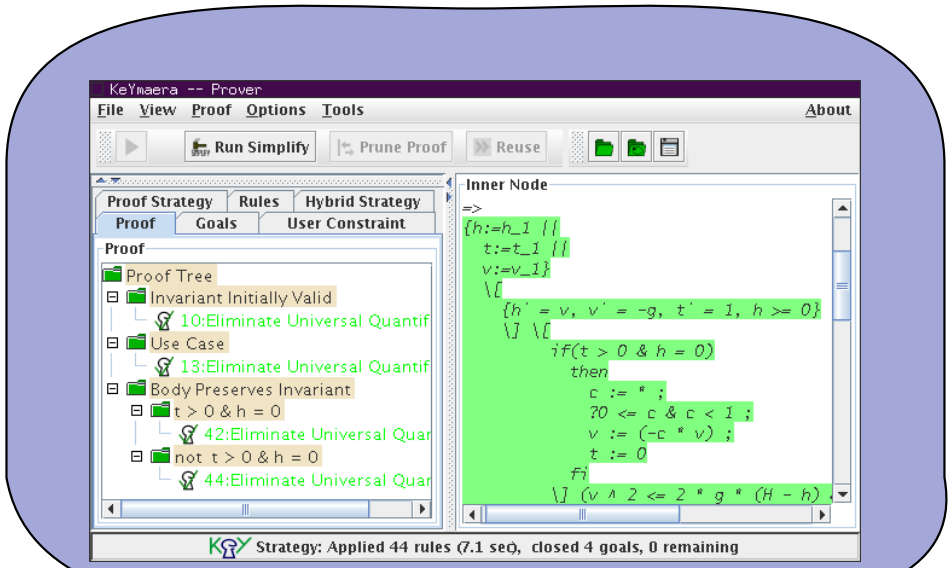
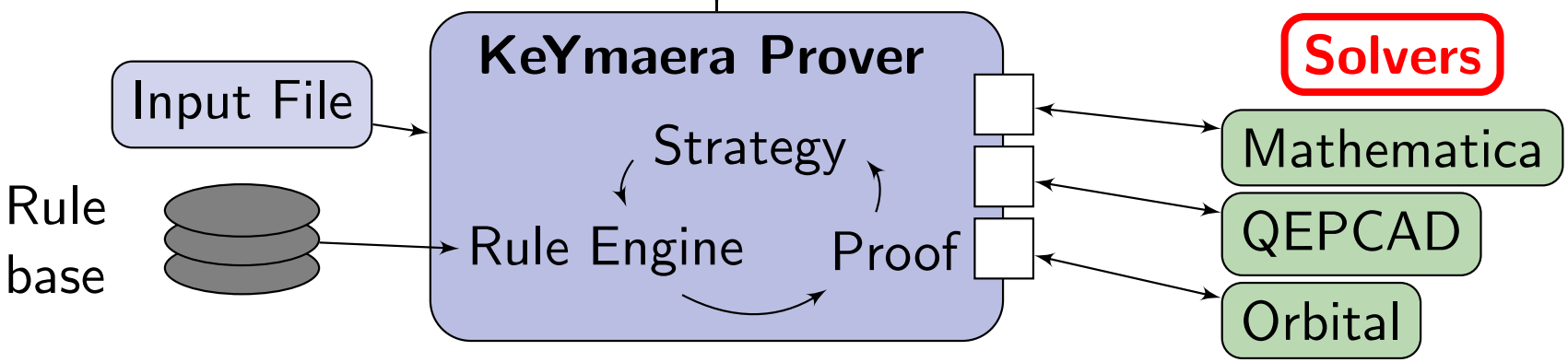
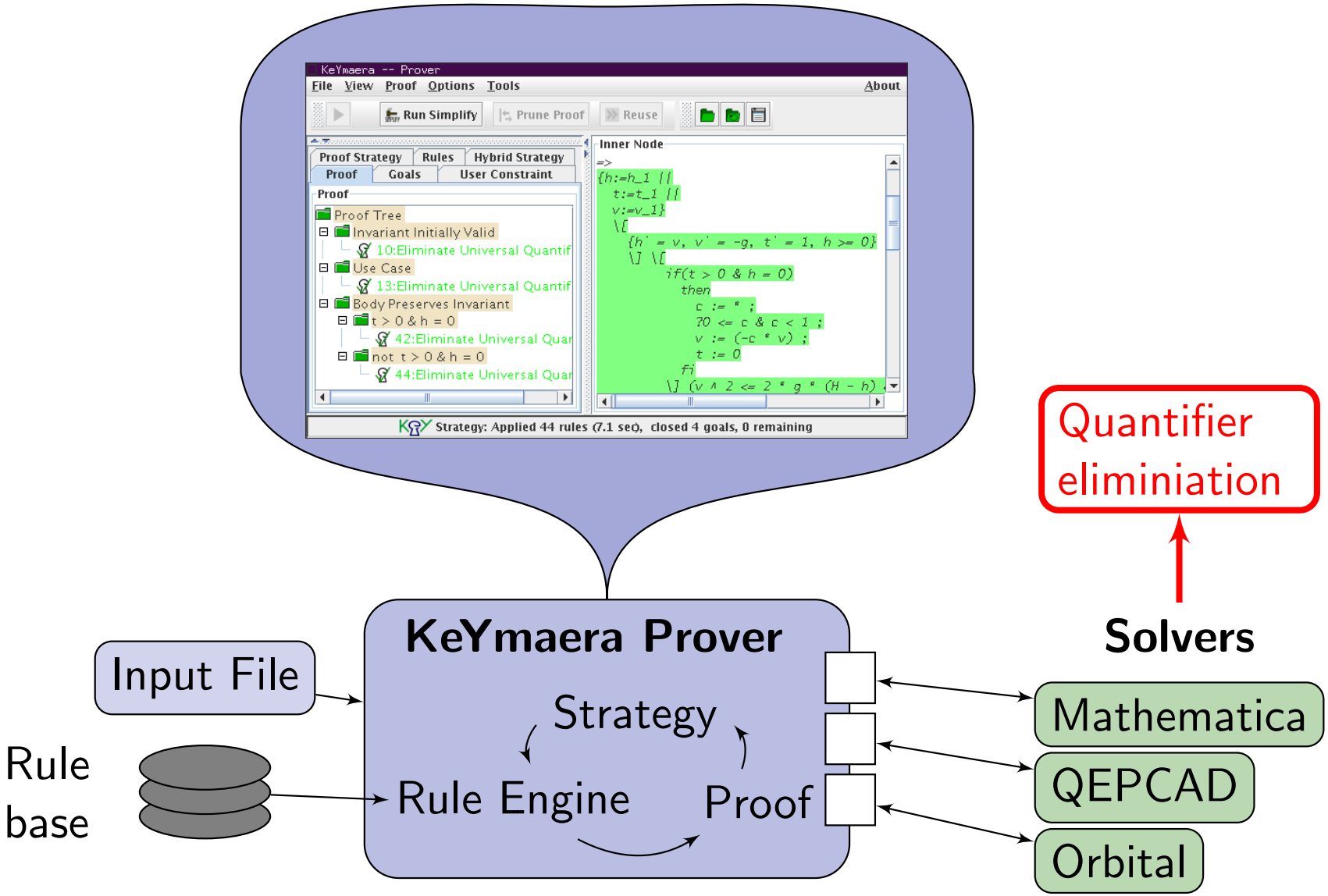
KeYmaera Architecture





KeYmaera Architecture



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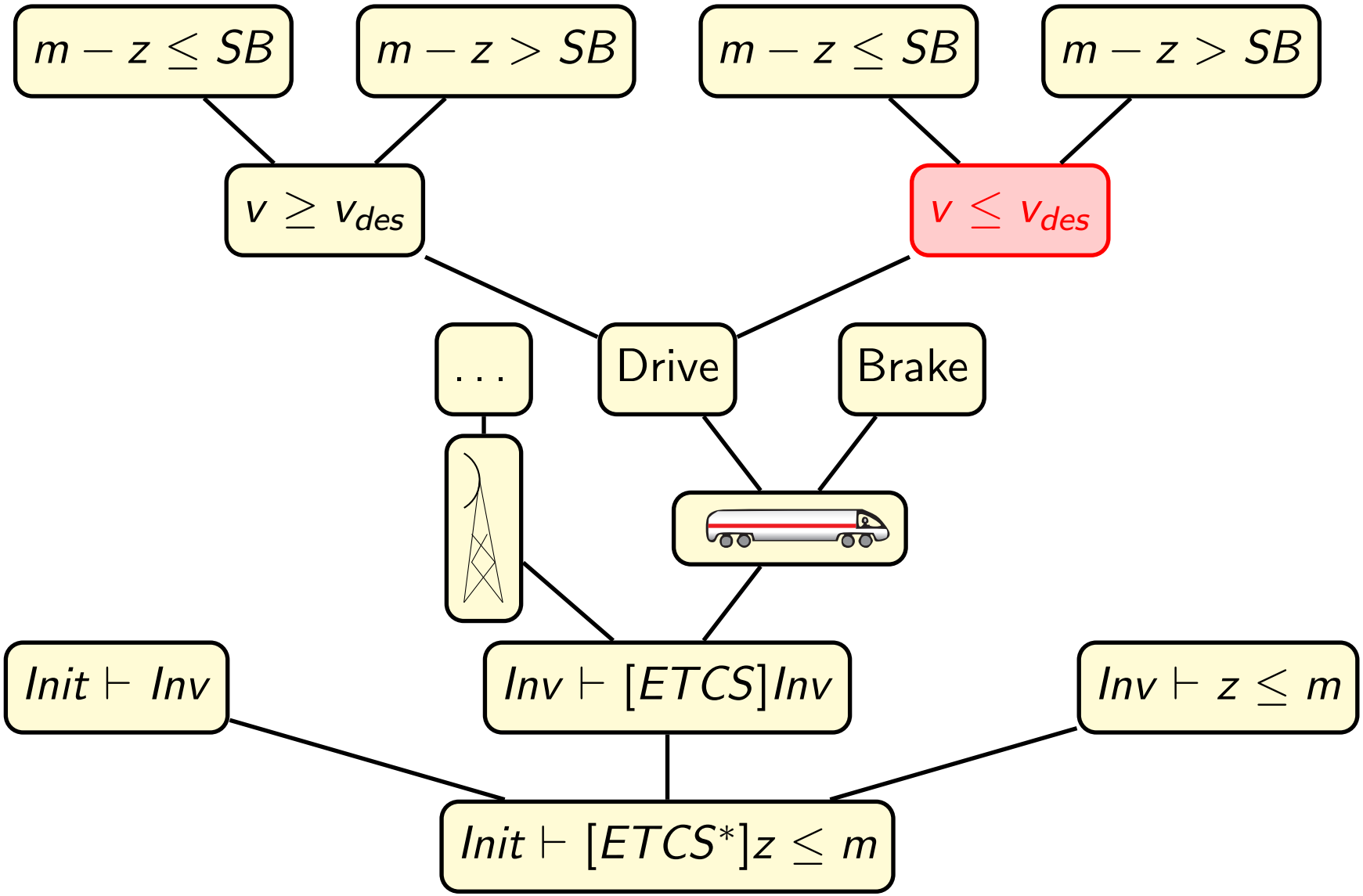

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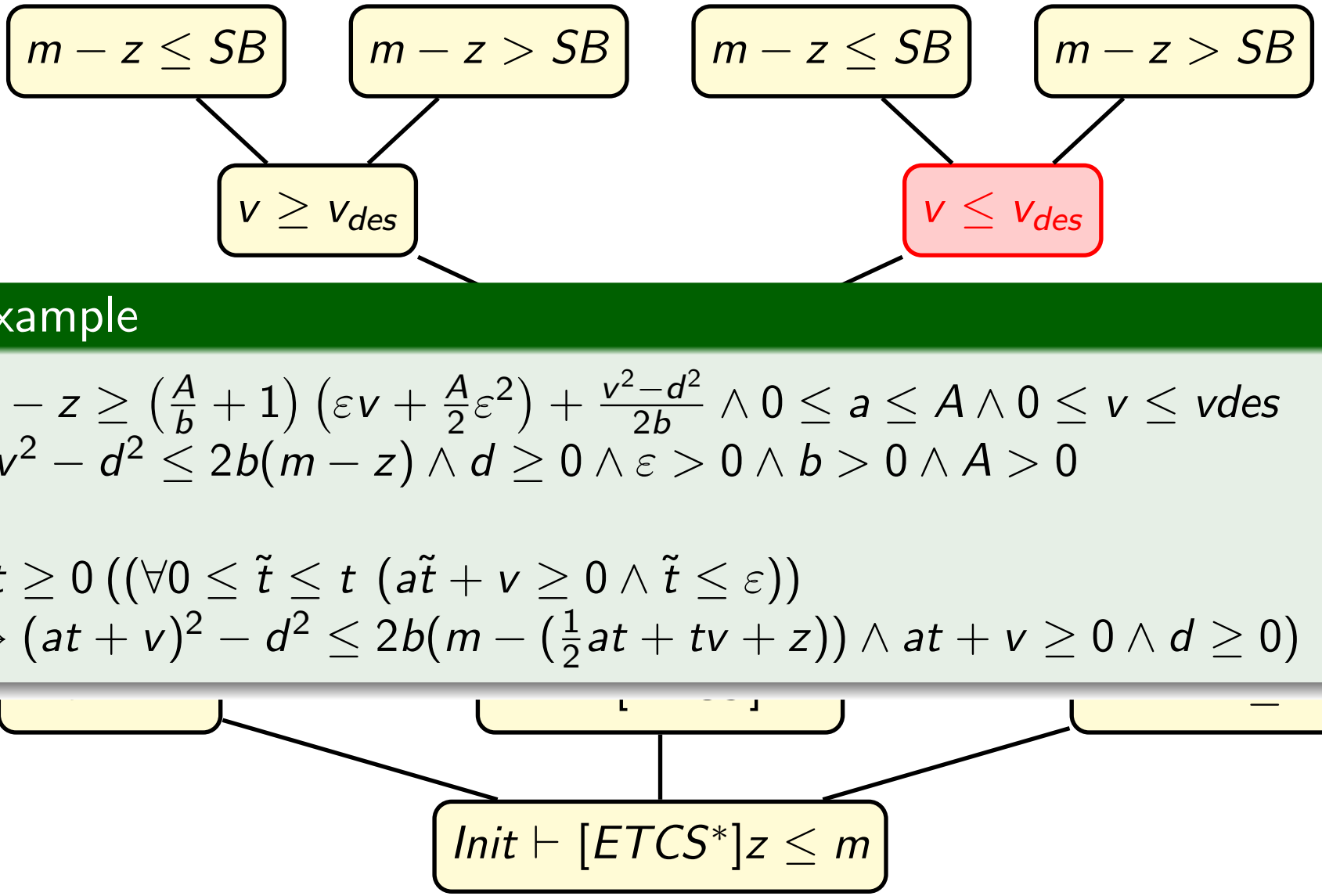


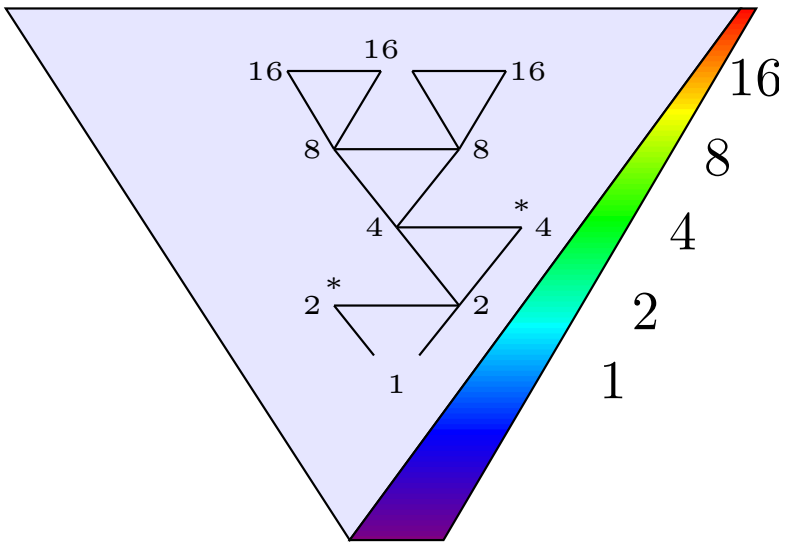
Example

$$m - z \geq \left(\frac{A}{b} + 1\right)\left(\varepsilon v + \frac{A}{2}\varepsilon^2\right) + \frac{v^2 - d^2}{2b} \wedge 0 \leq a \leq A \wedge 0 \leq v \leq vdes$$
$$\wedge v^2 - d^2 \leq 2b(m - z) \wedge d \geq 0 \wedge \varepsilon > 0 \wedge b > 0 \wedge A > 0$$
$$\vdash$$
$$\forall t \geq 0 \left((\forall 0 \leq \tilde{t} \leq t \ (a\tilde{t} + v \geq 0 \wedge \tilde{t} \leq \varepsilon)\right)$$
$$\rightarrow (at + v)^2 - d^2 \leq 2b(m - (\tfrac{1}{2}at + tv + z)) \wedge at + v \geq 0 \wedge d \geq 0)$$

Iterative Background Closure

- Quantifier elimination is doubly exponential
- Choice conflict:
  1. Apply quantifier elimination
  2. Split using

$$\frac{\vdash F \qquad \vdash G}{\vdash F \wedge G}$$



Experimental Results

Case Study	Interact	Steps	IBC(s)	Eager QE(s)
ETCS essentials	0	46	47.8	∞
	1	46	6.6	8.8
ETCS complete	0	163	2045.2	∞
	1	168	23.3	∞
ETCS reactivity	0	49	76.2	∞
ETCS liveness	3	112	17.6	16.0
Aircraft TRM	0	94	10.9	∞
	1	94	1.2	1.2
TRM 3 Planes	0	187	171.8	∞
	1	187	21.2	∞
TRM 4 Planes	0	255	704.3	∞
	1	255	170	∞
Water tank	0	-	∞	∞
	1	375	2.0	2.0

$\infty \mathrel{\hat{=}}$ more than five hours

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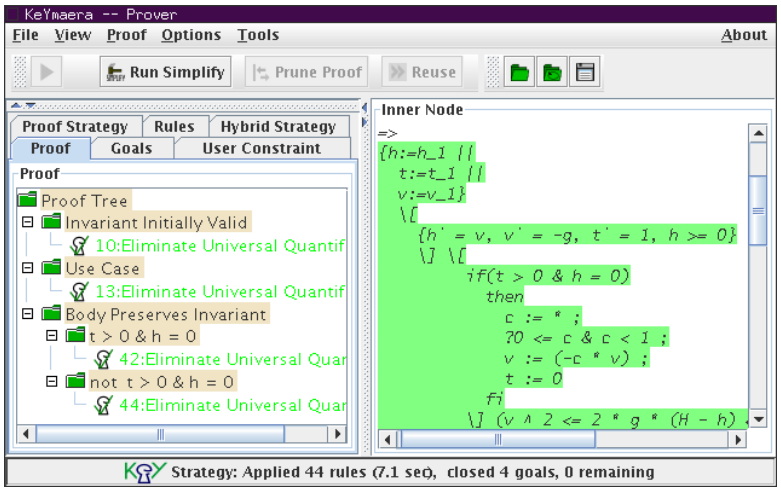
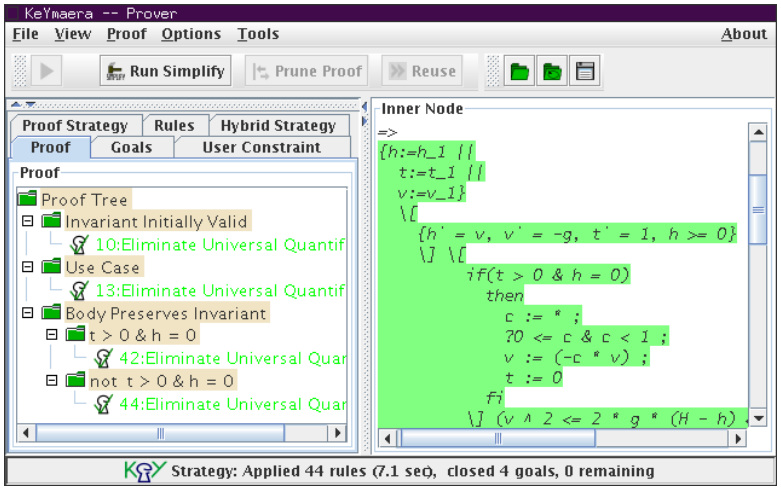
Conclusions

KeYmaera Summary

- Hybrid theorem prover for verifying hybrid systems
  - Differential Dynamic Logic (d$\mathcal{L}$) and Hybrid Programs
  - Sequent calculus
  - Quantifier elimination, computer algebra
- Automatic proof strategies (95 – 100%)
- Plugin-architecture for backends



Conclusions



Features

- Verify controllability, reactivity, safety and liveness properties
- Counterexample generation
- Automatic invariant discovery
- Handling differential inequalities ($z'' \leq a$)
- Support for proof annotations
- Equational Gröbner basis verification support


A. Platzer.
Differential Dynamic Logic for Verifying Parametric Hybrid Systems.
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