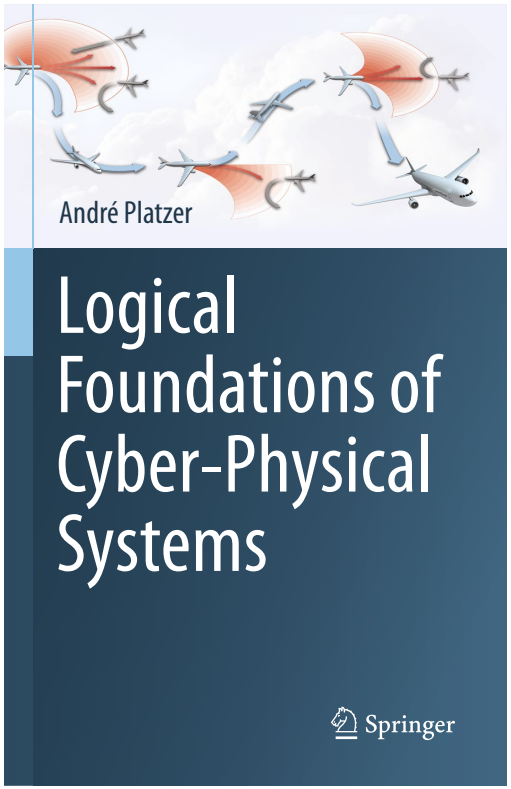
20: Virtual Substitution & Real Equations

Logical Foundations of Cyber-Physical Systems



André Platzer

Karlsruhe Institute of Technology
Department of Informatics

Computer Science Department
Carnegie Mellon University

Outline

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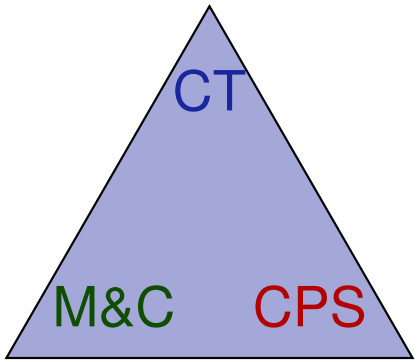
rigorous arithmetical reasoning
miracle of quantifier elimination
logical trinity for reals
switch between syntax & semantics at will
virtual substitution lemma
bridge gap between semantics and inexpressibles



analytic complexity
modeling tradeoffs

verifying CPS at scale

Outline

Evaluating Real Arithmetic Formulas

$$x^2 > 2 \land 2x < 3 \lor x^3 \leq x^2$$

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When $\omega(x) = 2$

$$\omega[\![x^2 > 2 \land 2x < 3 \lor x^3 \leq x^2]\!]$$

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Are the following formulas valid, i.e., true in all states?

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Evaluating Real Arithmetic Formulas

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Framing the Miracle: Quiz

Is validity of formulas

decidable/semidecidable/undecidable/not semidecidable for:

1. Propositional logic [no variables]
2. FOL$[p, f, \ldots]$ uninterpreted
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×	FOL$_{\mathbb{R}}[+, \cdot, \sin, =, <]$ $\sin x = 0$	not semidecidable [Richardson'68]

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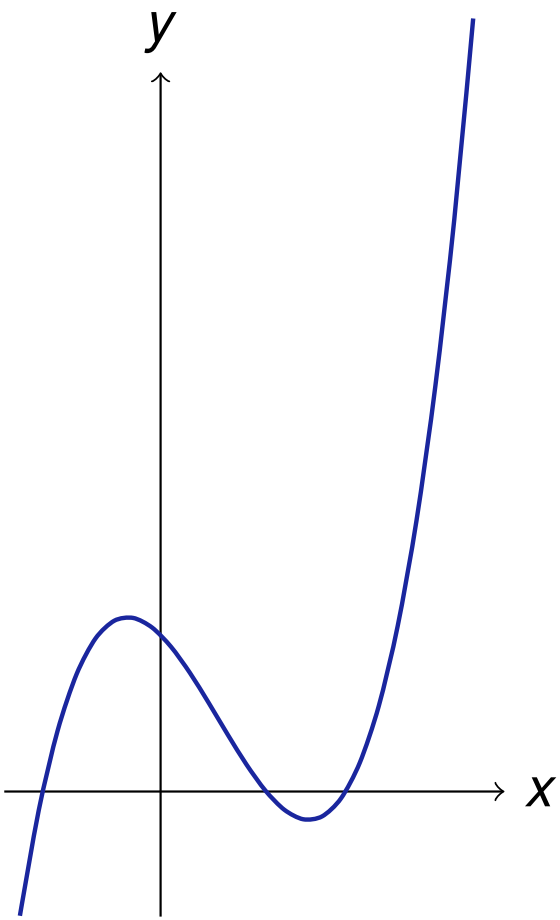
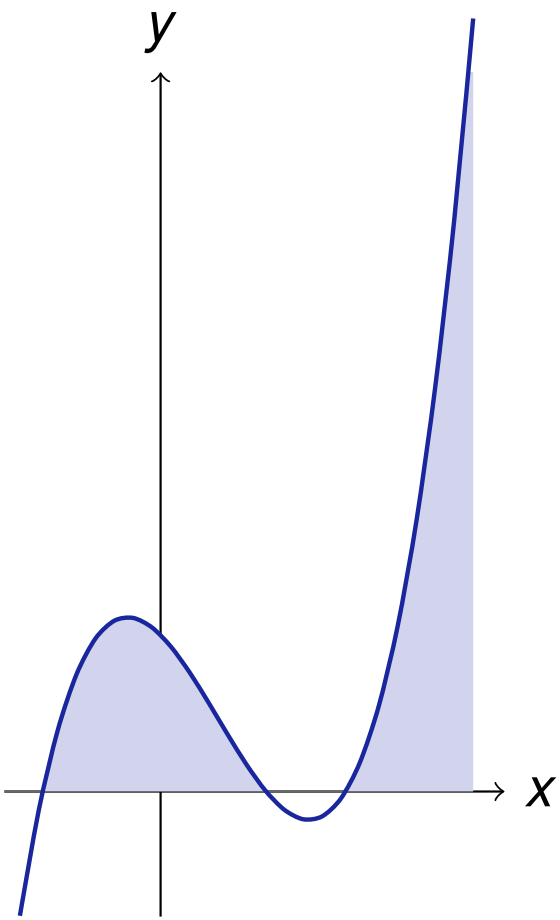
Quantifier Elimination ⇜⇝ Projection



$$F \equiv \exists y\,(y \geq 0 \wedge 1 - x - 1.83x^2 + 1.66x^3 > y)$$

Quantifier Elimination ⇜⇝ Projection



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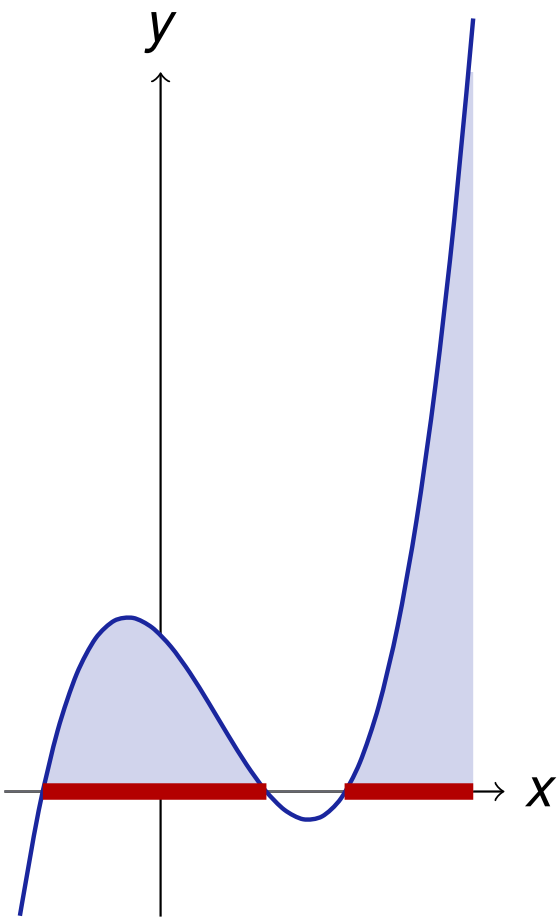

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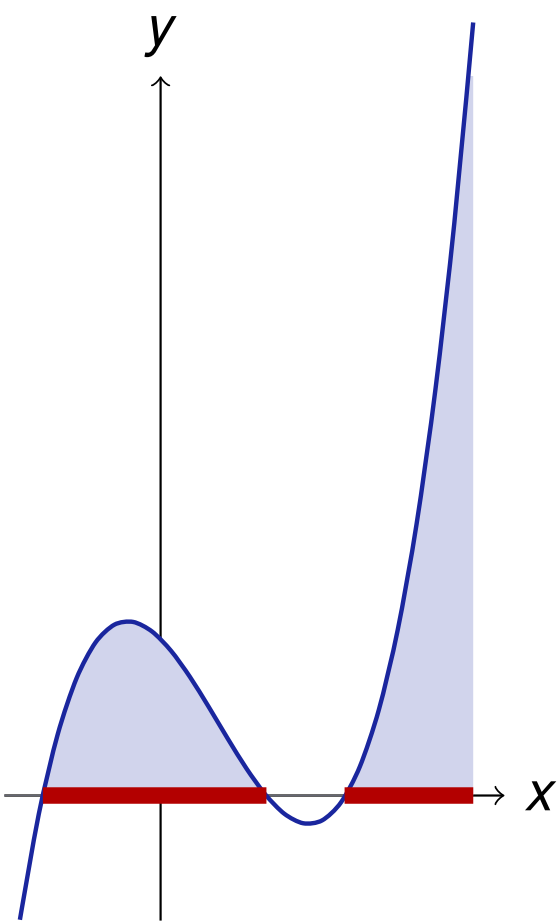

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QE

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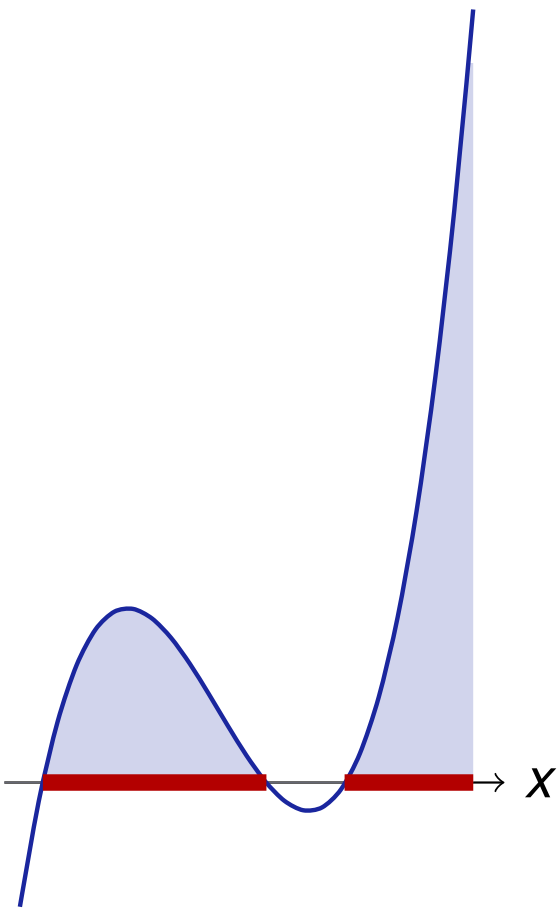

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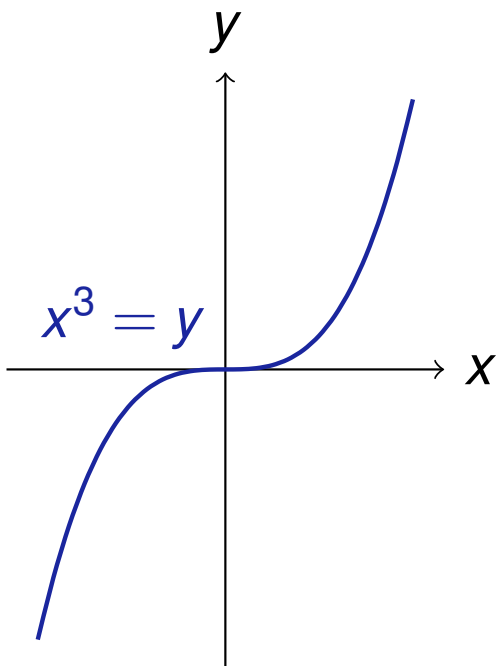
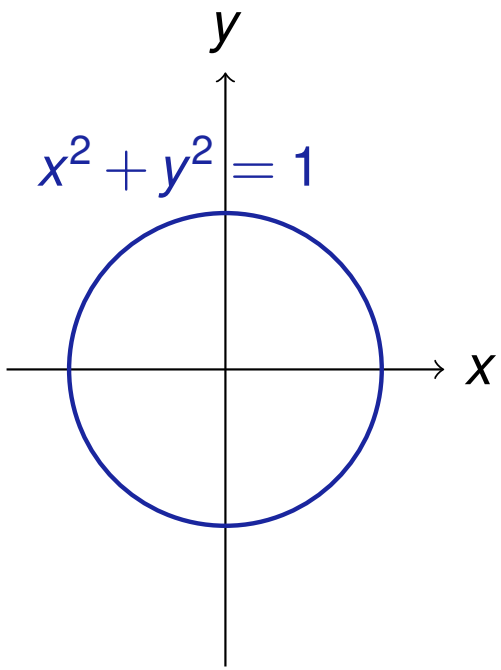
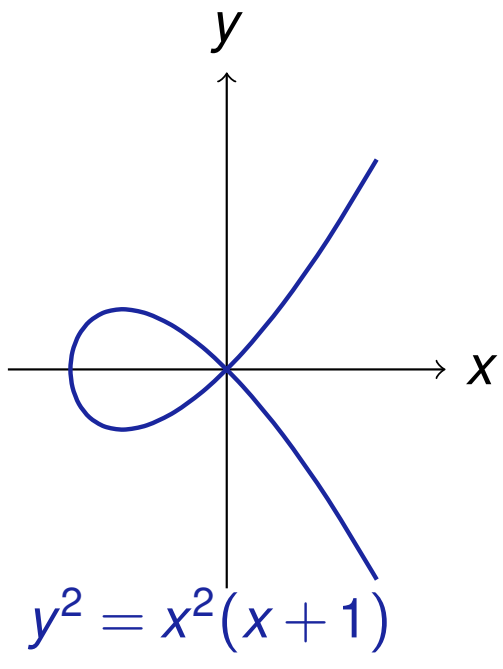
If all but one variable has fixed value: Finite union of intervals.

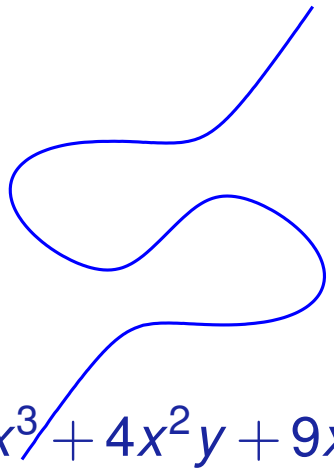
Univariate polynomials have finitely many roots. Signs change finitely often.

Polynomial Equations ⟺ Algebraic Varieties



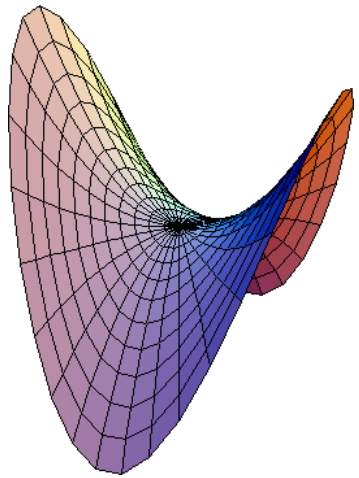






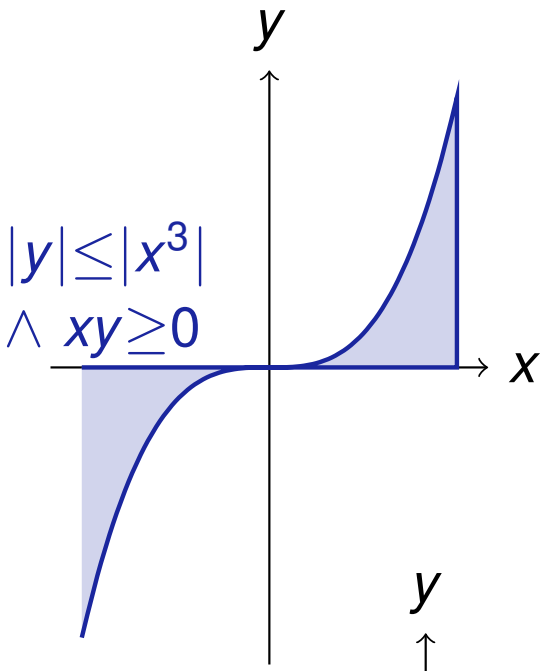
$4x^3 + 4x^2y + 9xy^2 - 9y^3 - 36x + 36y = 0$

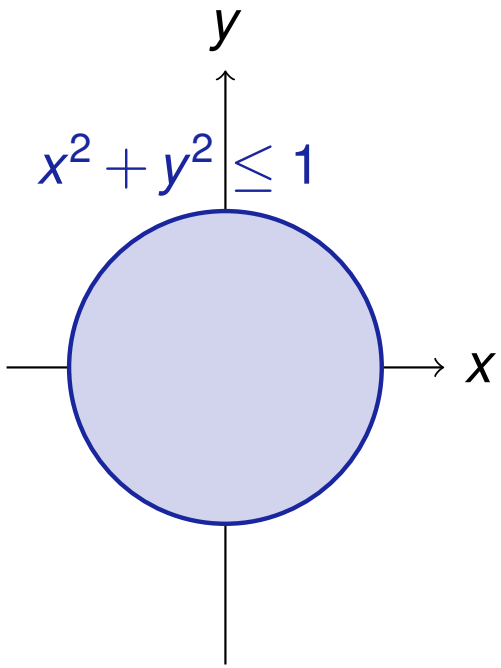
$z = x^2 - y^2$

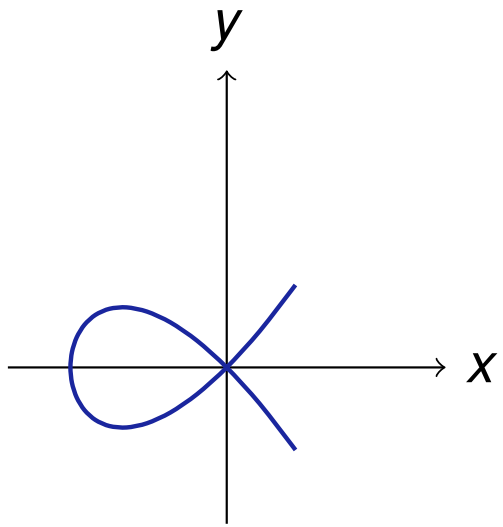


Algebraic variety: defined by conjunction of polynomial equations

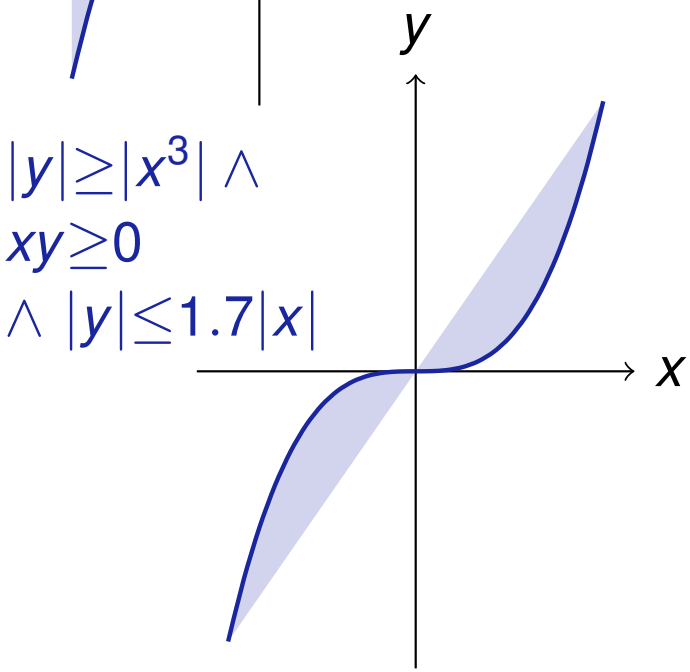
Polynomial Inequalities ⇜⇝ Semialgebraic Sets







$y^2 = x^2(x+1)$
$\wedge\ x \leq 0.44$



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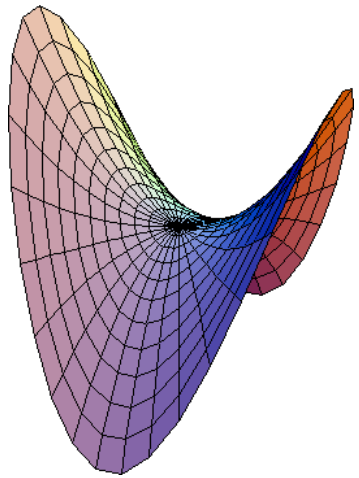

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Theorem (Tarski'31)

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Theorem (Complexity, Davenport&Heintz'88,Weispfenning'88)

(Time and space) complexity of QE for $\mathbb{R}$ is doubly exponential in the number n of quantifier (alternations).

$$2^{2^{O(n)}}$$

Quantifier Elimination in Real Arithmetic

Theorem (Tarski'31)

First-order logic of real arithmetic is decidable since it admits quantifier elimination, i.e., for each formula P, compute quantifier-free formula $\mathrm{QE}(P)$ that is equivalent, i.e., $P \leftrightarrow \mathrm{QE}(P)$ is valid.

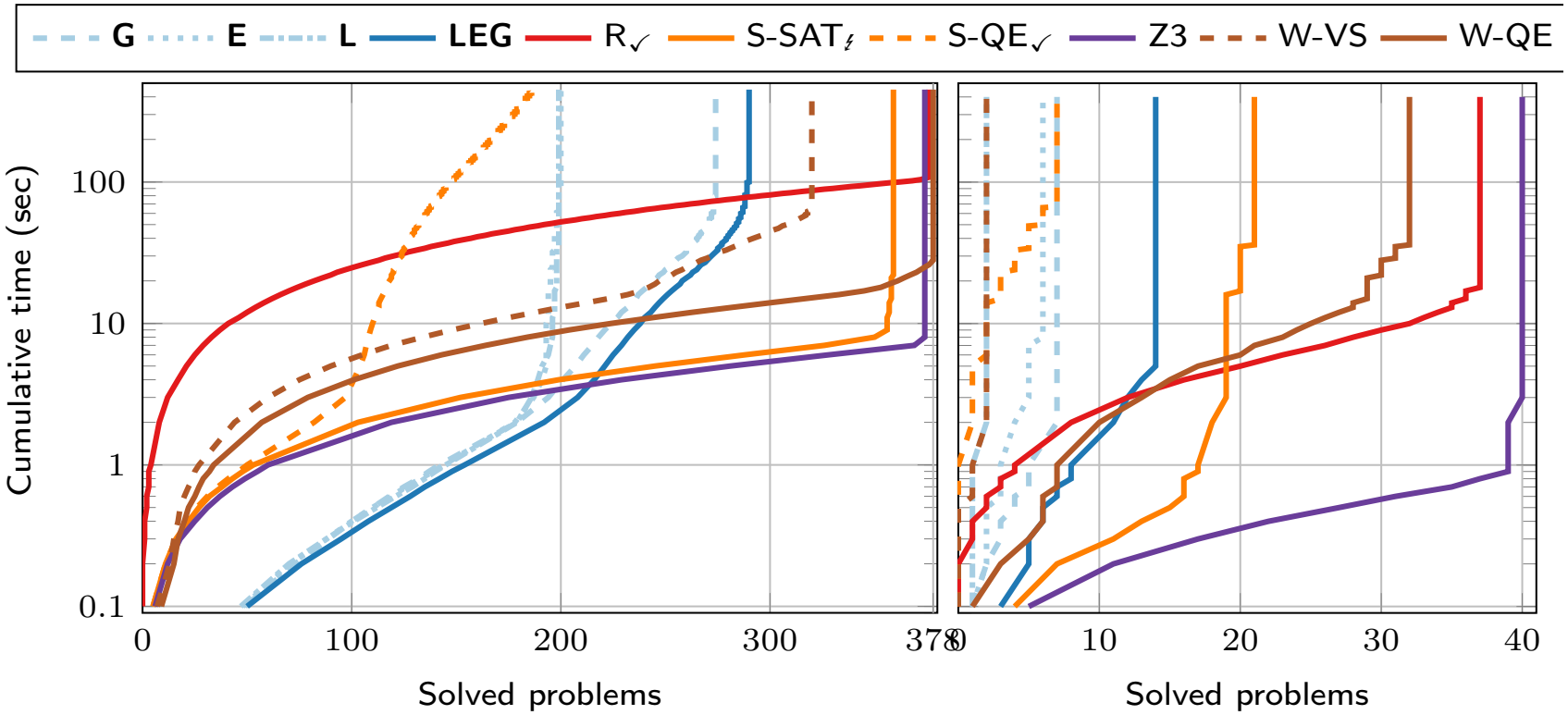
Theorem (Complexity, Davenport&Heintz'88,Weispfenning'88)

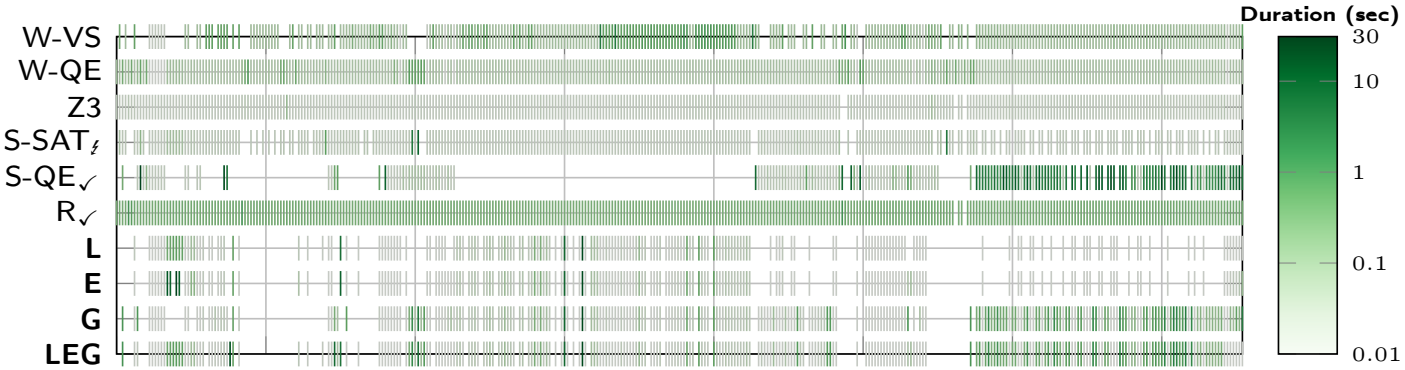
(Time and space) complexity of QE for $\mathbb{R}$ is doubly exponential in the number n of quantifier (alternations).

$$2^{2^{O(n)}}$$

Answer even for one *free* variable and only linear polynomials

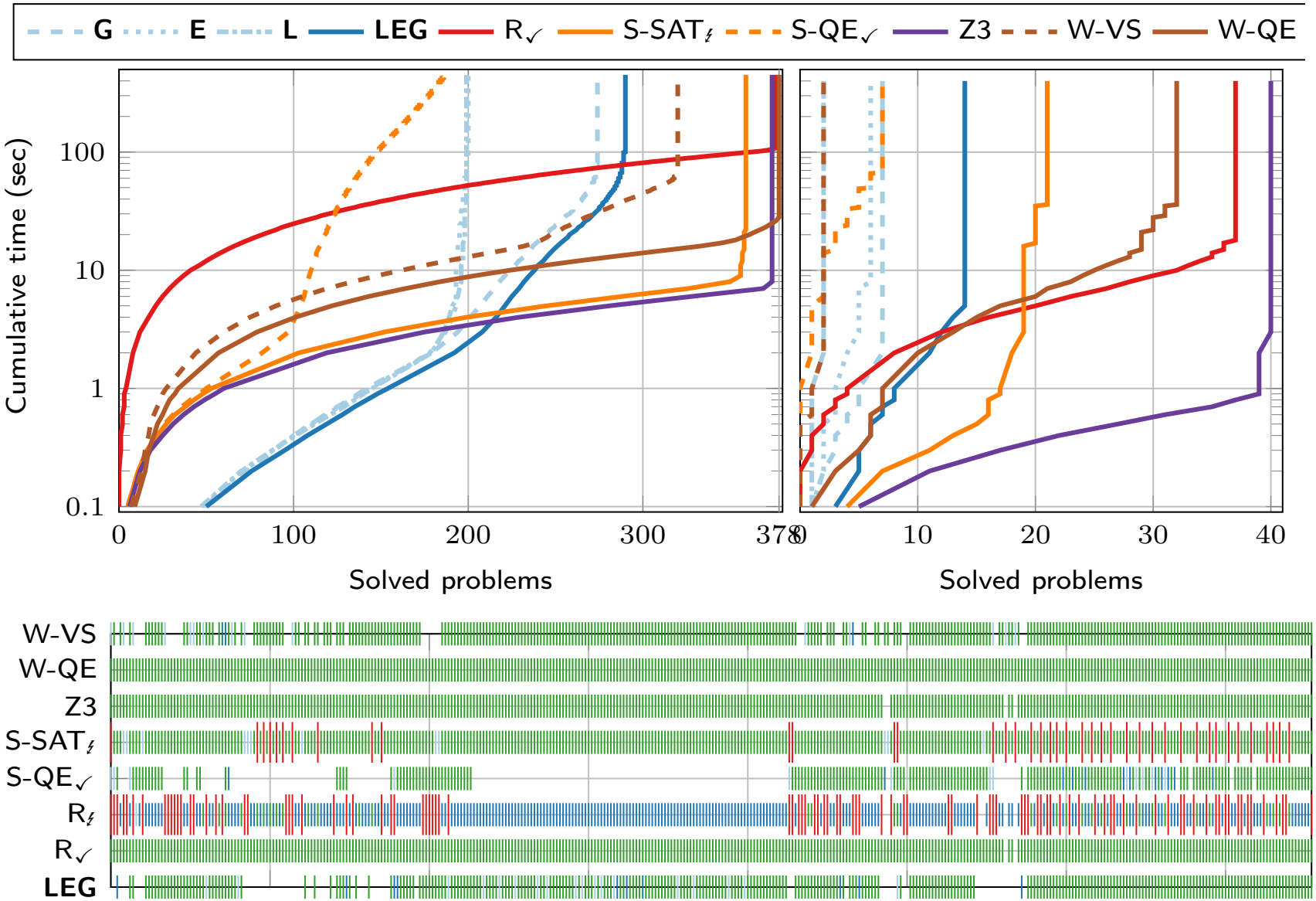
Quantifier Elimination Algorithms Make a Difference





FM'21

Quantifier Elimination Algorithms Make a Difference



QE has computational & correctness impact: |=single, |=inconsistent

FM'21

Quantifier Elimination Examples

$$\begin{aligned}
\mathrm{QE}(\exists x\,(2x^2 + c \leq 5)) &\equiv \\
\mathrm{QE}(\forall c \exists x\,(2x^2 + c \leq 5)) &\equiv
\end{aligned}$$

$$\begin{aligned}
\mathrm{QE}(\exists x\,(a = b + x^2)) &\equiv \\
\mathrm{QE}(\exists x\,(x^2 = 2)) &\equiv \\
\mathrm{QE}(\exists x\,(x^2 = 2 \land y = x)) &\equiv
\end{aligned}$$

Quantifier Elimination Examples

$$\mathrm{QE}(\exists x\,(2x^2 + c \leq 5)) \quad \equiv c \leq 5$$
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Framework: Logical Normalization for QE

$$\begin{aligned}
\mathrm{QE}(A \wedge B) &\equiv \\
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\mathrm{QE}(\neg A) &\equiv \\
\mathrm{QE}(\forall x\, A) &\equiv \\
\mathrm{QE}(\exists x\, A) &\equiv
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A has quantifiers

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\mathrm{QE}(\exists x\, (A \wedge (B \vee C))) &\equiv \mathrm{QE}(\exists x\, ((A \wedge B) \vee (A \wedge C))) && \text{expensive} \\
\mathrm{QE}(\exists x\, ((A \vee B) \wedge C)) &\equiv \mathrm{QE}(\exists x\, ((A \wedge C) \vee (B \wedge C))) && \text{expensive}
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Framework: Logical Normalization for QE

Normal Form $\mathrm{QE}(\exists x\,(A_1 \wedge \ldots \wedge A_k))$ with atomic A_i

$$\mathrm{QE}(A \wedge B) \equiv \mathrm{QE}(A) \wedge \mathrm{QE}(B)$$
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Framework: Arithmetical Normalization for QE

Normal Form $\mathrm{QE}(\exists x\,(p_1 \sim_i 0 \wedge \ldots \wedge p_k \sim_k 0))$ and $\sim_i \in \{>, =, \geq, \neq\}$

$$p = q \equiv p - q = 0$$
$$p \geq q \equiv p - q \geq 0$$
$$p > q \equiv p - q > 0$$
$$p \neq q \equiv p - q \neq 0$$
$$p \leq q \equiv q - p \geq 0$$
$$p < q \equiv q - p > 0$$
$$\neg(p \geq q) \equiv p < q$$
$$\neg(p > q) \equiv p \leq q$$
$$\neg(p = q) \equiv p \neq q$$
$$\neg(p \neq q) \equiv p = q$$

Quantifier Elimination by Virtual Substitution

Virtual Substitution

$$\exists x\, F \leftrightarrow \bigvee_{t \in T} A_t \wedge F_x^t$$

where terms T substituted (virtually) into F depend on F
where A_t are quantifier-free additional compatibility conditions

Scalability requires simplifier for intermediate results

Quantifier Elimination by Virtual Substitution

Virtual Substitution

Quantifier $\exists x\, F \leftrightarrow \bigvee_{t \in T} A_t \wedge F_x^t$ Quantifier-free

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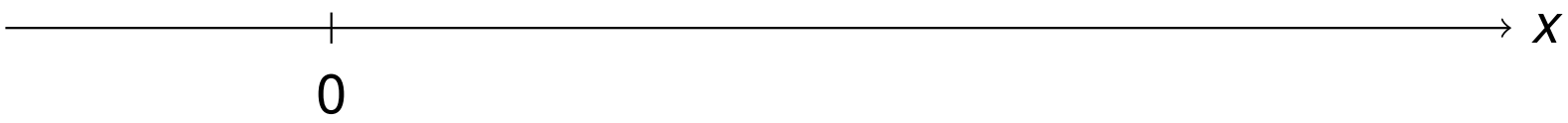
Naïve Virtual Substitution by Example



Can we get rid of the quantifier without changing the semantics?

$$\exists x(x > 2 \land x < \tfrac{17}{5})$$

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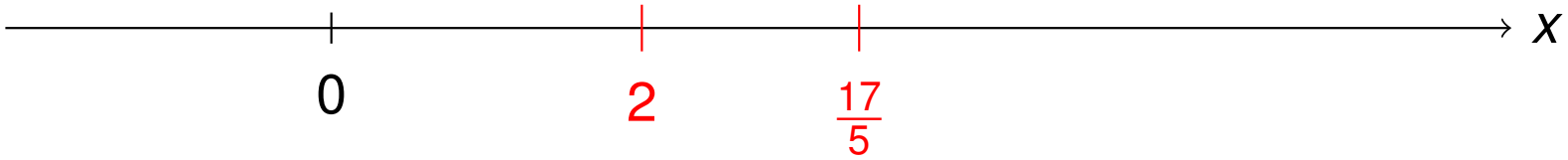

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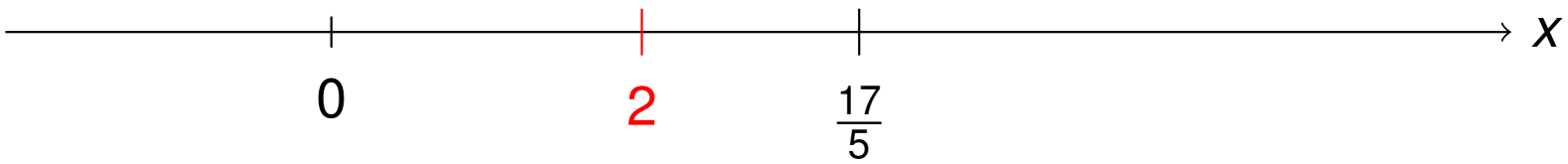

Can we get rid of the quantifier without changing the semantics?

$$\begin{array}{lll} & \exists x(x > 2 \land x < \frac{17}{5}) & \\ \equiv & (2 > 2 \land 2 < \frac{17}{5}) & \text{boundary case “}x = 2\text{”} \end{array}$$

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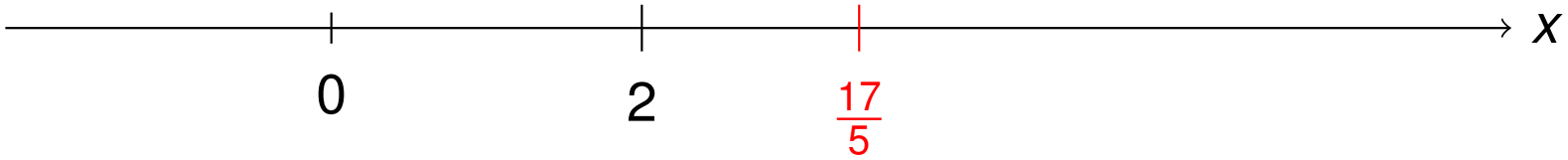

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\end{array}
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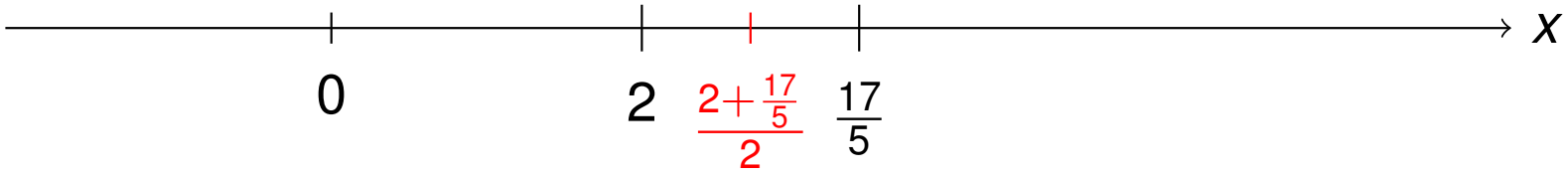

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\lor & (\frac{2+\frac{17}{5}}{2} > 2 \land \frac{2+\frac{17}{5}}{2} < \frac{17}{5}) & \text{intermediate case “}x = \frac{2+\frac{17}{5}}{2}\text{”}
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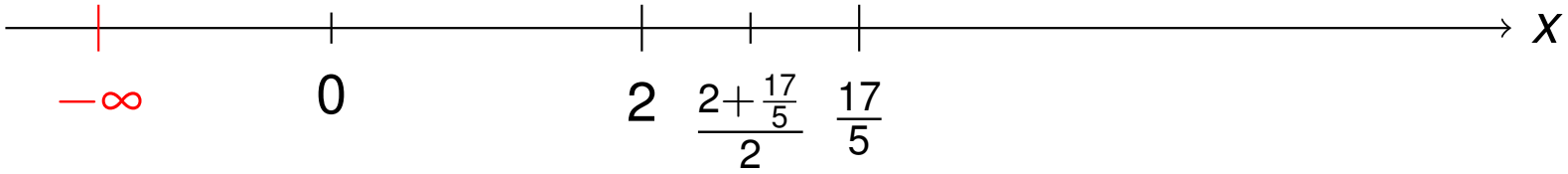

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\lor & (-\infty > 2 \land -\infty < \frac{17}{5}) & \text{extremal case “}x = -\infty\text{”}
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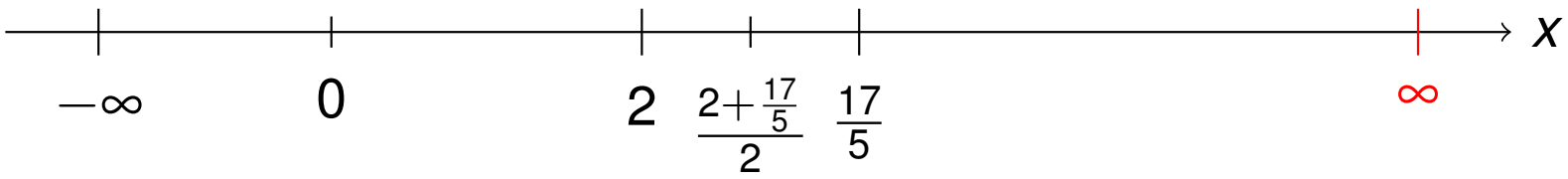

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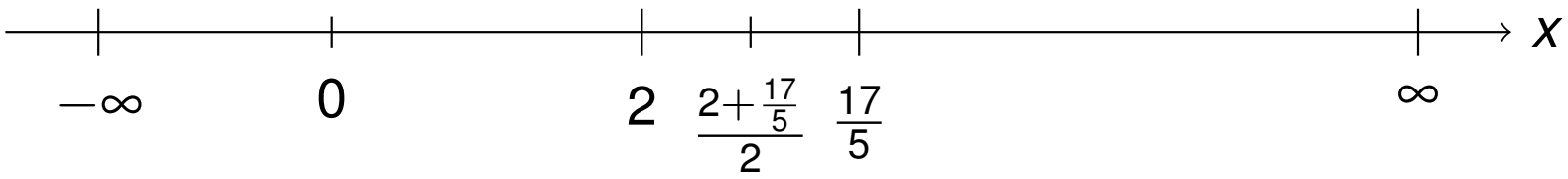

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\equiv & \mathit{true} & \text{evaluate}
\end{array}
$$

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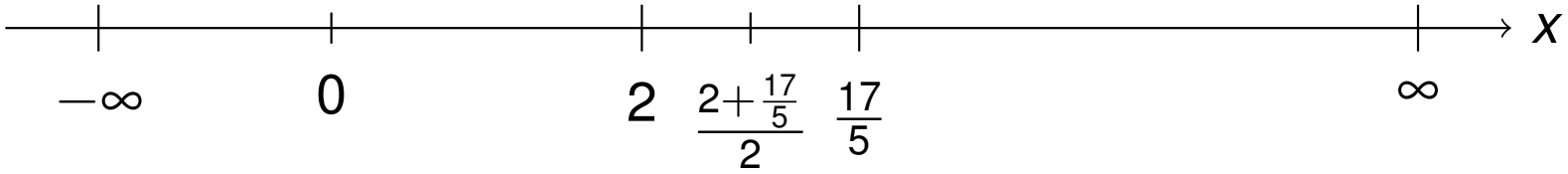

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\equiv & (2 > 2 \land 2 < \frac{17}{5}) & \text{boundary case “}x = 2\text{”} \\
\lor & (\frac{17}{5} > 2 \land \frac{17}{5} < \frac{17}{5}) & \text{boundary case “}x = \frac{17}{5}\text{”} \\
\lor & (\frac{2+\frac{17}{5}}{2} > 2 \land \frac{2+\frac{17}{5}}{2} < \frac{17}{5}) & \text{intermediate case “}x = \frac{2+\frac{17}{5}}{2}\text{”} \\
\lor & (-\infty > 2 \land -\infty < \frac{17}{5}) & \text{extremal case “}x = -\infty\text{”} \\
\lor & (\infty > 2 \land \infty < \frac{17}{5}) & \text{extremal case “}x = \infty\text{”} \\
\equiv & \mathit{true} & \text{evaluate}
\end{array}
$$

- ∞ is not in $\text{FOL}_{\mathbb{R}}$
- Interior points aren't always terms in $\text{FOL}_{\mathbb{R}}$ if nonlinear
- Substituting them into formulas requires attention

Linear Virtual Substitution

Theorem (Virtual Substitution: Linear Equation)

$$\exists x\,(bx + c = 0 \wedge F) \;\leftrightarrow$$

Linear Virtual Substitution

Theorem (Virtual Substitution: Linear Equation)

$$\exists x\,(bx + c = 0 \wedge F) \;\leftrightarrow\; F_x^{-c/b}$$

Linear solution

Linear Virtual Substitution

Theorem (Virtual Substitution: Linear Equation)

$$\exists x\,(bx + c = 0 \wedge F) \;\leftrightarrow\; b \neq 0 \wedge F_x^{-c/b}$$

Don't divide by 0

Linear Virtual Substitution

Theorem (Virtual Substitution: Linear Equation)

$$b \neq 0 \rightarrow \big(\exists x\,(bx + c = 0 \wedge F) \;\leftrightarrow\; b \neq 0 \wedge F_x^{-c/b}\big)$$

Only actually linear solution if $b \neq 0$

Linear Virtual Substitution

Theorem (Virtual Substitution: Linear Equation $x \notin b, c$)

$$b \neq 0 \rightarrow \big(\exists x\,(bx + c = 0 \wedge F) \;\leftrightarrow\; b \neq 0 \wedge F_x^{-c/b}\big) \quad \text{if } x \notin b, c$$

Only linear if no x in b, c

Linear Virtual Substitution

Theorem (Virtual Substitution: Linear Equation $x \notin b, c$)

$$b \neq 0 \rightarrow \big(\exists x\,(bx + c = 0 \wedge F) \;\leftrightarrow\; b \neq 0 \wedge F_x^{-c/b}\big) \quad \text{if } x \notin b, c$$

Conditional equivalence, so conditions may need to be checked or case-split

Linear Virtual Substitution

Theorem (Virtual Substitution: Linear Equation $x \notin b, c$)

$$b \neq 0 \rightarrow \big(\exists x\,(bx + c = 0 \wedge F) \;\leftrightarrow\; b \neq 0 \wedge F_x^{-c/b}\big) \quad \text{if } x \notin b, c$$

Lemma (Uniform substitution of linear equations)

The linear equation axiom is sound (b, c are arity 0 function symbols):

$$\exists lin\; b \neq 0 \rightarrow \big(\exists x\,(b \cdot x + c = 0 \wedge q(x)) \leftrightarrow q(-c/b)\big)$$

$$\exists x\,\big(\underbrace{(y^2+4)}_{b} \cdot x + \underbrace{(yz-1)}_{c} = 0 \wedge x^3 + x \geq 0\big) \leftrightarrow \left(-\frac{yz-1}{y^2+4}\right)^3 + \left(-\frac{yz-1}{y^2+4}\right) \geq 0$$

Outline

Quadratic Virtual Substitution

Theorem (Virtual Substitution: Quadratic Equation)

$$\exists x\,(ax^2 + bx + c = 0 \wedge F) \leftrightarrow$$

Quadratic Virtual Substitution

Theorem (Virtual Substitution: Quadratic Equation)

$$\exists x\,(ax^2 + bx + c = 0 \wedge F) \leftrightarrow$$

$$F_x^{(-b+\sqrt{b^2-4ac})/(2a)}$$

Quadratic solution

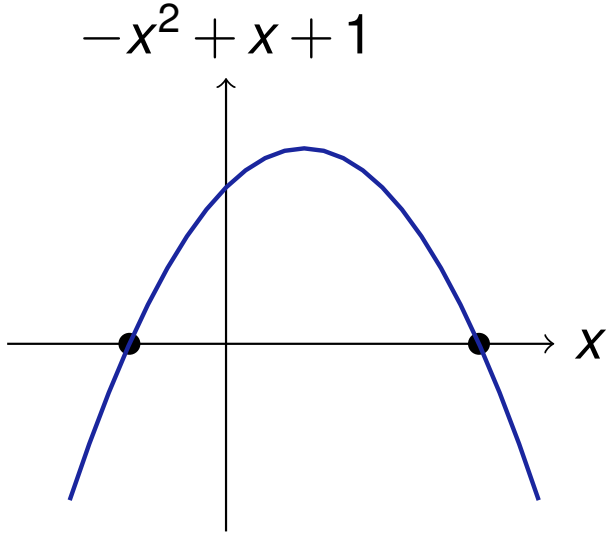
Quadratic Virtual Substitution

Theorem (Virtual Substitution: Quadratic Equation)

$$\exists x\,(ax^2+bx+c=0 \wedge F) \leftrightarrow$$

$$\left(F_x^{(-b+\sqrt{b^2-4ac})/(2a)} \vee F_x^{(-b-\sqrt{b^2-4ac})/(2a)}\right)$$

Or negative square root solution



Quadratic Virtual Substitution

Theorem (Virtual Substitution: Quadratic Equation)

$$\exists x\,(ax^2 + bx + c = 0 \wedge F) \leftrightarrow$$

$$a \neq 0 \wedge \left(F_x^{(-b+\sqrt{b^2-4ac})/(2a)} \vee F_x^{(-b-\sqrt{b^2-4ac})/(2a)}\right)$$

Don't divide by 0

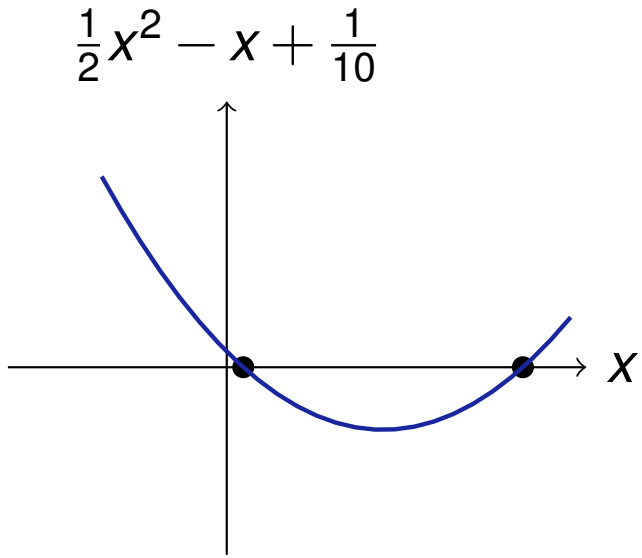
Quadratic Virtual Substitution

Theorem (Virtual Substitution: Quadratic Equation)

$$\exists x\,(ax^2 + bx + c = 0 \wedge F) \leftrightarrow$$

$$a \neq 0 \wedge b^2 - 4ac \geq 0 \wedge \left(F_x^{(-b+\sqrt{b^2-4ac})/(2a)} \vee F_x^{(-b-\sqrt{b^2-4ac})/(2a)}\right)$$

Real solution if $\sqrt{\cdot}$ exists by discriminant



Quadratic Virtual Substitution

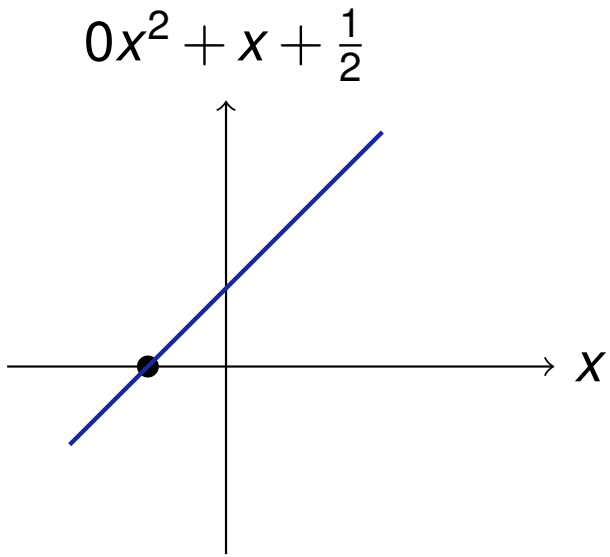
Theorem (Virtual Substitution: Quadratic Equation)

$$\exists x\,(ax^2 + bx + c = 0 \wedge F) \leftrightarrow$$

$$a = 0 \wedge b \neq 0 \wedge F_x^{-c/b}$$

$$\vee\, a \neq 0 \wedge b^2 - 4ac \geq 0 \wedge \left(F_x^{(-b+\sqrt{b^2-4ac})/(2a)} \vee F_x^{(-b-\sqrt{b^2-4ac})/(2a)}\right)$$

Instead linear solution if $a = 0$ (may case-split)



Quadratic Virtual Substitution

Theorem (Virtual Substitution: Quadratic Equation)

$$a \neq 0 \vee b \neq 0 \vee c \neq 0 \rightarrow$$
$$\Big(\exists x\,(ax^2 + bx + c = 0 \wedge F) \leftrightarrow$$
$$a = 0 \wedge b \neq 0 \wedge F_x^{-c/b}$$
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Only equivalent if not all 0 which gives trivial equation (else use F)

Quadratic Virtual Substitution

Theorem (Virtual Substitution: Quadratic Equation $x \notin a, b, c$)

$$a \neq 0 \vee b \neq 0 \vee c \neq 0 \rightarrow$$
$$\Big(\exists x\,(ax^2 + bx + c = 0 \wedge F) \leftrightarrow$$
$$a = 0 \wedge b \neq 0 \wedge F_x^{-c/b}$$
$$\vee\, a \neq 0 \wedge b^2 - 4ac \geq 0 \wedge \left(F_x^{(-b+\sqrt{b^2-4ac})/(2a)} \vee F_x^{(-b-\sqrt{b^2-4ac})/(2a)}\right)\Big)$$

Only linear or quadratic if no x in a, b, c

Quadratic Virtual Substitution

Theorem (Virtual Substitution: Quadratic Equation $x \notin a, b, c$)

$$a \neq 0 \vee b \neq 0 \vee c \neq 0 \rightarrow$$
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1. Quantifier-free equivalent

Quadratic Virtual Substitution

Theorem (Virtual Substitution: Quadratic Equation $x \notin a, b, c$)

$$a \neq 0 \vee b \neq 0 \vee c \neq 0 \rightarrow$$
$$\Big(\exists x\,(ax^2 + bx + c = 0 \wedge F) \leftrightarrow$$
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1. Quantifier-free equivalent
2. Just not a formula . . .

Quadratic Virtual Substitution

Theorem (Virtual Substitution: Quadratic Equation $x \notin a, b, c$)

$$a \neq 0 \vee b \neq 0 \vee c \neq 0 \rightarrow$$
$$\Big(\exists x\,(ax^2 + bx + c = 0 \wedge F) \leftrightarrow$$
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1. Quantifier-free equivalent
2. Just not a formula ...
3. $(-b+\sqrt{b^2-4ac})/(2a)$ is not in $\mathrm{FOL}_{\mathbb{R}}$ and neither is $-c/b$

Quadratic Virtual Substitution

Theorem (Virtual Substitution: Quadratic Equation $x \notin a, b, c$)

$$
\begin{aligned}
&a \neq 0 \vee b \neq 0 \vee c \neq 0 \rightarrow \\
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&\vee a \neq 0 \wedge b^2 - 4ac \geq 0 \wedge \left(F_x^{(-b+\sqrt{b^2-4ac})/(2a)} \vee F_x^{(-b-\sqrt{b^2-4ac})/(2a)}\right)\Big)
\end{aligned}
$$

1. Quantifier-free equivalent
2. Just not a formula ...
3. $(-b+\sqrt{b^2-4ac})/(2a)$ is not in $\mathrm{FOL}_{\mathbb{R}}$ and neither is $-c/b$
4. Virtual substitution $F_{\overline{x}}^{(a+b\sqrt{c})/d}$ acts as if it were to substitute $(a+b\sqrt{c})/d$ for x in F

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5. $\exists r\,(r^2 = c)$ would do it for $\sqrt{c}$

Quadratic Virtual Substitution

Theorem (Virtual Substitution: Quadratic Equation $x \notin a, b, c$)

$$a \neq 0 \vee b \neq 0 \vee c \neq 0 \rightarrow$$
$$\Big(\exists x\,(ax^2 + bx + c = 0 \wedge F) \leftrightarrow$$
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5. $\exists r\,(r^2 = c)$ would do it for $\sqrt{c}$ but that's going in circles

Quadratic Virtual Substitution

Theorem (Virtual Substitution: Quadratic Equation $x \notin a, b, c$)

$$
\begin{aligned}
&a \neq 0 \lor b \neq 0 \lor c \neq 0 \rightarrow \\
&\Big(\exists x \left(ax^2 + bx + c = 0 \land F\right) \leftrightarrow \\
&\quad a = 0 \land b \neq 0 \land F_{\overline{x}}^{-c/b} \\
&\lor a \neq 0 \land b^2 - 4ac \geq 0 \land \left(F_{\overline{x}}^{(-b+\sqrt{b^2-4ac})/(2a)} \lor F_{\overline{x}}^{(-b-\sqrt{b^2-4ac})/(2a)}\right)\Big)
\end{aligned}
$$

Virtual Substitution into Polynomial

Virtually substitute $(a+b\sqrt{c})/d$ into a polynomial p:

$$p_{\overline{x}}^{(a+b\sqrt{c})/d} \stackrel{\text{def}}{=}$$

Convention: On this slide c' is not a derivative but just another name ...

Virtual Substitution into Polynomial

Virtually substitute $(a+b\sqrt{c})/d$ into a polynomial p:

$$p_{\overline{x}}^{(a+b\sqrt{c})/d} \stackrel{\text{def}}{=} p((a+b\sqrt{c})/d)$$

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Virtual Substitution into Polynomial

Virtually substitute $(a+b\sqrt{c})/d$ into a polynomial p:

$$p_{\overline{x}}^{(a+b\sqrt{c})/d} \overset{\text{def}}{=} p((a+b\sqrt{c})/d) \quad \text{by algebraic evaluation of } +,\cdot$$

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Square Root Algebra

Virtual Substitution into Polynomial

Virtually substitute $(a+b\sqrt{c})/d$ into a polynomial p:

$$p_{\overline{x}}^{(a+b\sqrt{c})/d} \stackrel{\text{def}}{=} p((a+b\sqrt{c})/d) \quad \text{by algebraic evaluation of } +,\cdot$$

$\sqrt{c}$-algebra

Algebra of terms $(a+b\sqrt{c})/d$ with polynomials $a,b,c,d \in \mathbb{Q}[x_1,..,x_n]$:

$$((a+b\sqrt{c})/d)+((a'+b'\sqrt{c})/d') =$$
$$((a+b\sqrt{c})/d)\cdot((a'+b'\sqrt{c})/d') =$$

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Square Root Algebra

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Algebra of terms $(a+b\sqrt{c})/d$ with polynomials $a,b,c,d \in \mathbb{Q}[x_1,..,x_n]$:

$$((a+b\sqrt{c})/d)+((a'+b'\sqrt{c})/d') = ((ad'+da')+(bd'+db')\sqrt{c})/(dd')$$
$$((a+b\sqrt{c})/d)\cdot((a'+b'\sqrt{c})/d') =$$

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Square Root Algebra

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$$((a+b\sqrt{c})/d)\cdot((a'+b'\sqrt{c})/d') = ((aa'+bb'c)+(ab'+ba')\sqrt{c})/(dd')$$

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$\sqrt{c}$-algebra

Algebra of terms $(a+b\sqrt{c})/d$ with polynomials $a,b,c,d \in \mathbb{Q}[x_1,..,x_n]$:
where $c \geq 0, d, d' \neq 0$

$$((a+b\sqrt{c})/d)+((a'+b'\sqrt{c})/d') = ((ad'+da')+(bd'+db')\sqrt{c})/(dd')$$
$$((a+b\sqrt{c})/d)\cdot((a'+b'\sqrt{c})/d') = ((aa'+bb'c)+(ab'+ba')\sqrt{c})/(dd')$$

Convention: On this slide c' is not a derivative but just another name ...

Virtual Substitution into Comparisons

Virtually substitute $(a+b\sqrt{c})/d$ into a comparison $p \sim 0$:

$$(p \sim 0)^{(a+b\sqrt{c})/d}_{\overline{x}} \equiv$$

Virtual Substitution into Comparisons

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Virtual $\sqrt{\cdot}$ Substitution

Virtual Substitution into Comparisons

Virtually substitute $(a+b\sqrt{c})/d$ into a comparison $p \sim 0$:

$$(p \sim 0)^{(a+b\sqrt{c})/d}_{\overline{x}} \equiv (p^{(a+b\sqrt{c})/d}_{\overline{x}} \sim 0)$$

$\sqrt{c}$-comparisons $d \neq 0 \land c \geq 0$

$$(a+0\sqrt{c})/d = 0 \equiv$$
$$(a+0\sqrt{c})/d \leq 0 \equiv$$
$$(a+0\sqrt{c})/d < 0 \equiv$$
$$(a+b\sqrt{c})/d = 0 \equiv$$
$$(a+b\sqrt{c})/d \leq 0 \equiv$$
$$(a+b\sqrt{c})/d < 0 \equiv$$

Virtual $\sqrt{\cdot}$ Substitution

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Virtually substitute $(a+b\sqrt{c})/d$ into a comparison $p \sim 0$:

$$(p \sim 0)^{(a+b\sqrt{c})/d}_{\overline{x}} \equiv (p^{(a+b\sqrt{c})/d}_{\overline{x}} \sim 0)$$

$\sqrt{c}$-comparisons $d \neq 0 \land c \geq 0$

$$(a+0\sqrt{c})/d = 0 \equiv a = 0$$

$$(a+0\sqrt{c})/d \leq 0 \equiv ad \leq 0$$

$$(a+0\sqrt{c})/d < 0 \equiv ad < 0$$

$$(a+b\sqrt{c})/d = 0 \equiv ab \leq 0 \land a^2 - b^2c = 0$$

$$(a+b\sqrt{c})/d \leq 0 \equiv ad \leq 0 \land a^2 - b^2c \geq 0 \lor bd \leq 0 \land a^2 - b^2c \leq 0$$

$$(a+b\sqrt{c})/d < 0 \equiv ad < 0 \land a^2 - b^2c > 0 \lor bd \leq 0 \land (ad < 0 \lor a^2 - b^2c < 0)$$

Virtual $\sqrt{\cdot}$ Substitution

Virtual Substitution into Comparisons

Virtually substitute $(a+b\sqrt{c})/d$ into a comparison $p \sim 0$:

$$(p \sim 0)_{\overline{x}}^{(a+b\sqrt{c})/d} \equiv (p_{\overline{x}}^{(a+b\sqrt{c})/d} \sim 0) \quad \text{accordingly for } \wedge, \vee, \ldots$$

$\sqrt{c}$-comparisons $d \neq 0 \wedge c \geq 0$

$$(a+0\sqrt{c})/d = 0 \equiv a = 0$$

$$(a+0\sqrt{c})/d \leq 0 \equiv ad \leq 0$$

$$(a+0\sqrt{c})/d < 0 \equiv ad < 0$$

$$(a+b\sqrt{c})/d = 0 \equiv ab \leq 0 \wedge a^2 - b^2c = 0$$

$$(a+b\sqrt{c})/d \leq 0 \equiv ad \leq 0 \wedge a^2 - b^2c \geq 0 \vee bd \leq 0 \wedge a^2 - b^2c \leq 0$$

$$(a+b\sqrt{c})/d < 0 \equiv ad < 0 \wedge a^2 - b^2c > 0$$
$$\vee\, bd \leq 0 \wedge (ad < 0 \vee a^2 - b^2c < 0)$$

Quadratic Virtual Substitution

Theorem (Virtual Substitution: Quadratic Equation $x \notin a, b, c$)

$$
\begin{aligned}
&a \neq 0 \vee b \neq 0 \vee c \neq 0 \rightarrow \\
&\Big(\exists x \, (ax^2 + bx + c = 0 \wedge F) \leftrightarrow \\
&\quad a = 0 \wedge b \neq 0 \wedge F_{\overline{x}}^{-c/b} \\
&\vee a \neq 0 \wedge b^2 - 4ac \geq 0 \wedge \Big(F_{\overline{x}}^{(-b+\sqrt{b^2-4ac})/(2a)} \vee F_{\overline{x}}^{(-b-\sqrt{b^2-4ac})/(2a)}\Big)\Big)
\end{aligned}
$$

Lemma (Virtual Substitution Lemma for $\sqrt{\cdot}$)

$$F_x^{(a+b\sqrt{c})/d} \equiv F_{\overline{x}}^{(a+b\sqrt{c})/d}$$

Quadratic Virtual Substitution

Theorem (Virtual Substitution: Quadratic Equation $x \notin a, b, c$)

$$
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&a \neq 0 \vee b \neq 0 \vee c \neq 0 \rightarrow \\
&\Big(\exists x \, (ax^2 + bx + c = 0 \wedge F) \leftrightarrow \\
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\end{aligned}
$$

Lemma (Virtual Substitution Lemma for $\sqrt{\cdot}$)

Extended logic $\quad F_x^{(a+b\sqrt{c})/d} \equiv F_{\overline{x}}^{(a+b\sqrt{c})/d} \quad$ $\mathrm{FOL}_{\mathbb{R}}$

Quadratic Virtual Substitution

Theorem (Virtual Substitution: Quadratic Equation $x \notin a, b, c$)

$$a \neq 0 \vee b \neq 0 \vee c \neq 0 \rightarrow$$
$$\Big(\exists x\,(ax^2 + bx + c = 0 \wedge F) \leftrightarrow$$
$$a = 0 \wedge b \neq 0 \wedge F_{\overline{x}}^{-c/b}$$
$$\vee\, a \neq 0 \wedge b^2 - 4ac \geq 0 \wedge \Big(F_{\overline{x}}^{(-b+\sqrt{b^2-4ac})/(2a)} \vee F_{\overline{x}}^{(-b-\sqrt{b^2-4ac})/(2a)}\Big)\Big)$$

Lemma (Virtual Substitution Lemma for $\sqrt{\cdot}$)

Extended logic $\quad F_x^{(a+b\sqrt{c})/d} \equiv F_{\overline{x}}^{(a+b\sqrt{c})/d} \quad$ $\mathrm{FOL}_{\mathbb{R}}$

$\omega_x^r \in [\![F]\!]$ *iff* $\omega \in [\![F_{\overline{x}}^{(a+b\sqrt{c})/d}]\!]$ *where* $r = (\omega[\![a]\!] + \omega[\![b]\!]\sqrt{\omega[\![c]\!]})/(\omega[\![d]\!]) \in \mathbb{R}$

Example: Quadratic Curiosity

$$a \neq 0 \to (\exists x\,(ax^2+bx+c = 0 \land ax^2+bx+c \leq 0) \leftrightarrow b^2-4ac \geq 0 \land \mathit{true})$$

$$(ax^2+bx+c)^{(-b+\sqrt{b^2-4ac})/(2a)}_{\bar{x}}$$

$$= a((-b+\sqrt{b^2-4ac})/(2a))^2 + b((-b+\sqrt{b^2-4ac})/(2a)) + c$$

$$= a((b^2+b^2-4ac+(-b-b)\sqrt{b^2-4ac})/(4a^2)) + (-b^2+b\sqrt{b^2-4ac})/(2a) + c$$

$$= (ab^2+ab^2-4a^2c+(-ab-ab)\sqrt{b^2-4ac})/(4a^2) + (-b^2+2ac+b\sqrt{b^2-4ac})/(2a)$$

$$= ((ab^2+ab^2-4a^2c)2a + (-b^2+2ac)4a^2 + ((-ab-ab)2a+b4a^2)\sqrt{b^2-4ac})/(8a^3)$$

$$= (\cancel{2a^2b^2} + \cancel{2a^2b^2} - \cancel{8a^3c} - \cancel{4a^2b^2} + \cancel{8a^3c} + (-\cancel{2a^2b} - \cancel{2a^2b} + \cancel{4a^2b})\sqrt{b^2-4ac})/(8a^3)$$

$$= (0+0\sqrt{b^2-4ac})/(8a^3) = (0+0\sqrt{..})/1 = 0$$

$$(ax^2+bx+c = 0)^{(-b+\sqrt{b^2-4ac})/(2a)}_{\bar{x}} \equiv ((0+0\sqrt{..})/1 = 0) \equiv (0 \cdot 1 = 0) \equiv \mathit{true}$$

$$(ax^2+bx+c \leq 0)^{(-b+\sqrt{b^2-4ac})/(2a)}_{\bar{x}} \equiv (\underbrace{(0+0\sqrt{..})/1}_{0} \leq 0) \equiv (0 \cdot 1 \leq 0) \equiv \mathit{true}$$

Example: Nonnegative Roots of Quadratic Polynomials

$$a \neq 0 \to \big(\exists x\,(ax^2+bx+c=0 \land x \geq 0)$$
$$\leftrightarrow b^2-4ac \geq 0 \land (2ba \leq 0 \land 4ac \geq 0 \lor -2a \leq 0 \land 4ac \leq 0$$
$$\lor 2ba \leq 0 \land 4ac \geq 0 \lor 2a \leq 0 \land 4ac \leq 0)\big)$$

$$-(-b+\sqrt{b^2-4ac})/(2a) = ((-1+0\sqrt{b^2-4ac})/1)\cdot((-b+\sqrt{b^2-4ac})/(2a)$$
$$= (b-\sqrt{b^2-4ac})/(2a)$$

$$(-x \leq 0)_{\overline{x}}^{(-b+\sqrt{b^2-4ac})/(2a)} \equiv ((b-\sqrt{b^2-4ac})/(2a) \leq 0)_{\overline{x}}^{x}$$
$$\equiv b2a \leq 0 \land \cancel{b^2} - (-1)^2(\cancel{b^2}-4ac) \geq 0 \lor -1\cdot 2a \leq 0 \land \cancel{b^2} - (-1)^2(\cancel{b^2}-4ac) \leq 0$$
$$\equiv 2ba \leq 0 \land 4ac \geq 0 \lor -2a \leq 0 \land 4ac \leq 0$$

$$(-x \leq 0)_{\overline{x}}^{(-b-\sqrt{b^2-4ac})/(2a)} \equiv ((b+\sqrt{b^2-4ac})/(2a) \leq 0)_{\overline{x}}^{x}$$
$$\equiv b2a \leq 0 \land \cancel{b^2} - 1^2(\cancel{b^2}-4ac) \geq 0 \lor 1\cdot 2a \leq 0 \land \cancel{b^2} - 1^2(\cancel{b^2}-4ac) \leq 0$$
$$\equiv 2ba \leq 0 \land 4ac \geq 0 \lor 2a \leq 0 \land 4ac \leq 0$$

Outline

Virtual Substitution of $(a+b\sqrt{c})/d$ into Comparisons

$$(p \sim 0)^{(a+b\sqrt{c})/d}_{\overline{x}} \equiv (p^{(a+b\sqrt{c})/d}_{\overline{x}} \sim 0) \quad \text{accordingly for } \wedge, \vee, \ldots$$

$\sqrt{c}$-algebra $d \neq 0 \wedge c \geq 0$

$$((a+b\sqrt{c})/d) + ((a'+b'\sqrt{c})/d') = ((ad'+da')+(bd'+db')\sqrt{c})/(dd')$$

$$((a+b\sqrt{c})/d) \cdot ((a'+b'\sqrt{c})/d') = ((aa'+bb'c)+(ab'+ba')\sqrt{c})/(dd')$$

$\sqrt{c}$-comparisons $d \neq 0 \wedge c \geq 0$

$$(a+b\sqrt{c})/d = 0 \equiv ab \leq 0 \wedge a^2 - b^2c = 0$$

$$(a+b\sqrt{c})/d \leq 0 \equiv ad \leq 0 \wedge a^2 - b^2c \geq 0 \vee bd \leq 0 \wedge a^2 - b^2c \leq 0$$

$$(a+b\sqrt{c})/d < 0 \equiv ad < 0 \wedge a^2 - b^2c > 0$$
$$\vee bd \leq 0 \wedge (ad < 0 \vee a^2 - b^2c < 0)$$

Quadratic Virtual Substitution

Theorem (Virtual Substitution: Quadratic Equation $x \notin a, b, c$)

$$a \neq 0 \vee b \neq 0 \vee c \neq 0 \rightarrow$$
$$\Big(\exists x\,(ax^2 + bx + c = 0 \wedge F) \leftrightarrow$$
$$a = 0 \wedge b \neq 0 \wedge F_{\overline{x}}^{-c/b}$$
$$\vee\, a \neq 0 \wedge b^2 - 4ac \geq 0 \wedge \Big(F_{\overline{x}}^{(-b+\sqrt{b^2-4ac})/(2a)} \vee F_{\overline{x}}^{(-b-\sqrt{b^2-4ac})/(2a)}\Big)\Big)$$

Lemma (Virtual Substitution Lemma for $\sqrt{\cdot}$)

Extended logic $\quad F_x^{(a+b\sqrt{c})/d} \equiv F_{\overline{x}}^{(a+b\sqrt{c})/d} \quad$ $\mathrm{FOL}_{\mathbb{R}}$

$\omega_x^r \in [\![F]\!]$ *iff* $\omega \in [\![F_{\overline{x}}^{(a+b\sqrt{c})/d}]\!]$ *where* $r = (\omega[\![a]\!] + \omega[\![b]\!]\sqrt{\omega[\![c]\!]})/(\omega[\![d]\!]) \in \mathbb{R}$


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