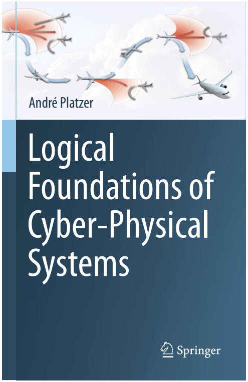
14: Hybrid Systems & Games

Logical Foundations of Cyber-Physical Systems



André Platzer

Logical Foundations of Cyber-Physical Systems

Springer

André Platzer

Karlsruhe Institute of Technology
Department of Informatics

Computer Science Department
Carnegie Mellon University

Outline

1. Learning Objectives
2. Motivation
3. A Gradual Introduction to Hybrid Games
    - Choices & Nondeterminism
    - Control & Dual Control
    - Demon's Derived Controls
4. Differential Game Logic
    - Syntax of Hybrid Games
    - Syntax of Differential Game Logic Formulas
    - Examples
    - Push-around Cart
    - Robot Dance
    - Example: Robot Soccer
5. An Informal Operational Game Tree Semantics
6. Summary

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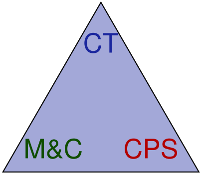
Learning Objectives

Hybrid Systems & Games

fundamental principles of computational thinking
logical extensions
PL modularity principles
compositional extensions
differential game logic
best/worst-case analysis
models of alternating computation



CT

M&C

CPS

adversarial dynamics
conflicting actions
multi-agent systems
angelic/demonic choice

multi-agent state change
CPS semantics
reflections on choices

Outline

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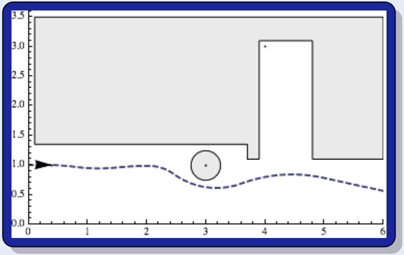
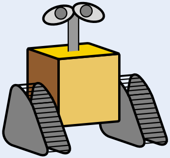
CPS Analysis: Robot Control

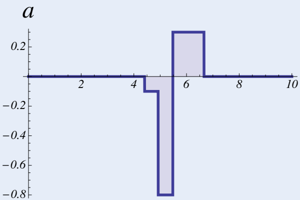
Challenge (Hybrid Systems)

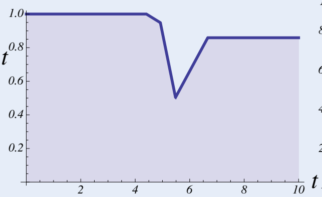
Fixed rule describing state evolution with both

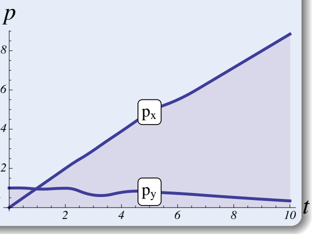
- Discrete dynamics (control decisions)
- Continuous dynamics (differential equations)











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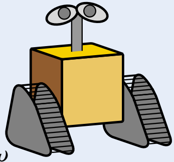
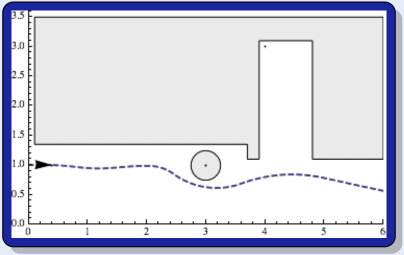
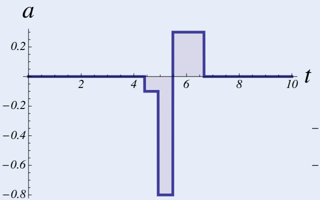
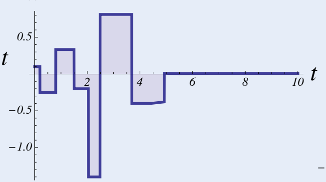
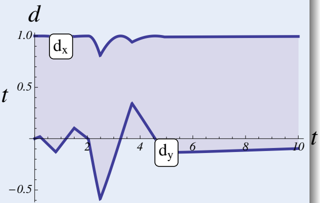








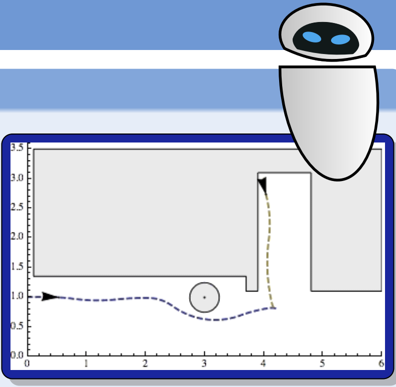


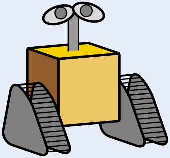
CPS Analysis: Robot Control

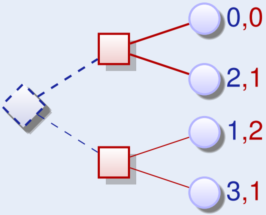
Challenge (Games)

Game rules describing play evolution with both

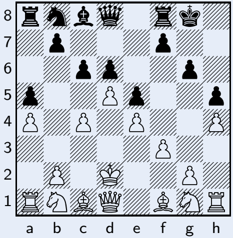
- Angelic choices (player ◇ Angel)
- Demonic choices (player □ Demon)





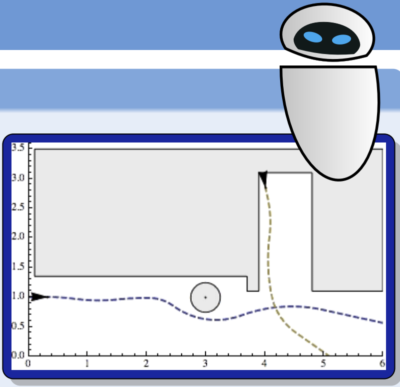


◇\□	Tr	Pl
Trash	1,2	0,0
Plant	0,0	2,1



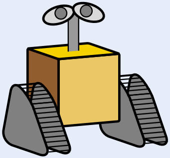
CPS Analysis: Robot Control

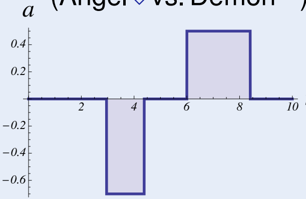
Challenge (Hybrid Games)

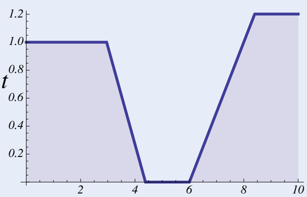


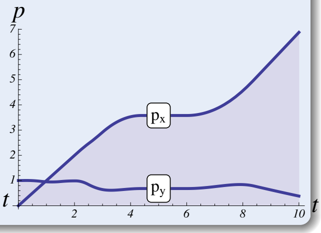
Game rules describing play evolution with

- Discrete dynamics (control decisions)
- Continuous dynamics (differential equations)
- Adversarial dynamics (Angel ◇ vs. Demon □)









CPS Analysis: Robot Control

Challenge (Hybrid Games)

Game rules describing play evolution with

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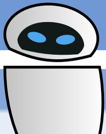
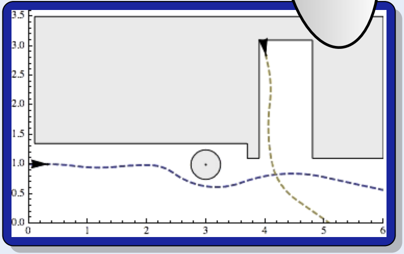
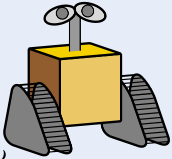
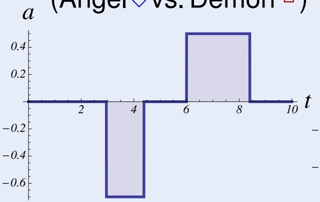
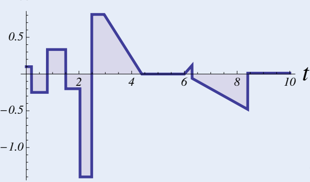
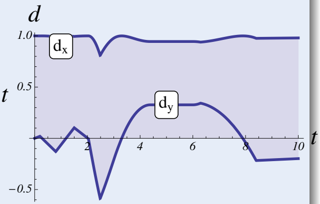










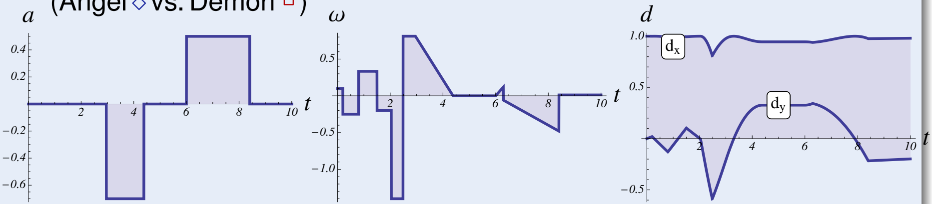


CPS Analysis: RoboCup Soccer

Challenge (Hybrid Games)

Game rules describing play evolution with

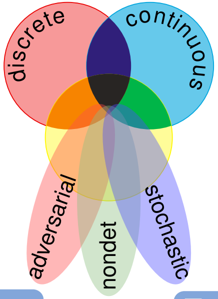
- Discrete dynamics (control decisions)
- Continuous dynamics (differential equations)
- Adversarial dynamics (Angel ◇ vs. Demon □)



CPSs are Multi-Dynamical Systems

CPS Dynamics

CPS are characterized by multiple facets of dynamical systems.



CPS Compositions

CPS combines multiple simple dynamical effects.

Descriptive simplification

Tame Parts

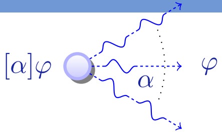
Exploiting compositionality tames CPS complexity.

Analytic simplification

Dynamic Logics for Dynamical Systems

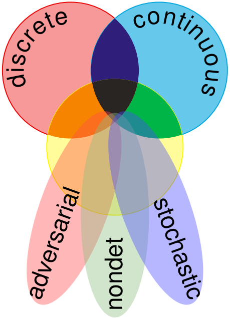
differential dynamic logic

$\mathsf{dL} = \mathsf{DL} + \mathsf{HP}$



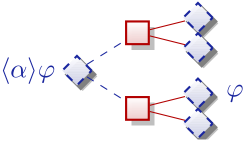
differential game logic

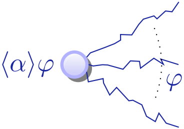
$\mathsf{dGL} = \mathsf{GL} + \mathsf{HG}$



stochastic differential DL

$\mathsf{SdL} = \mathsf{DL} + \mathsf{SHP}$





quantified differential DL

$\mathsf{QdL} = \mathsf{FOL} + \mathsf{DL} + \mathsf{QHP}$

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Differential Dynamic Logic dL: Syntax

Definition (Hybrid program α)

$$x := e \mid ?Q \mid x' = f(x) \,\&\, Q \mid \alpha \cup \beta \mid \alpha ; \beta \mid \alpha^*$$

Definition (dL Formula P)

$$e \geq \tilde{e} \mid \neg P \mid P \wedge Q \mid \forall x\, P \mid \exists x\, P \mid [\alpha]P \mid \langle \alpha \rangle P$$

Differential Dynamic Logic dL: Syntax

Discrete Assign
Test Condition
Differential Equation
Nondet. Choice
Seq. Compose
Nondet. Repeat

Definition (Hybrid program α)

$$x := e \mid ?Q \mid x' = f(x) \,\&\, Q \mid \alpha \cup \beta \mid \alpha ; \beta \mid \alpha^*$$

Definition (dL Formula P)

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All Reals
Some Reals
All Runs
Some Runs

Differential Dynamic Logic dL: Nondeterminism

Nondet. Choice

Definition (Hybrid program α)

$$x := e \mid ?Q \mid x' = f(x) \,\&\, Q \mid \alpha \cup \beta \mid \alpha;\beta \mid \alpha^*$$

Definition (dL Formula P)

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Nondeterminism during HP runs

Differential Dynamic Logic dL: Nondeterminism

Differential Equation

Nondet. Choice

Nondet. Repeat

Definition (Hybrid program α)

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Differential Equation

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Nondet. Repeat

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All Choices

Some Choice

Differential Dynamic Logic dL: Nondeterminism

All choices resolved in one way

Differential Equation

Nondet. Choice

Nondet. Repeat

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Modality decides the mode: help/hurt

All Choices

Some Choice

Differential Dynamic Logic dL: Nondeterminism

All choices resolved in one way

Differential Equation

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Modality decides the mode: help/hurt

All Choices

Some Choice

$[\alpha_1]\langle\alpha_2\rangle[\alpha_3]\langle\alpha_4\rangle P$ only fixed interaction depth

Control & Dual Control Operators

◇ Angel Ops

$\cup$	choice
$*$	repeat
$x' = f(x)$	evolve
$?Q$	challenge

Let Angel be one player

Control & Dual Control Operators

◇ Angel Ops

$\cup$	choice
$*$	repeat
$x' = f(x)$	evolve
$?Q$	challenge

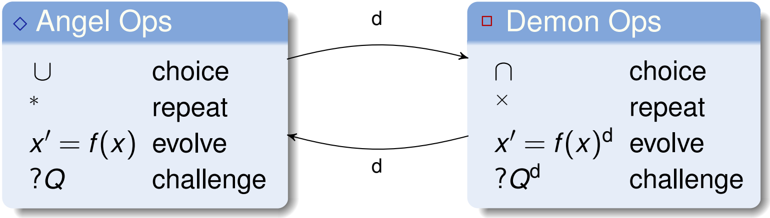
□ Demon Ops

$\cap$	choice
$\times$	repeat
$x' = f(x)^{\mathrm{d}}$	evolve
$?Q^{\mathrm{d}}$	challenge

Let Angel be one player

Let Demon be another player

Control & Dual Control Operators



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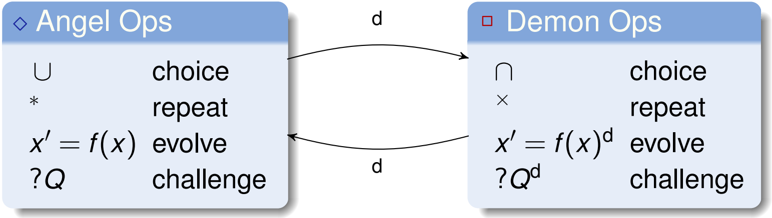
$\cap$	choice
$^\times$	repeat
$x' = f(x)^{\mathrm{d}}$	evolve
$?Q^{\mathrm{d}}$	challenge

Duality operator $^{\mathrm{d}}$ passes control between players

Game Operators



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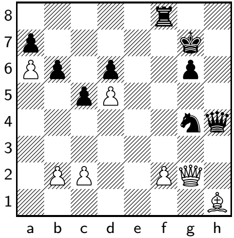
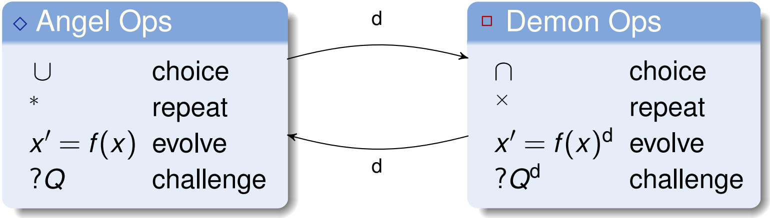
◻ Demon Ops

$\cap$	choice
$\times$	repeat
$x' = f(x)^{d}$	evolve
$?Q^{d}$	challenge

Duality operator ^{d} passes control between players



Game Operators



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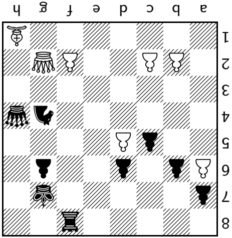
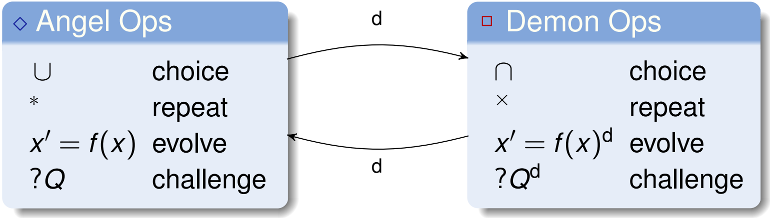
Duality operator ^{d} passes control between players



Game Operators



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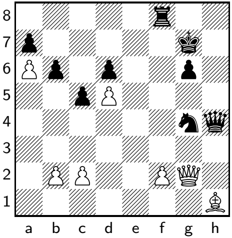
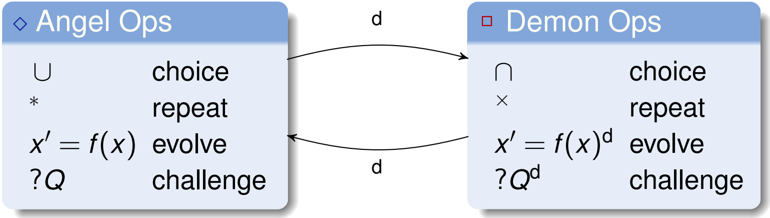


Definable Game Operators



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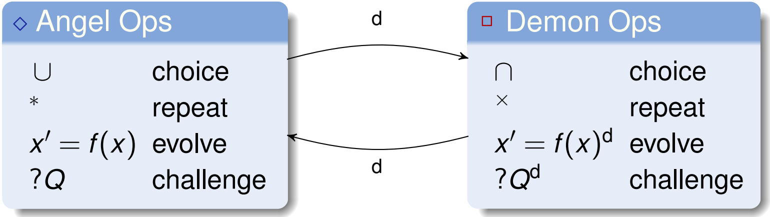
$?Q$	challenge

□ Demon Ops

$\cap$	choice
$\times$	repeat
$x' = f(x)^d$	evolve
$?Q^d$	challenge

$$\begin{aligned}
\text{if}(Q)\,\alpha\,\text{else}\,\beta &\equiv \\
\text{while}(Q)\,\alpha &\equiv \\
\alpha \cap \beta &\equiv \\
\alpha^{\times} &\equiv \\
(x' = f(x)\,\&\,Q)^d &\quad x' = f(x)\,\&\,Q \\
(x := e)^d &\quad x := e \\
?Q^d &\quad ?Q
\end{aligned}$$

Definable Game Operators



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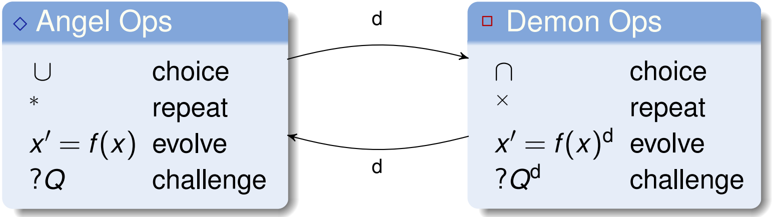
$$\begin{aligned}
\text{if}(Q)\,\alpha\,\text{else}\,\beta &\equiv (?Q;\alpha)\cup(?\neg Q;\beta)\\
\text{while}(Q)\,\alpha &\equiv\\
\alpha\cap\beta &\equiv\\
\alpha^{\times} &\equiv
\end{aligned}$$

$$\begin{array}{rl}
(x' = f(x)\,\&\,Q)^d & x' = f(x)\,\&\,Q\\
(x := e)^d & x := e\\
?Q^d & ?Q
\end{array}$$

Definable Game Operators



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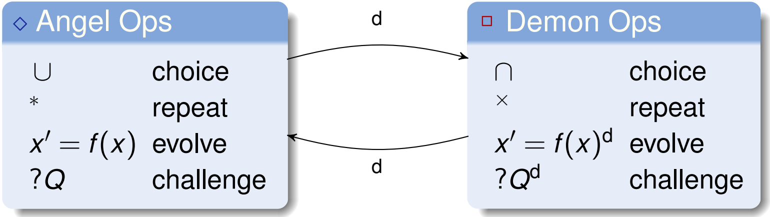
$$\begin{aligned}
\mathrm{if}(Q)\,\alpha\,\mathrm{else}\,\beta &\equiv (?Q;\alpha)\cup(?\neg Q;\beta)\\
\mathrm{while}(Q)\,\alpha &\equiv (?Q;\alpha)^*;?\neg Q\\
\alpha\cap\beta &\equiv\\
\alpha^{\times} &\equiv\\
(x' = f(x)\,\&\,Q)^{d} &\quad x' = f(x)\,\&\,Q\\
(x := e)^{d} &\quad x := e\\
?Q^{d} &\quad ?Q
\end{aligned}$$

Definable Game Operators



◇ **Angel Ops**

$\cup$	choice
$*$	repeat
$x' = f(x)$	evolve
$?Q$	challenge

□ **Demon Ops**

$\cap$	choice
$\times$	repeat
$x' = f(x)^d$	evolve
$?Q^d$	challenge

$$\begin{aligned}
\text{if}(Q)\,\alpha\,\text{else}\,\beta &\equiv (?Q;\alpha)\cup(?\neg Q;\beta)\\
\text{while}(Q)\,\alpha &\equiv (?Q;\alpha)^*;?\neg Q\\
\alpha\cap\beta &\equiv\\
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(x'=f(x)\,\&\,Q)^d &\quad x'=f(x)\,\&\,Q\\
(x:=e)^d &\quad x:=e\\
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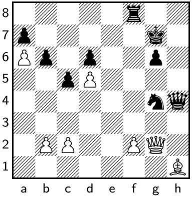
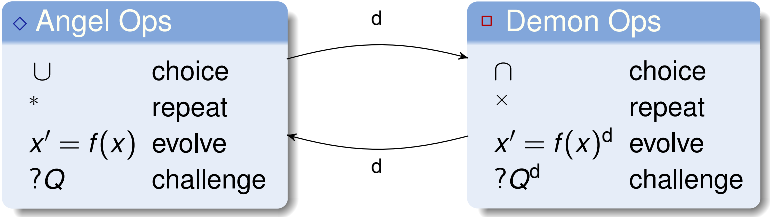
\end{aligned}$$



Definable Game Operators



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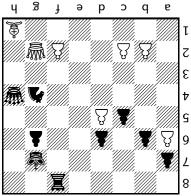
$$\mathrm{if}(Q)\,\alpha\,\mathrm{else}\,\beta \equiv (?Q;\alpha) \cup (?\neg Q;\beta)$$

$$\mathrm{while}(Q)\,\alpha \equiv (?Q;\alpha)^{*};?\neg Q$$

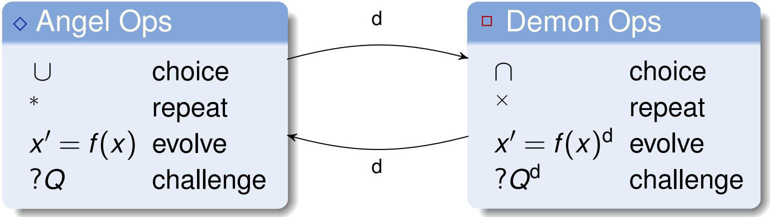
$$\alpha \cap \beta \equiv (\alpha^{d} \cup \beta^{d})^{d}$$

$$\alpha^{\times} \equiv$$

$(x' = f(x)\,\&\,Q)^{d}$	$x' = f(x)\,\&\,Q$
$(x := e)^{d}$	$x := e$
$?Q^{d}$	$?Q$



Definable Game Operators



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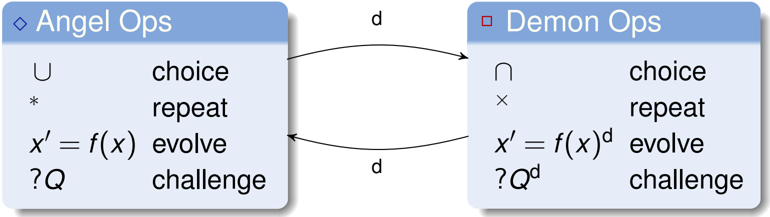
\end{aligned}$$

Definable Game Operators



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$$\alpha \cap \beta \equiv (\alpha^{d} \cup \beta^{d})^{d}$$

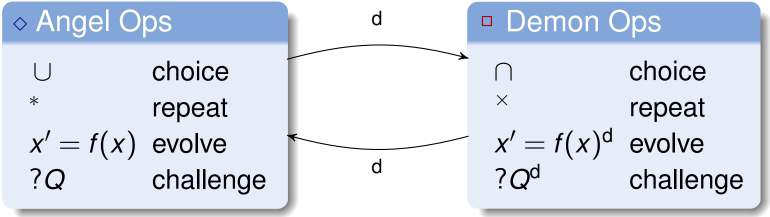
$$\alpha^{\times} \equiv ((\alpha^{d})^{*})^{d}$$

$$(x' = f(x)\,\&\,Q)^{d} \not\equiv x' = f(x)\,\&\,Q$$

$$(x := e)^{d} \qquad x := e$$

$$?Q^{d} \qquad ?Q$$

Definable Game Operators



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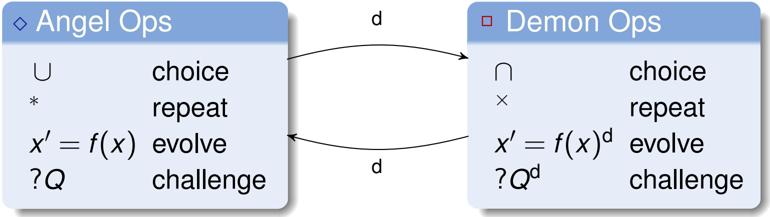
Definable Game Operators



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(x' = f(x)\,\&\,Q)^d &\not\equiv x' = f(x)\,\&\,Q \\
(x := e)^d &\equiv x := e \\
?Q^d &\not\equiv ?Q
\end{aligned}$$

Outline

1. Learning Objectives
2. Motivation
3. A Gradual Introduction to Hybrid Games
   - Choices & Nondeterminism
   - Control & Dual Control
   - Demon's Derived Controls
4. Differential Game Logic
   - Syntax of Hybrid Games
   - Syntax of Differential Game Logic Formulas
   - Examples
   - Push-around Cart
   - Robot Dance
   - Example: Robot Soccer
5. An Informal Operational Game Tree Semantics
6. Summary

Hybrid Games: Syntax

Definition (Hybrid game α)

$$\alpha,\beta \ ::= \ x:=e \mid ?Q \mid x' = f(x)\,\&\,Q \mid \alpha \cup \beta \mid \alpha;\beta \mid \alpha^* \mid \alpha^{\mathrm{d}}$$

Hybrid Games: Syntax

Discrete Assign

Test Game

Differential Equation

Choice Game

Seq. Game

Repeat Game

Definition (Hybrid game α)

$$\alpha,\beta ::= x:=e \mid ?Q \mid x'=f(x)\,\&\,Q \mid \alpha\cup\beta \mid \alpha;\beta \mid \alpha^* \mid \alpha^d$$

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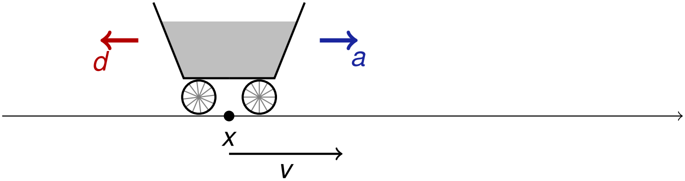
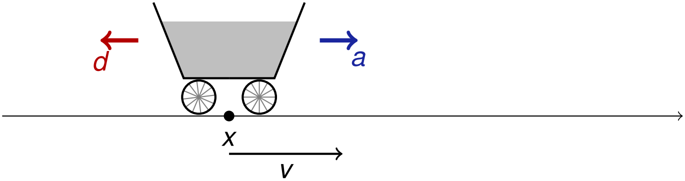
Example: Push-around Cart

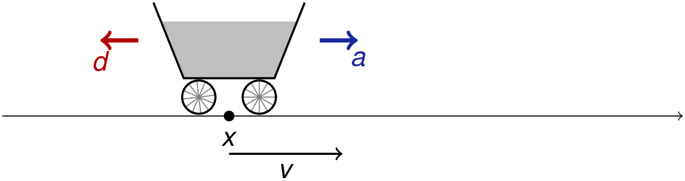


Example: Push-around Cart



$$\big((a:=1 \cup a:=-1);(d:=1 \cup d:=-1)^{\mathrm{d}};\{x'=v,v'=a+d\}\big)^*$$

Example: Push-around Cart



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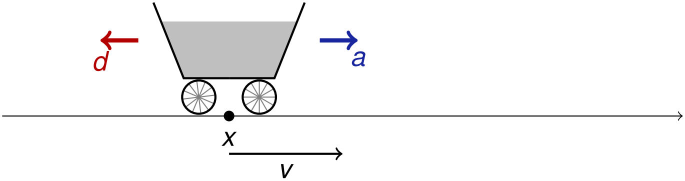

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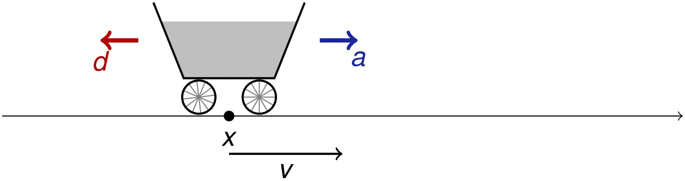

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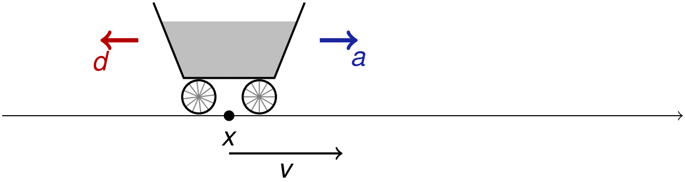
HP $\big((d:=1 \cup d:=-1);(a:=1 \cup a:=-1);\{x'=v,v'=a+d\}\big)^*$

Example: Push-around Cart



$$\big((a:=1 \cup a:=-1);(d:=1 \cap d:=-1);\{x'=v,v'=a+d\}\big)^*$$

$$\big((d:=1 \cap d:=-1);(a:=1 \cup a:=-1);\{x'=v,v'=a+d\}\big)^*$$

$$\text{HP } \big((d:=1 \cup d:=-1);(a:=1 \cup a:=-1);\{x'=v,v'=a+d\}\big)^*$$

Hybrid systems can't say that a is Angel's choice and d is Demon's.
Only that there are choices.

Differential Game Logic: Syntax

Definition (Hybrid game α)

$$\alpha, \beta \ ::= \ x := e \mid ?Q \mid x' = f(x) \,\&\, Q \mid \alpha \cup \beta \mid \alpha ; \beta \mid \alpha^* \mid \alpha^{\mathrm{d}}$$

Differential Game Logic: Syntax

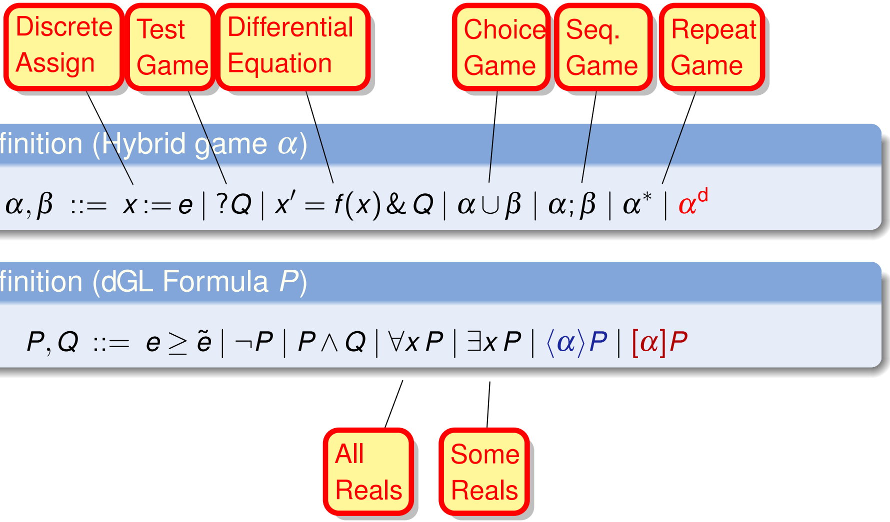
Definition (Hybrid game α)

$$\alpha,\beta ::= x:=e \mid ?Q \mid x'=f(x)\,\&\,Q \mid \alpha\cup\beta \mid \alpha;\beta \mid \alpha^* \mid \alpha^{\mathrm{d}}$$

Definition (dGL Formula P)

$$P,Q ::= e\geq\tilde{e} \mid \neg P \mid P\wedge Q \mid \forall x\,P \mid \exists x\,P \mid \langle\alpha\rangle P \mid [\alpha]P$$

Differential Game Logic: Syntax



Discrete Assign

Test Game

Differential Equation

Choice Game

Seq. Game

Repeat Game

Definition (Hybrid game α)

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All Reals

Some Reals

Differential Game Logic: Syntax

Discrete Assign | Test Game | Differential Equation | Choice Game | Seq. Game | Repeat Game | Dual Game

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$$\alpha, \beta \ ::= \ x := e \mid ?Q \mid x' = f(x) \,\&\, Q \mid \alpha \cup \beta \mid \alpha ; \beta \mid \alpha^* \mid \alpha^d$$

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All Reals | Some Reals | Angel Wins

Differential Game Logic: Syntax

Discrete Assign

Test Game

Differential Equation

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Dual Game

Definition (Hybrid game α)

$$\alpha,\beta \ ::= \ x:=e \mid ?Q \mid x'=f(x)\,\&\,Q \mid \alpha\cup\beta \mid \alpha;\beta \mid \alpha^* \mid \alpha^d$$

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All Reals

Some Reals

Angel Wins

Demon Wins

Simple Examples

$$\langle (x := x+1; (x' = 1)^{\mathrm{d}} \cup x := x-1)^{*} \rangle\, (0 \leq x < 1)$$

$$\langle (x := x+1; (x' = 1)^{\mathrm{d}} \cup (x := x-1 \cap x := x-2))^{*} \rangle (0 \leq x < 1)$$

Simple Examples

$$\vDash \langle (x := x+1; (x' = 1)^{d} \cup x := x-1)^{*} \rangle (0 \leq x < 1)$$

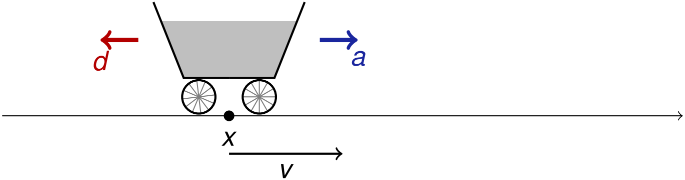
$$\langle (x := x+1; (x' = 1)^{d} \cup (x := x-1 \cap x := x-2))^{*} \rangle (0 \leq x < 1)$$

Simple Examples

$$\vDash \langle(x := x+1; (x' = 1)^{d} \cup x := x-1)^{*}\rangle\, (0 \leq x < 1)$$

$$\not\vDash \langle(x := x+1; (x' = 1)^{d} \cup (x := x-1 \cap x := x-2))^{*}\rangle (0 \leq x < 1)$$

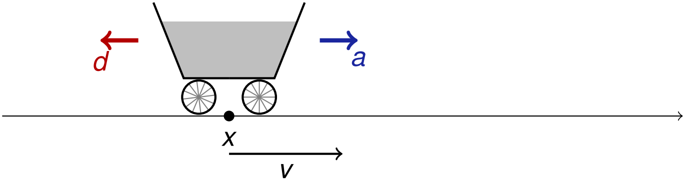
Example: Push-around Cart



$$v \geq 1 \rightarrow$$

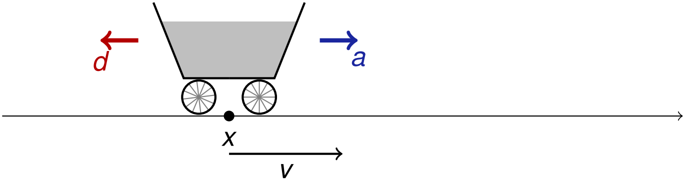
$$\left[\left((d := 1 \cup d := -1)^{\mathrm{d}}; (a := 1 \cup a := -1); \{x' = v, v' = a + d\}\right)^*\right] v \geq 0$$

Example: Push-around Cart



$$\vDash v \geq 1 \rightarrow$$
$$\left[\left((d := 1 \cap d := -1); (a := 1 \cup a := -1); \{x' = v, v' = a + d\}\right)^*\right] v \geq 0$$

Example: Push-around Cart



$$\vDash v \geq 1 \rightarrow$$

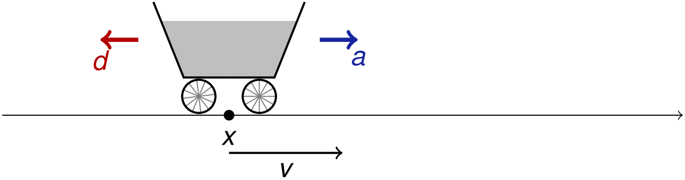
d before a can compensate

$$\left[\left((d := 1 \cap d := -1); (a := 1 \cup a := -1); \{x' = v, v' = a + d\}\right)^{*}\right] v \geq 0$$

$$x \geq 0 \land v \geq 0 \rightarrow$$

$$\left[\left((d := 1 \cap d := -1); (a := 1 \cup a := -1); \{x' = v, v' = a + d\}\right)^{*}\right] x \geq 0$$

Example: Push-around Cart



$\vDash v \geq 1 \rightarrow$ d before a can compensate

$$\big[\big((d:=1 \cap d:=-1);(a:=1 \cup a:=-1);\{x'=v, v'=a+d\}\big)^*\big]\, v \geq 0$$

$\nvDash x \geq 0 \land v \geq 0 \rightarrow$ d before a can compensate

$$\big[\big((d:=1 \cap d:=-1);(a:=1 \cup a:=-1);\{x'=v, v'=a+d\}\big)^*\big]\, x \geq 0$$

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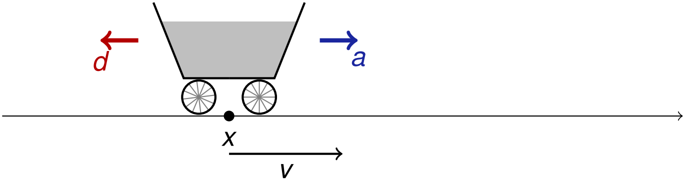

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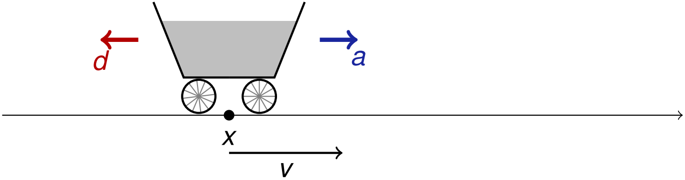
$$[((d:=1 \cap d:=-1);(a:=1 \cup a:=-1);\{x'=v, v'=a+d\})^*]\, v \geq 0$$

$$x \geq 0 \qquad \rightarrow$$

$$\langle((d:=1 \cap d:=-1);(a:=1 \cup a:=-1);\{x'=v, v'=a+d\})^*\rangle\, x \geq 0$$

Example: Push-around Cart



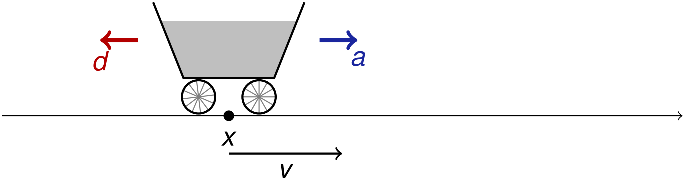
$\vDash v \geq 1 \rightarrow$ d before a can compensate

$$[((d := 1 \cap d := -1); (a := 1 \cup a := -1); \{x' = v, v' = a + d\})^*] v \geq 0$$

$\vDash x \geq 0 \rightarrow$ boring by skip

$$\langle((d := 1 \cap d := -1); (a := 1 \cup a := -1); \{x' = v, v' = a + d\})^*\rangle x \geq 0$$

Example: Push-around Cart



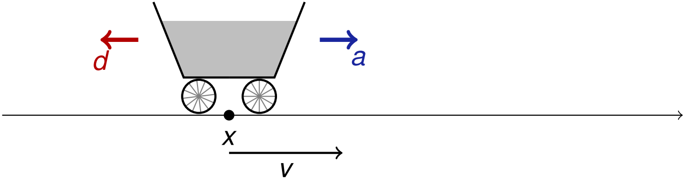
$\vDash v \geq 1 \rightarrow$

d before a can compensate

$$\Big[\big((d := 1 \cap d := -1); (a := 1 \cup a := -1); \{x' = v, v' = a + d\}\big)^*\Big]\, v \geq 0$$

$$\Big\langle\big((d := 1 \cap d := -1); (a := 1 \cup a := -1); \{x' = v, v' = a + d\}\big)^*\Big\rangle\, x \geq 0$$

Example: Push-around Cart



$\vDash v \geq 1 \rightarrow$ d before a can compensate

$$[((d := 1 \cap d := -1); (a := 1 \cup a := -1); \{x' = v, v' = a + d\})^*]\, v \geq 0$$

$\not\vDash$ counterstrategy $d := -1$

$$\langle((d := 1 \cap d := -1); (a := 1 \cup a := -1); \{x' = v, v' = a + d\})^*\rangle\, x \geq 0$$

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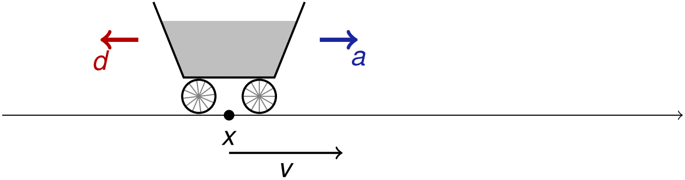


$\vDash v \geq 1 \rightarrow$ d before a can compensate

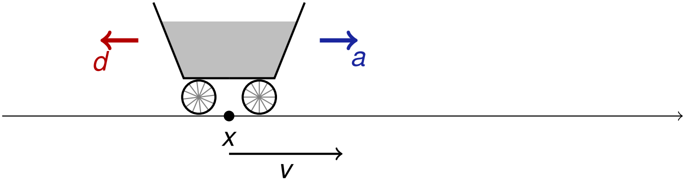
$$[((d := 1 \cap d := -1); (a := 1 \cup a := -1); \{x' = v, v' = a + d\})^*] v \geq 0$$

$\nvDash$ counterstrategy $d := -1$

$$\langle((d := 1 \cap d := -1); (a := 1 \cup a := -1); \{x' = v, v' = a + d\})^*\rangle x \geq 0$$

$$\langle((d := 1 \cap d := -1); (a := 2 \cup a := -2); \{x' = v, v' = a + d\})^*\rangle x \geq 0$$

Example: Push-around Cart



$\vDash v \geq 1 \rightarrow$ d before a can compensate

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$\nvDash$ counterstrategy $d := -1$

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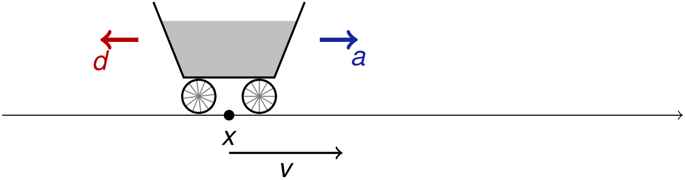

$\vDash v \geq 1 \rightarrow$ d before a can compensate

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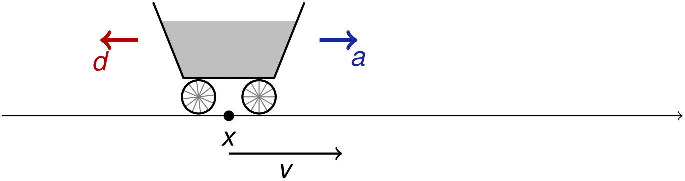
$\nvDash$ counterstrategy $d:=-1$

$$\langle((d:=1 \cap d:=-1);(a:=1 \cup a:=-1);\{x'=v, v'=a+d\})^*\rangle\, x \geq 0$$

$$\vDash \langle((d:=1 \cap d:=-1);\,(a:=2 \cup a:=-2);\,\{x'=v, v'=a+d\})^*\rangle\, x \geq 0$$

$$\langle((d:=2 \cap d:=-2);\,(a:=2 \cup a:=-2);$$
$$t:=0;\,\{x'=v, v'=a+d, t'=1 \,\&\, t \leq 1\})^*\rangle\, x^2 \geq 100$$

Example: Push-around Cart



$\vDash v \geq 1 \rightarrow$ d before a can compensate

$$[((d := 1 \cap d := -1); (a := 1 \cup a := -1); \{x' = v, v' = a + d\})^*] v \geq 0$$

$\not\vDash$ counterstrategy $d := -1$

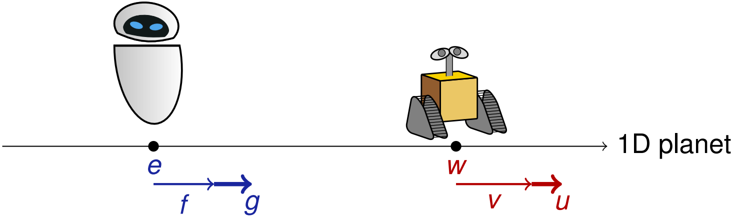
$$\langle((d := 1 \cap d := -1); (a := 1 \cup a := -1); \{x' = v, v' = a + d\})^*\rangle x \geq 0$$

$$\vDash \langle((d := 1 \cap d := -1); (a := 2 \cup a := -2); \{x' = v, v' = a + d\})^*\rangle x \geq 0$$

$\vDash \langle((d := 2 \cap d := -2); (a := 2 \cup a := -2);$ $a := d$ then $a := 2 \operatorname{sign} v$

$$t := 0; \{x' = v, v' = a + d, t' = 1 \,\&\, t \leq 1\})^*\rangle x^2 \geq 100$$

Example: WALL·E and EVE Robot Dance

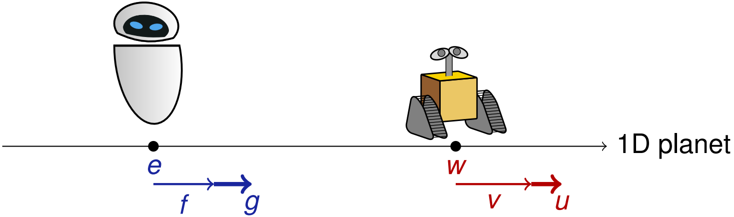


$$(w-e)^2 \leq 1 \land v = f \rightarrow$$
$$\langle ((u := 1 \cap u := -1);$$
$$(g := 1 \cup g := -1);$$
$$t := 0;$$
$$\{w' = v, v' = u, e' = f, f' = g, t' = 1 \,\&\, t \leq 1\}^{d}$$
$$)^{\times} \rangle \, (w-e)^2 \leq 1$$

EVE at e plays Angel's part controlling g

WALL·E at w plays Demon's part controlling u

Example: WALL·E and EVE Robot Dance and the World

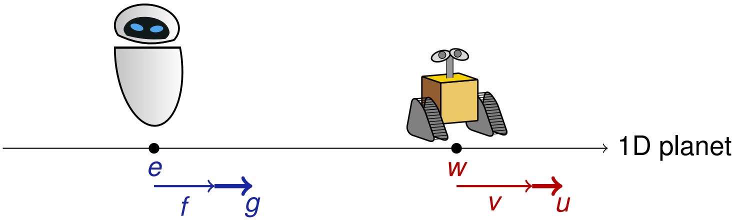


$$
\begin{aligned}
&(w-e)^2 \le 1 \land v = f \rightarrow \\
&\langle \big((u := 1 \cap u := -1); \\
&\quad (g := 1 \cup g := -1); \\
&\quad t := 0; \\
&\quad \{w' = v, v' = u, e' = f, f' = g, t' = 1 \,\&\, t \le 1\}^{d} \\
&\big)^{\times} \rangle \, (w-e)^2 \le 1
\end{aligned}
$$

EVE at e plays Angel's part controlling g

WALL·E at w plays Demon's part controlling u and world time

Example: EVE and WALL·E Robot Dance

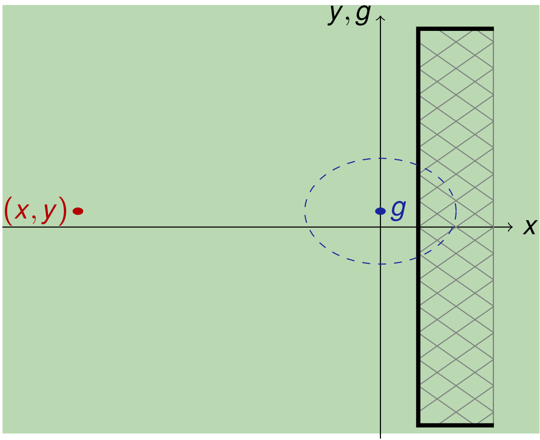


$$
\begin{aligned}
&(w-e)^2 \le 1 \land v = f \to \\
&[((u := 1 \cap u := -1);\\
&\quad (g := 1 \cup g := -1);\\
&\quad t := 0;\\
&\quad \{w' = v, v' = u, e' = f, f' = g, t' = 1 \,\&\, t \le 1\}\\
&)^{\times}]\ (w-e)^2 > 1
\end{aligned}
$$

WALL·E at w plays Demon's part controlling u and world time

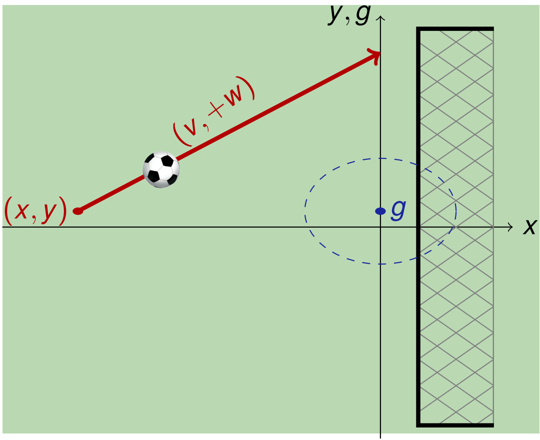
EVE at e plays Angel's part controlling g

Example: Goalie in Robot Soccer



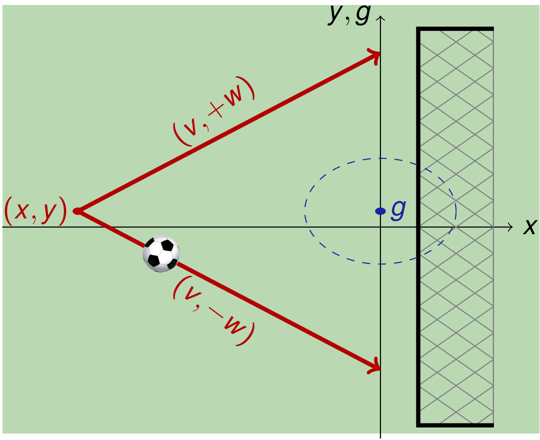
$$x < 0 \wedge v > 0 \wedge y = g \rightarrow \\ \big\langle (w := +w \cap w := -w); \\ \big((u := +u \cup u := -u); \{x' = v, y' = w, g' = u\} \big)^* \big\rangle \, x^2 + (y-g)^2 \leq 1$$

Example: Goalie in Robot Soccer



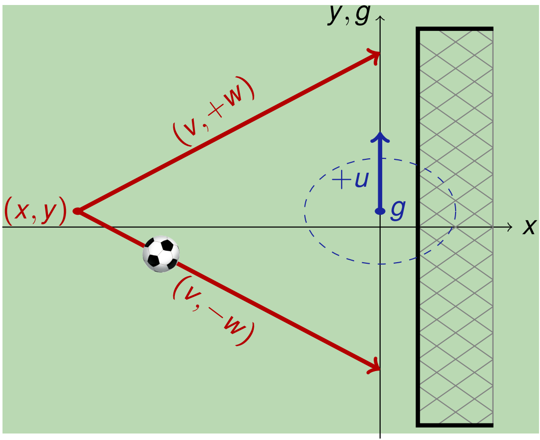
$$x < 0 \land v > 0 \land y = g \rightarrow$$
$$\langle (w := +w \cap w := -w);$$
$$((u := +u \cup u := -u); \{x' = v, y' = w, g' = u\})^* \rangle\, x^2 + (y-g)^2 \leq 1$$

Example: Goalie in Robot Soccer



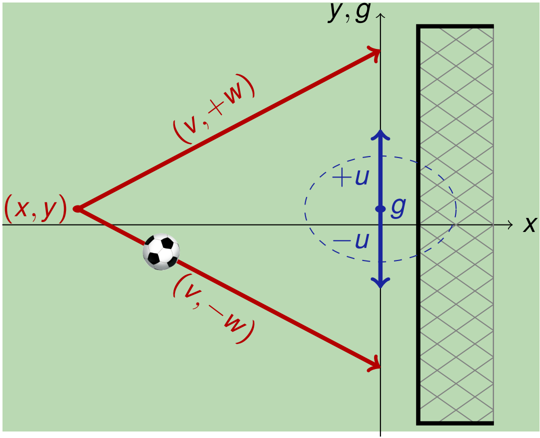
$$x<0 \land v>0 \land y=g \rightarrow \langle (w:=+w \cap w:=-w); ((u:=+u \cup u:=-u); \{x'=v, y'=w, g'=u\})^* \rangle\, x^2+(y-g)^2 \le 1$$

Example: Goalie in Robot Soccer



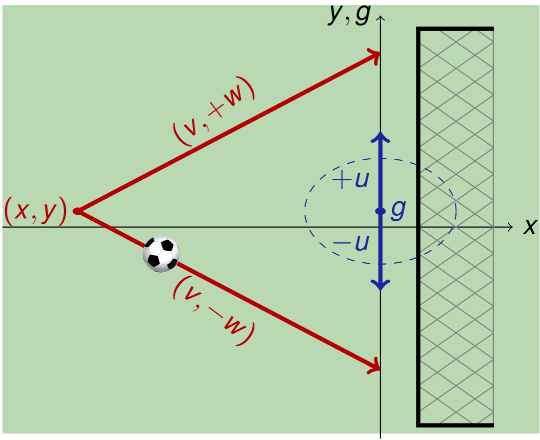
$$x < 0 \land v > 0 \land y = g \rightarrow$$
$$\langle (w := +w \cap w := -w);$$
$$\big((u := +u \cup u := -u); \{x' = v, y' = w, g' = u\}\big)^* \rangle\, x^2 + (y - g)^2 \leq 1$$

Example: Goalie in Robot Soccer



$$x < 0 \land v > 0 \land y = g \to \\ \langle (w := +w \cap w := -w); \\ ((u := +u \cup u := -u); \{x' = v, y' = w, g' = u\})^* \rangle\, x^2 + (y-g)^2 \le 1$$

Example: Goalie in Robot Soccer



Goalie's Secret

$$\left(\frac{x}{v}\right)^2 (u-w)^2 \leq 1 \land$$

$$x < 0 \land v > 0 \land y = g \rightarrow$$
$$\langle (w := +w \cap w := -w);$$
$$\left((u := +u \cup u := -u); \{x' = v, y' = w, g' = u\}\right)^* \rangle\, x^2 + (y-g)^2 \leq 1$$

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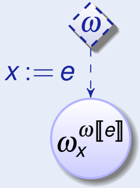
5. An Informal Operational Game Tree Semantics
6. Summary

Definition (Hybrid game α: operational semantics)

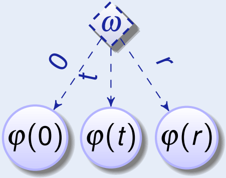
$x := e$



Differential Game Logic: Operational Semantics

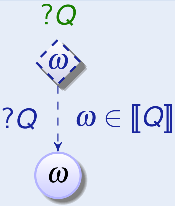
Definition (Hybrid game α: operational semantics)

$x' = f(x) \,\&\, Q$



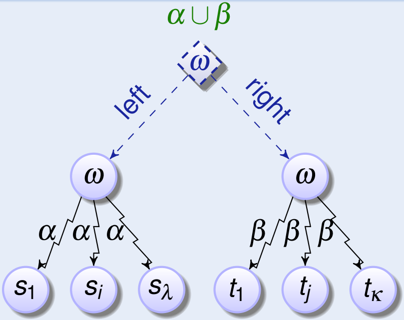
Differential Game Logic: Operational Semantics

Definition (Hybrid game α: operational semantics)



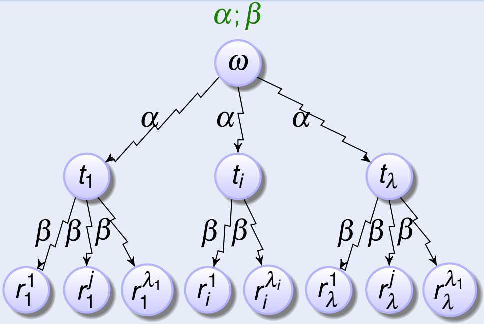
Differential Game Logic: Operational Semantics

Definition (Hybrid game α: operational semantics)



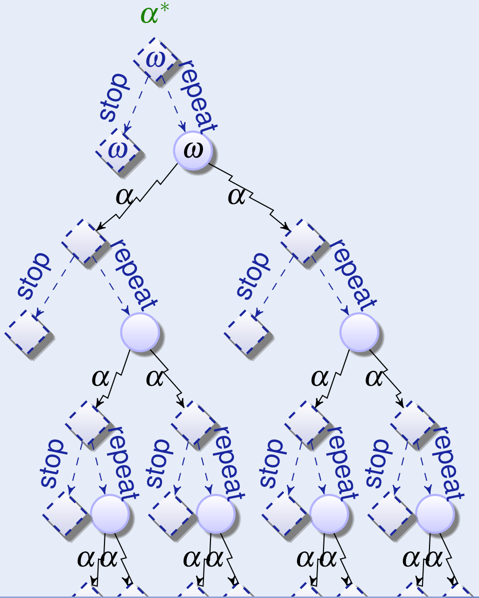
Differential Game Logic: Operational Semantics

Definition (Hybrid game α: operational semantics)



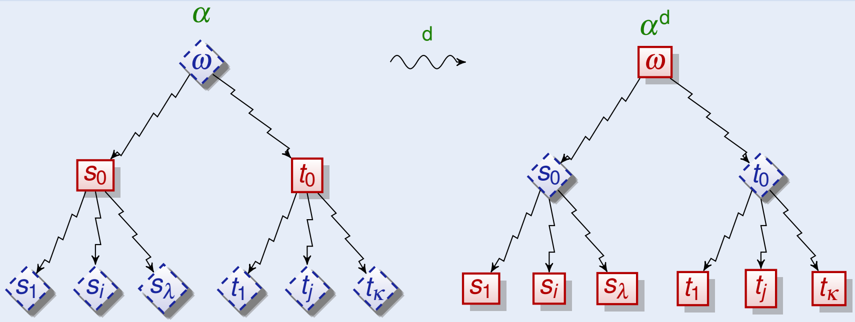
Differential Game Logic: Operational Semantics

Definition (Hybrid game α: operational semantics)



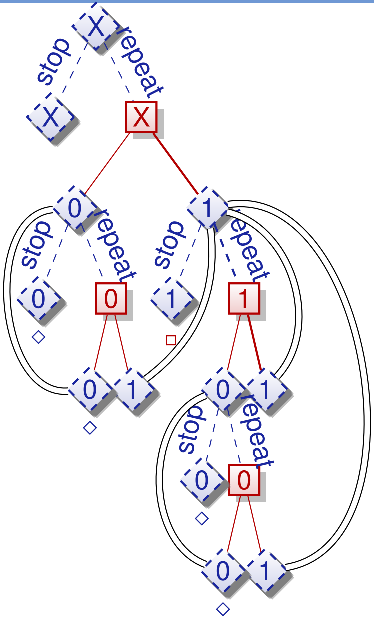
Differential Game Logic: Operational Semantics

Definition (Hybrid game α: operational semantics)



Filibusters

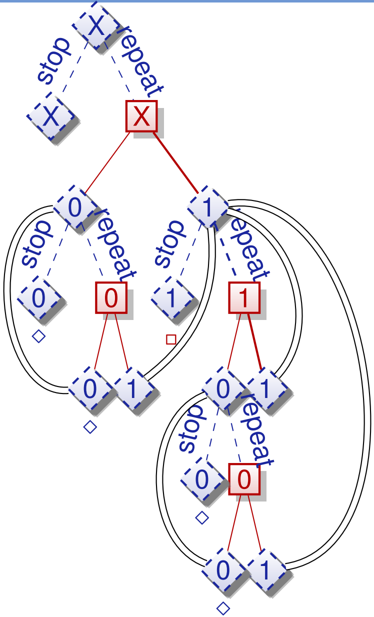
$\langle (x := 0 \cap x := 1)^* \rangle x = 0$



Filibusters & The Significance of Finitude

$$\langle (x := 0 \cap x := 1)^* \rangle x = 0$$

$\overset{\text{wfd}}{\rightsquigarrow}$ false unless $x = 0$



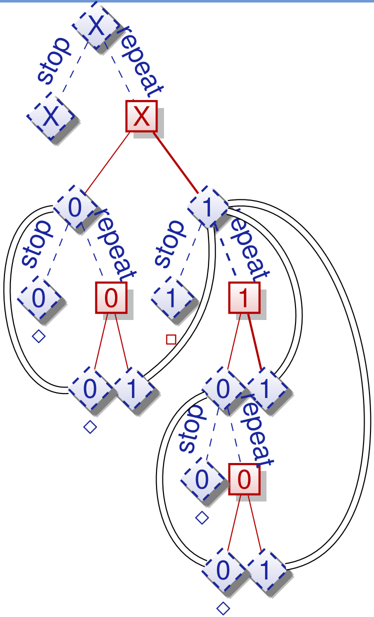
Filibusters & The Significance of Finitude

$$\langle (x' = 1^{\mathrm{d}}; x := 0)^{*} \rangle x = 0$$

$$\langle (x := 0; x' = 1^{\mathrm{d}})^{*} \rangle x = 0$$

$$\langle (x := 0 \cap x := 1)^{*} \rangle x = 0$$

$\overset{\text{wfd}}{\rightsquigarrow}$ false unless $x = 0$



Filibusters & The Significance of Finitude

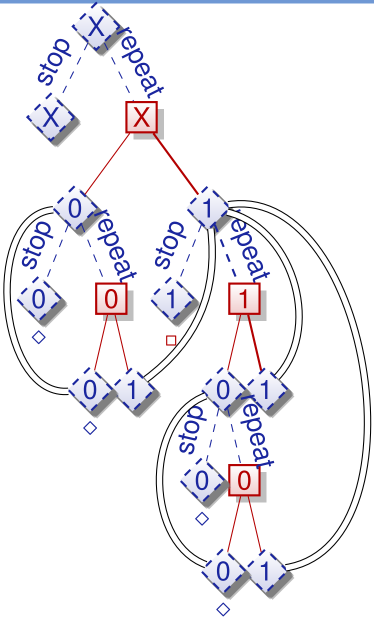
$\overset{<\infty}{\rightsquigarrow}$ true

$\langle (x' = 1^{\mathrm{d}}; x := 0)^{*} \rangle x = 0$

$\langle (x := 0; x' = 1^{\mathrm{d}})^{*} \rangle x = 0$

$\langle (x := 0 \cap x := 1)^{*} \rangle x = 0$

$\overset{\text{wfd}}{\rightsquigarrow}$ false unless $x = 0$



Filibusters & The Significance of Finitude

$\overset{<\infty}{\rightsquigarrow}$ true

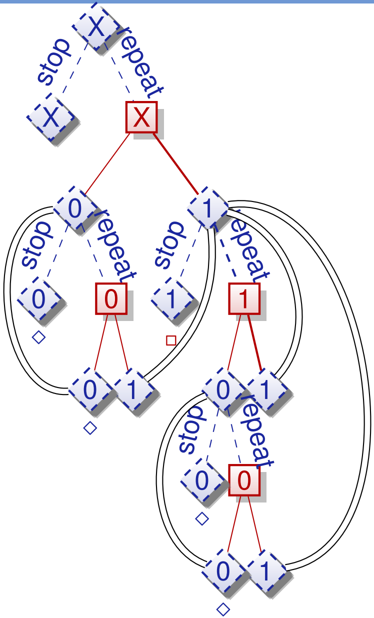
$\langle(x' = 1^{\mathrm{d}}; x := 0)^*\rangle x = 0$

$\langle(x := 0; x' = 1^{\mathrm{d}})^*\rangle x = 0$

$\langle(x := 0 \cap x := 1)^*\rangle x = 0$

$\overset{\text{wfd}}{\rightsquigarrow}$ false unless $x = 0$

Well-defined games
can't be postponed forever!



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Differential Game Logic: Syntax

Discrete Assign | Test Game | Differential Equation | Choice Game | Seq. Game | Repeat Game | Dual Game

Definition (Hybrid game α)

$$\alpha,\beta ::= x:=e \mid ?Q \mid x'=f(x)\,\&\,Q \mid \alpha\cup\beta \mid \alpha;\beta \mid \alpha^* \mid \alpha^d$$

Definition (dGL Formula P)

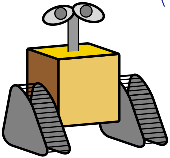
$$P,Q ::= e\geq\tilde{e} \mid \neg P \mid P\wedge Q \mid \forall x\,P \mid \exists x\,P \mid \langle\alpha\rangle P \mid [\alpha]P$$

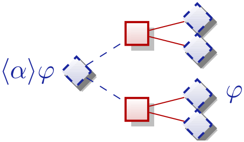
All Reals | Some Reals | Angel Wins | Demon Wins

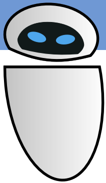
Summary

differential game logic

$\text{dGL} = \text{GL} + \text{HG} = \text{dL} + {}^{d}$



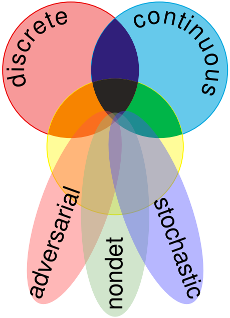




- Differential game logic
- Logic for hybrid games
- Compositional PL + logic
- Discrete + continuous + adversarial
- Operational semantics (informally)

Next chapter

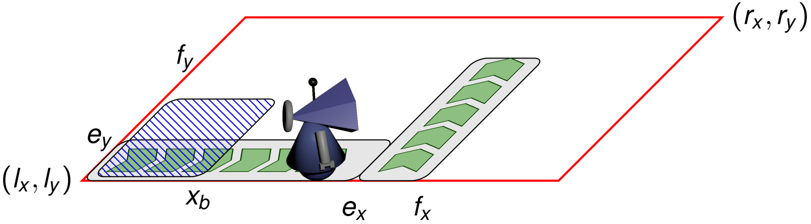
1. Formal semantics



Outline

7 Example: Robot Factory

Example: Robot Factory Decentralized Automation



Model

- (x, y) robot coordinates
- (v_x, v_y) velocities
- conveyor belts may instantaneously increase robot's velocity by (c_x, c_y)

Primary objectives of the robot

- Leave ▨ within time ε
- Never leave outer □

Challenges

- Distributed, physical environment
- Possibly conflicting secondary objectives

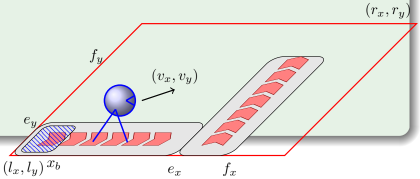
Robot Factory Automation (*RF*)

Example (Robot-Demon vs. Angel-Factory Environment)

$$\Big(\big(?true \cup (?(x < e_x \land y < e_y \land \mathrm{eff}_1 = 1);\ v_x := v_x + c_x;\ \mathrm{eff}_1 := 0) \qquad \text{// belt}$$

$$\cup (?(e_x \le x \land y \le f_y \land \mathrm{eff}_2 = 1);\ v_y := v_y + c_y;\ \mathrm{eff}_2 := 0)\big);$$

$$\Big)^*$$



Robot Factory Automation (*RF*)

Example (Robot-Demon vs. Angel-Factory Environment)

$$\Big(\big(\,?true \cup (?(x < e_x \land y < e_y \land \mathrm{eff}_1 = 1);\ v_x := v_x + c_x;\ \mathrm{eff}_1 := 0) \qquad \text{// belt}$$

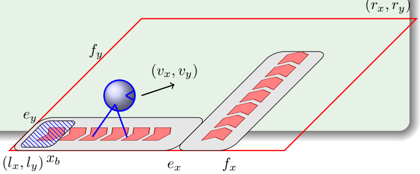
$$\cup (?(e_x \le x \land y \le f_y \land \mathrm{eff}_2 = 1);\ v_y := v_y + c_y;\ \mathrm{eff}_2 := 0)\ \big);$$

$$(\,a_x := *;\ ?(-A \le a_x \le A);$$

$$a_y := *;\ ?(-A \le a_y \le A); \qquad \text{// “independent” robot acceleration}$$

$$t_s := 0\,)^{\mathrm{d}};$$

$$\Big)^*$$



Robot Factory Automation (RF)

Example (Robot-Demon vs. Angel-Factory Environment)

$$\Big(\big(?true \cup (?(x < e_x \wedge y < e_y \wedge \mathrm{eff}_1 = 1);\ v_x := v_x + c_x;\ \mathrm{eff}_1 := 0) \qquad \text{// belt}$$

$$\cup (?(e_x \le x \wedge y \le f_y \wedge \mathrm{eff}_2 = 1);\ v_y := v_y + c_y;\ \mathrm{eff}_2 := 0) \big);$$

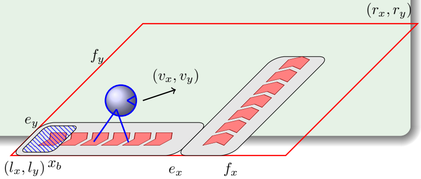
$$(a_x := *;\ ?(-A \le a_x \le A);$$

$$a_y := *;\ ?(-A \le a_y \le A); \qquad \text{// “independent” robot acceleration}$$

$$t_s := 0)^d;$$

$$(x' = v_x, y' = v_y, v_x' = a_x, v_y' = a_y, t' = 1, t_s' = 1 \ \&\ t_s \le \varepsilon);$$

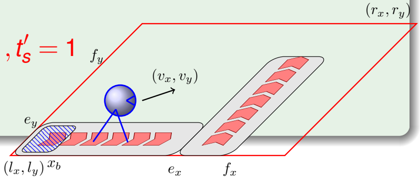
$$\Big)^*$$



Robot Factory Automation (*RF*)

Example (Robot-Demon vs. Angel-Factory Environment)

$$
\begin{aligned}
&\Big(\big(?\mathit{true} \cup (?(x < e_x \land y < e_y \land \mathrm{eff}_1 = 1);\ v_x := v_x + c_x;\ \mathrm{eff}_1 := 0) && \text{// belt}\\
&\qquad\quad \cup (?(e_x \le x \land y \le f_y \land \mathrm{eff}_2 = 1);\ v_y := v_y + c_y;\ \mathrm{eff}_2 := 0)\ \big);\\
&\ (\ a_x := *;\ ?(-A \le a_x \le A);\\
&\quad a_y := *;\ ?(-A \le a_y \le A); && \text{// “independent” robot acceleration}\\
&\quad t_s := 0\ )^d;\\
&\big((\ x' = v_x, y' = v_y, v_x' = a_x, v_y' = a_y, t' = 1, t_s' = 1\ \&\ t_s \le \varepsilon\ );\\
&\ \cap (?(a_x v_x \le 0 \land a_y v_y \le 0)^d; && \text{// brake}\\
&\quad \text{if } v_x = 0 \text{ then } a_x := 0 \text{ fi}; && \text{// per direction: no time lock}\\
&\quad \text{if } v_y = 0 \text{ then } a_y := 0 \text{ fi};\\
&\quad (x' = v_x, y' = v_y, v_x' = a_x, v_y' = a_y, t' = 1, t_s' = 1\\
&\quad \&\ t_s \le \varepsilon \land a_x v_x \le 0 \land a_y v_y \le 0))\big)\Big)^*
\end{aligned}
$$



Robot Factory Automation (*RF*)

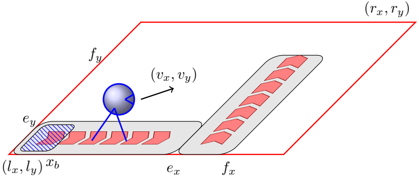
Proposition (Robot stays in □)

$$\vDash (x = y = 0 \land v_x = v_y = 0 \land \text{ ▸ Controllability Assumptions }) \\ \to [RF](x \in [l_x, r_x] \land y \in [l_y, r_y])$$

Proposition (Stays in □ and leaves ▨ on time)

$RF|_x$: RF projected to the x-axis

$$\vDash (x = 0 \land v_x = 0 \land \text{ ▸ Controllability Assumptions }) \\ \to [RF|_x](x \in [l_x, r_x] \land (t \geq \varepsilon \to x \geq x_b))$$



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