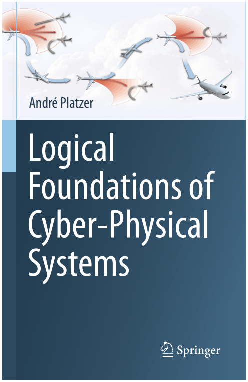
12: Ghosts & Differential Ghosts

Logical Foundations of Cyber-Physical Systems



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Outline

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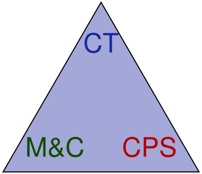
Learning Objectives

rigorous reasoning about ODEs
extra dimensions for extra invariants
invent dark energy
intuition for differential invariants
states and proofs
verify CPS models at scale



CT

M&C

CPS

none: ghosts are for proofs!

relations of state
extra ghost state
CPS semantics

Learning Objectives

Ghosts & Differential Ghosts

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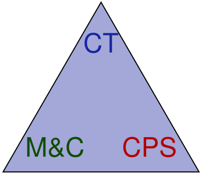


mark ghosts in models
syntax of models
solutions of ODEs

relations of state
extra ghost state
CPS semantics

Outline

1. Learning Objectives
2. Recap: Proofs for Differential Equations
3. A Gradual Introduction to Ghost Variables
   - Discrete Ghosts
   - Proving Bouncing Balls with Sneaky Solutions
   - Differential Ghosts of Time
   - Constructing Differential Ghosts
4. Differential Ghosts
   - Substitute Ghosts
   - Solvable Ghosts
   - Limit Velocity of an Aerodynamic Ball
5. Summary

Differential Invariants for Differential Equations

Differential Weakening

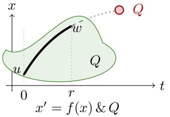
$$\frac{Q \vdash F}{P \vdash [x' = f(x) \,\&\, Q]F}$$

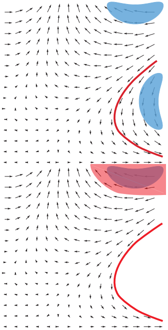
Differential Invariant

$$\frac{Q \vdash [x' := f(x)](F)'}{F \vdash [x' = f(x) \,\&\, Q]F}$$

Differential Cut

$$\frac{F \vdash [x' = f(x) \,\&\, Q]C \quad F \vdash [x' = f(x) \,\&\, Q \land C]F}{F \vdash [x' = f(x) \,\&\, Q]F}$$





DW $[x' = f(x) \,\&\, Q]F \leftrightarrow [x' = f(x) \,\&\, Q](Q \rightarrow F)$

DI $[x' = f(x) \,\&\, Q]F \leftarrow \big(Q \rightarrow F \land [x' = f(x) \,\&\, Q](F)'\big)$

DC $\big([x' = f(x) \,\&\, Q]F \leftrightarrow [x' = f(x) \,\&\, Q \land C]F\big) \leftarrow [x' = f(x) \,\&\, Q]C$

Differential Invariants for Differential Equations

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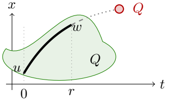
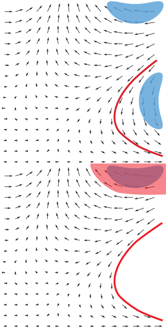
DC $\big([x' = f(x) \,\&\, Q]F \leftrightarrow [x' = f(x) \,\&\, Q \land C]F\big) \leftarrow [x' = f(x) \,\&\, Q]C$

DE $[x' = f(x) \,\&\, Q]F \leftrightarrow [x' = f(x) \,\&\, Q][x' := f(x)]F$



$x' = f(x) \,\&\, Q$



Outline

Discrete Ghosts

$$\text{iG}\ \frac{\Gamma \vdash [y := e]p, \Delta}{\Gamma \vdash p, \Delta} \quad (y \text{ new})$$

$$\rightarrow\text{R}\ \frac{}{\vdash xy - 1 = 0 \rightarrow [x' = x, y' = -y]xy = 1}$$

Clou: Ask a ghost to remember some auxiliary state for the proof.

Discrete Ghosts

$$\text{iG}\ \frac{\Gamma \vdash [y := e]p, \Delta}{\Gamma \vdash p, \Delta}\quad (y \text{ new})$$

$$\to\text{R}\ \dfrac{\text{iG}\ \dfrac{}{xy - 1 = 0 \vdash [x' = x, y' = -y]xy = 1}}{\vdash xy - 1 = 0 \to [x' = x, y' = -y]xy = 1}$$

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Discrete Ghosts

$$\text{iG}\ \frac{\Gamma \vdash [y := e]p, \Delta}{\Gamma \vdash p, \Delta}\quad (y \text{ new})$$

discrete ghost c remembers function of old state

$$
\begin{array}{rr}
[:=]_= & \overline{xy - 1 = 0 \vdash [c := xy][x' = x, y' = -y]xy = 1} \\
\text{iG} & \overline{xy - 1 = 0 \vdash [x' = x, y' = -y]xy = 1} \\
\rightarrow\text{R} & \overline{\vdash xy - 1 = 0 \rightarrow [x' = x, y' = -y]xy = 1}
\end{array}
$$

Clou: Ask a ghost to remember some auxiliary state for the proof.

Discrete Ghosts

$$\text{iG}\ \frac{\Gamma \vdash [y := e]p, \Delta}{\Gamma \vdash p, \Delta} \quad (y \text{ new})$$

$p \leftrightarrow [y := e]p$ by $[:=]$

$$[:=]_= \ \frac{\Gamma, y = e \vdash p(y), \Delta}{\Gamma \vdash [x := e]p(x), \Delta}$$

$$\begin{array}{ll}
\text{MR} & \overline{xy - 1 = 0, c = xy \vdash [x' = x, y' = -y]xy = 1} \\
{[:=]_=} & \overline{xy - 1 = 0 \vdash [c := xy][x' = x, y' = -y]xy = 1} \\
\text{iG} & \overline{xy - 1 = 0 \vdash [x' = x, y' = -y]xy = 1} \\
\rightarrow\text{R} & \overline{\vdash xy - 1 = 0 \rightarrow [x' = x, y' = -y]xy = 1}
\end{array}$$

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Discrete Ghosts

$$\text{iG}\ \dfrac{\Gamma \vdash [y := e]p, \Delta}{\Gamma \vdash p, \Delta}\ (y \text{ new}) \qquad\qquad p \leftrightarrow [y := e]p \text{ by } [:=]$$

$$[:=]_=\ \dfrac{\Gamma, y = e \vdash p(y), \Delta}{\Gamma \vdash [x := e]p(x), \Delta}(y \text{ new})$$

$$\text{→R}\ \dfrac{\text{iG}\ \dfrac{[:=]_=\ \dfrac{\text{MR}\ \dfrac{\text{dI}\ \dfrac{}{xy-1=0, c=xy \vdash [x'=x, y'=-y]c=xy}\ \triangleright}{xy-1=0, c=xy \vdash [x'=x, y'=-y]xy=1}}{xy-1=0 \vdash [c := xy][x'=x, y'=-y]xy=1}}{xy-1=0 \vdash [x'=x, y'=-y]xy=1}}{\vdash xy-1=0 \to [x'=x, y'=-y]xy=1}$$

Clou: Ask a ghost to remember some auxiliary state for the proof.

Discrete Ghosts

$$\text{iG}\ \frac{\Gamma \vdash [y := e]p, \Delta}{\Gamma \vdash p, \Delta}\ (y \text{ new}) \qquad\qquad p \leftrightarrow [y := e]p \text{ by } [:=]$$

$$[:=]_{=}\ \frac{\Gamma, y = e \vdash p(y), \Delta}{\Gamma \vdash [x := e]p(x), \Delta}(y \text{ new})$$

$$\begin{array}{l|c}
\hline
[:=] & \vdash [{\color{red}x'}:=x][{\color{red}y'}:=-y]0 = {\color{red}x'}y + x{\color{red}y'} \\
\hline
\text{dI} & xy - 1 = 0, c = xy \vdash [x' = x, y' = -y]c = xy \qquad \triangleright \\
\hline
\text{MR} & xy - 1 = 0, c = xy \vdash [x' = x, y' = -y]xy = 1 \\
\hline
[:=]_{=} & xy - 1 = 0 \vdash [c := xy][x' = x, y' = -y]xy = 1 \\
\hline
\text{iG} & xy - 1 = 0 \vdash [x' = x, y' = -y]xy = 1 \\
\hline
\rightarrow\text{R} & \vdash xy - 1 = 0 \rightarrow [x' = x, y' = -y]xy = 1
\end{array}$$

Clou: Ask a ghost to remember some auxiliary state for the proof.

Discrete Ghosts

$$\text{iG}\ \frac{\Gamma \vdash [y := e]p, \Delta}{\Gamma \vdash p, \Delta}\quad (y \text{ new})$$

$$p \leftrightarrow [y := e]p \text{ by } [:=]$$

$$[:=]_=\ \frac{\Gamma, y = e \vdash p(y), \Delta}{\Gamma \vdash [x := e]p(x), \Delta}(y \text{ new})$$

$$
\begin{array}{ll}
\mathbb{R} & \dfrac{}{\vdash 0 = xy + x(-y)} \\[6pt]
[:=] & \dfrac{\vdash 0 = xy + x(-y)}{\vdash [x':=x][y':=-y]0 = x'y + xy'} \\[6pt]
\text{dI} & \dfrac{\vdash [x':=x][y':=-y]0 = x'y + xy'}{xy - 1 = 0, c = xy \vdash [x' = x, y' = -y]c = xy \qquad \triangleright} \\[6pt]
\text{MR} & \dfrac{xy - 1 = 0, c = xy \vdash [x' = x, y' = -y]c = xy}{xy - 1 = 0, c = xy \vdash [x' = x, y' = -y]xy = 1} \\[6pt]
[:=]_= & \dfrac{xy - 1 = 0, c = xy \vdash [x' = x, y' = -y]xy = 1}{xy - 1 = 0 \vdash [c := xy][x' = x, y' = -y]xy = 1} \\[6pt]
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\end{array}
$$

Clou: Ask a ghost to remember some auxiliary state for the proof.

Discrete Ghosts

$$\text{iG}\ \frac{\Gamma \vdash [y := e]p, \Delta}{\Gamma \vdash p, \Delta}\ (y \text{ new}) \qquad\qquad p \leftrightarrow [y := e]p \text{ by } [:=]$$

$$[:=]_{=}\ \frac{\Gamma, y = e \vdash p(y), \Delta}{\Gamma \vdash [x := e]p(x), \Delta}(y \text{ new})$$

$$\begin{array}{lc}
 & * \\ \hline
\mathbb{R} & \vdash 0 = xy + x(-y) \\ \hline
[:=] & \vdash [x' := x][y' := -y]0 = x'y + xy' \\ \hline
\text{dI} & xy - 1 = 0, c = xy \vdash [x' = x, y' = -y]c = xy \qquad \triangleright \\ \hline
\text{MR} & xy - 1 = 0, c = xy \vdash [x' = x, y' = -y]xy = 1 \\ \hline
[:=]_{=} & xy - 1 = 0 \vdash [c := xy][x' = x, y' = -y]xy = 1 \\ \hline
\text{iG} & xy - 1 = 0 \vdash [x' = x, y' = -y]xy = 1 \\ \hline
\to\text{R} & \vdash xy - 1 = 0 \to [x' = x, y' = -y]xy = 1
\end{array}$$

Clou: Ask a ghost to remember some auxiliary state for the proof.

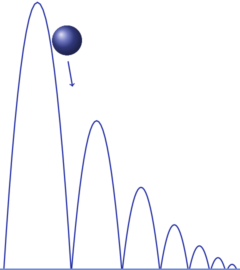
Quantum's Back for a Differential Invariant Proof

$$[]\wedge\,\dfrac{\text{dI}\,\dfrac{[:=]\,\dfrac{\mathbb{R}\,\dfrac{*}{x\geq 0 \vdash 2gv = -2v(-g)}}{x\geq 0 \vdash [x':=v][v':=-g]2gx' = -2vv'}}{2gx=2gH-v^2 \vdash [x''=-g\,\&\,x\geq 0]2gx=2gH-v^2} \quad \text{dW}\,\dfrac{\text{id}\,\dfrac{*}{x\geq 0 \vdash x\geq 0}}{\vdash [x''=-g\,\&\,x\geq 0]x\geq 0}}{2gx=2gH-v^2 \vdash [x''=-g\,\&\,x\geq 0](2gx=2gH-v^2 \wedge x\geq 0)}$$

No solutions but still a proof.

Simple proof with simple arithmetic.

Independent proofs for independent questions.



Quantum's Back for a Differential Invariant Proof

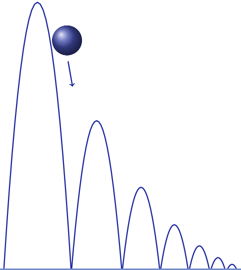
$$[]\wedge\,\dfrac{\text{dI}\,\dfrac{[:=]\,\dfrac{\mathbb{R}\,\dfrac{*}{x\geq0 \vdash 2gv=-2v(-g)}}{x\geq0\vdash [x':=v][v':=-g]2gx'=-2vv'}}{2gx=2gH-v^2 \vdash [x''=-g \,\&\, x\geq0]2gx=2gH-v^2} \quad \text{dW}\,\dfrac{\text{id}\,\dfrac{*}{x\geq0\vdash x\geq0}}{\vdash [x''=-g\,\&\,x\geq0]x\geq0}}{2gx=2gH-v^2 \vdash [x''=-g\,\&\,x\geq0](2gx=2gH-v^2\wedge x\geq0)}$$

No solutions but still a proof.

Simple proof with simple arithmetic.

Independent proofs for independent questions.

But need to have the right invariant.



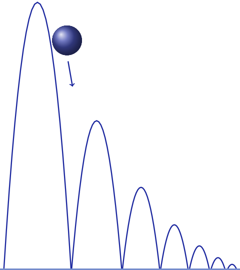
Quantum's Back for a Solution Proof w/o Solution Axiom

$$A \vdash [x'' = -g \,\&\, x \geq 0]B(x,v)$$

$$A \equiv 0 \leq x \land x = H \land v = 0 \land g > 0 \land 1 \geq c \geq 0$$

$$B(x,v) \equiv 0 \leq x \land x \leq H$$

$$x'' = -g \equiv \{x' = v, v' = -g\}$$



Quantum's Back for a Solution Proof w/o Solution Axiom

Solution:

$$x =$$

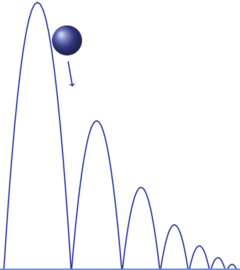
$$v =$$

$$A \vdash [x'' = -g \,\&\, x \geq 0] B(x,v)$$

$$A \equiv$$

$$B(x,v) \equiv$$

$$x'' = -g \equiv \{x' = v, v' = -g\}$$

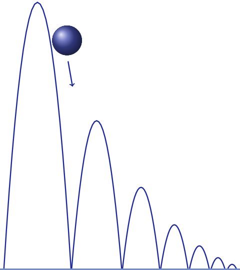


Quantum's Back for a Solution Proof w/o Solution Axiom

Solution:

$$x(t) = x + vt - \frac{g}{2}t^2$$
$$v(t) = v - gt$$

$$A \vdash [x'' = -g \,\&\, x \geq 0]B(x,v)$$
$$A \equiv \text{ redacted}$$
$$B(x,v) \equiv \text{ redacted}$$
$$x'' = -g \equiv \{x' = v, v' = -g\}$$



Quantum's Back for a Solution Proof w/o Solution Axiom

Solution: How to use a solution without really trying solution axiom $[']$

$$x(t) = x + vt - \frac{g}{2}t^2$$
$$v(t) = v - gt$$

$$A \vdash [x'' = -g \,\&\, x \geq 0]B(x,v)$$
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$$B(x,v) \equiv \text{ redacted}$$
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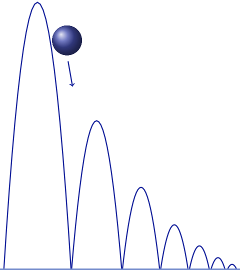


Quantum's Back for a Solution Proof w/o Solution Axiom

Solution: How to use a solution without really trying solution axiom $[']$

$$x(t) = x + vt - \frac{g}{2}t^2 \qquad \text{solution of ODE invariant along ODE}$$

$$v(t) = v - gt$$

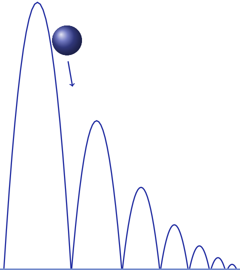
Can't just say $x(t)$

$$A \vdash [x'' = -g \,\&\, x \geq 0] B(x,v)$$

$$A \equiv \text{ redacted}$$

$$B(x,v) \equiv \text{ redacted}$$

$$x'' = -g \equiv \{x' = v, v' = -g\}$$



Quantum's Back for a Solution Proof w/o Solution Axiom

Solution: How to use a solution without really trying solution axiom $[']$

$$\begin{aligned} x &= x + vt - \tfrac{g}{2}t^2 \\ v &= v - gt \end{aligned}$$

$$A \vdash [x'' = -g \,\&\, x \geq 0]B(x,v)$$

$$A \equiv \text{ redacted}$$

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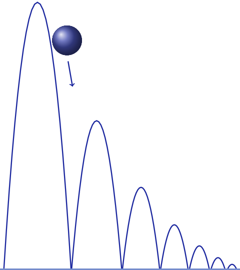
$$x'' = -g \equiv \{x' = v, v' = -g\}$$



Quantum's Back for a Solution Proof w/o Solution Axiom

Solution: How to use a solution without really trying solution axiom [′]

$$\begin{aligned} x &= x_0 + v_0 t - \tfrac{g}{2} t^2 \\ v &= v_0 - gt \end{aligned}$$

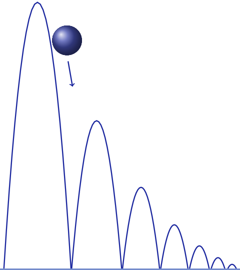
initial velocity v_0 before ODE

$$A \vdash [x'' = -g \,\&\, x \geq 0] B(x,v)$$

$$A \equiv \text{ redacted}$$

$$B(x,v) \equiv \text{ redacted}$$

$$x'' = -g \equiv \{x' = v, v' = -g\}$$



Quantum's Back for a Solution Proof w/o Solution Axiom

Solution: How to use a solution without really trying solution axiom $[']$

$$x = x_0 + v_0 t - \frac{g}{2} t^2$$

$$v = v_0 - gt$$

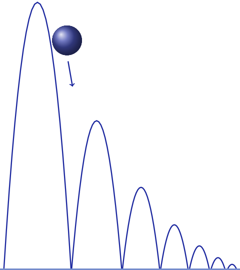
initial velocity v_0 before ODE How?

$$A \vdash [x'' = -g \,\&\, x \geq 0] B(x,v)$$

$$A \equiv \text{ redacted}$$

$$B(x,v) \equiv \text{ redacted}$$

$$x'' = -g \equiv \{x' = v, v' = -g\}$$



Proving Bouncing Balls with Sneaky Solutions

$$\text{iG}\ \frac{}{A \vdash [x'' = -g, t' = 1 \,\&\, x \geq 0]B(x,v)}$$

Proving Bouncing Balls with Sneaky Solutions

$$\dfrac{\dfrac{}{A \vdash [v_0 := v][x''=-g, t'=1 \,\&\, x \geq 0]B(x,v)}\,\text{dC}}{A \vdash [x'' = -g, t' = 1 \,\&\, x \geq 0]B(x,v)}\,\text{iG}$$

Proving Bouncing Balls with Sneaky Solutions

$$\text{dI}\ \dfrac{}{A \vdash [v_0 := v][x''{=}{-g}, t'{=}1 \,\&\, x{\geq}0]\, v = v_0 - tg}$$

$$\text{iG}\ \dfrac{\text{dC}\ \dfrac{\text{iG}\ \dfrac{\triangleleft}{A \vdash [v_0 := v][x'' = -g, t' = 1 \,\&\, x \geq 0 \land v = v_0 - tg]B(x,v)}}{A \vdash [v_0 := v][x''{=}{-g}, t'{=}1 \,\&\, x \geq 0]B(x,v)}}{A \vdash [x'' = -g, t' = 1 \,\&\, x \geq 0]B(x,v)}$$

Proving Bouncing Balls with Sneaky Solutions

$$\text{dI}\ \dfrac{\text{[:=]}\ \dfrac{}{x \geq 0 \vdash [v':=-g][t':=1]v' = -t'g}}{A \vdash [v_0 := v][x''=-g, t'=1 \,\&\, x \geq 0]\, v = v_0 - tg}$$

$$\text{iG}\ \dfrac{\text{dC}\ \dfrac{\text{iG}\ \dfrac{}{\triangleleft \qquad A \vdash [v_0 := v][x'' = -g, t' = 1 \,\&\, x \geq 0 \land v = v_0 - tg]B(x,v)}}{A \vdash [v_0 := v][x''=-g, t'=1 \,\&\, x \geq 0]B(x,v)}}{A \vdash [x'' = -g, t' = 1 \,\&\, x \geq 0]B(x,v)}$$

Proving Bouncing Balls with Sneaky Solutions

$$\text{dI}\ \dfrac{[:=]\ \dfrac{\mathbb{R}\ \dfrac{}{x \ge 0 \vdash -g = -1g}}{x \ge 0 \vdash [v' := -g][t' := 1]v' = -t'g}}{A \vdash [v_0 := v][x'' = -g, t' = 1 \,\&\, x \ge 0]v = v_0 - tg}$$

$$\text{iG}\ \dfrac{\text{dC}\ \dfrac{\text{iG}\ \dfrac{\triangleleft \qquad A \vdash [v_0 := v][x'' = -g, t' = 1 \,\&\, x \ge 0 \wedge v = v_0 - tg]B(x,v)}{A \vdash [v_0 := v][x'' = -g, t' = 1 \,\&\, x \ge 0]B(x,v)}}{}}{A \vdash [x'' = -g, t' = 1 \,\&\, x \ge 0]B(x,v)}$$

Proving Bouncing Balls with Sneaky Solutions

$$\text{dI}\;\dfrac{[:=]\;\dfrac{\mathbb{R}\;\dfrac{*}{x \geq 0 \vdash -g = -1g}}{x \geq 0 \vdash [v' {:=} {-g}][t' {:=} 1]v' = -t'g}}{A \vdash [v_0 := v][x''{=}{-g}, t'{=}1 \,\&\, x{\geq}0]v = v_0 - tg}$$

$$\text{iG}\;\dfrac{\text{dC}\;\dfrac{\text{iG}\;\dfrac{\triangleleft}{A \vdash [v_0 := v][x'' = -g, t' = 1 \,\&\, x \geq 0 \land v = v_0 - tg]B(x,v)}}{A \vdash [v_0 := v][x''{=}{-g}, t'{=}1 \,\&\, x \geq 0]B(x,v)}}{A \vdash [x'' = -g, t' = 1 \,\&\, x \geq 0]B(x,v)}$$

Proving Bouncing Balls with Sneaky Solutions

$$
\begin{array}{ll}
 & \qquad * \\ \cline{2-2}
\mathbb{R} & x \geq 0 \vdash -g = -1g \\ \cline{2-2}
[:=] & x \geq 0 \vdash [v' := -g][t' := 1]v' = -t'g \\ \cline{2-2}
\mathrm{dI} & A \vdash [v_0 := v][x'' = -g, t' = 1 \,\&\, x \geq 0]\, v = v_0 - tg
\end{array}
$$

$$
\begin{array}{ll}
 & \\ \cline{2-2}
\mathrm{dC} & A \vdash [{\color{red}x_0 := x}][v_0 := v][x'' = -g, t' = 1 \,\&\, x \geq 0 \land v = v_0 - tg]B(x,v) \\ \cline{2-2}
\mathrm{iG} \quad \triangleleft & A \vdash [v_0 := v][x'' = -g, t' = 1 \,\&\, x \geq 0 \land v = v_0 - tg]B(x,v) \\ \cline{2-2}
\mathrm{dC} & A \vdash [v_0 := v][x'' = -g, t' = 1 \,\&\, x \geq 0]B(x,v) \\ \cline{2-2}
\mathrm{iG} & A \vdash [x'' = -g, t' = 1 \,\&\, x \geq 0]B(x,v)
\end{array}
$$

Proving Bouncing Balls with Sneaky Solutions

$$A \vdash [x_0 := x][v_0 := v][x'' = -g, t' = 1 \,\&\, x \geq 0 \wedge v = v_0 - tg \wedge x = x_0 + v_0 t - \frac{g}{2} t^2] B(x,v)$$

$$\begin{array}{ll}
\hline
\text{dI} & A \vdash [x_0 := x][v_0 := v][x''=-g, t'=1 \,\&\, x\geq 0 \wedge v=v_0-tg]\, x=x_0+v_0t-\frac{g}{2}t^2 \;\triangleright \\
\hline
\text{dC} & A \vdash [x_0 := x][v_0 := v][x'' = -g, t' = 1 \,\&\, x\geq 0\wedge v=v_0-tg] B(x,v) \\
\hline
\text{iG} \quad \triangleleft & A \vdash [v_0 := v][x'' = -g, t' = 1 \,\&\, x \geq 0 \wedge v = v_0 - tg] B(x,v) \\
\hline
\text{dC} & A \vdash [v_0 := v][x''=-g, t'=1 \,\&\, x \geq 0] B(x,v) \\
\hline
\text{iG} & A \vdash [x'' = -g, t' = 1 \,\&\, x \geq 0] B(x,v)
\end{array}$$

Proving Bouncing Balls with Sneaky Solutions

$$\begin{array}{c}
\hline
A \vdash [x_0 := x][v_0 := v][x'' = -g, t' = 1 \,\&\, x \geq 0 \land v = v_0 - tg \land x = x_0 + v_0 t - \frac{g}{2} t^2] B(x,v)
\end{array}$$

$$\begin{array}{lr}
[:=] & \\
\hline
& x \geq 0 \land v = v_0 - tg \vdash [x' := v][t' := 1] x' = v_0 t' - 2 \frac{g}{2} t t' \\
\text{dI} & \\
\hline
& A \vdash [x_0 := x][v_0 := v][x'' = -g, t' = 1 \,\&\, x \geq 0 \land v = v_0 - tg] x = x_0 + v_0 t - \frac{g}{2} t^2 \triangleright \\
\text{dC} & \\
\hline
& A \vdash [x_0 := x][v_0 := v][x'' = -g, t' = 1 \,\&\, x \geq 0 \land v = v_0 - tg] B(x,v) \\
\text{iG} & \\
\hline
\triangleleft & A \vdash [v_0 := v][x'' = -g, t' = 1 \,\&\, x \geq 0 \land v = v_0 - tg] B(x,v) \\
\text{dC} & \\
\hline
& A \vdash [v_0 := v][x'' = -g, t' = 1 \,\&\, x \geq 0] B(x,v) \\
\text{iG} & \\
\hline
& A \vdash [x'' = -g, t' = 1 \,\&\, x \geq 0] B(x,v)
\end{array}$$

Proving Bouncing Balls with Sneaky Solutions

$$\overline{A \vdash [x_0 := x][v_0 := v][x'' = -g, t' = 1 \,\&\, x \ge 0 \land v = v_0 - tg \land x = x_0 + v_0 t - \tfrac{g}{2} t^2] B(x,v)}$$

$$\begin{array}{ll}
\text{id} & \overline{x \ge 0 \land v = v_0 - tg \vdash v = v_0 - 2\tfrac{g}{2} t} \\
{[:=]} & \overline{x \ge 0 \land v = v_0 - tg \vdash [x' := v][t' := 1] x' = v_0 t' - 2\tfrac{g}{2} t t'} \\
\text{dI} & \overline{A \vdash [x_0 := x][v_0 := v][x'' = -g, t' = 1 \,\&\, x \ge 0 \land v = v_0 - tg] x = x_0 + v_0 t - \tfrac{g}{2} t^2 \triangleright} \\
\text{dC} & \overline{A \vdash [x_0 := x][v_0 := v][x'' = -g, t' = 1 \,\&\, x \ge 0 \land v = v_0 - tg] B(x,v)} \\
\text{iG} & \overline{\triangleleft \quad A \vdash [v_0 := v][x'' = -g, t' = 1 \,\&\, x \ge 0 \land v = v_0 - tg] B(x,v)} \\
\text{dC} & \overline{A \vdash [v_0 := v][x'' = -g, t' = 1 \,\&\, x \ge 0] B(x,v)} \\
\text{iG} & \overline{A \vdash [x'' = -g, t' = 1 \,\&\, x \ge 0] B(x,v)}
\end{array}$$

Proving Bouncing Balls with Sneaky Solutions

$$\overline{A \vdash [x_0 := x][v_0 := v][x'' = -g, t' = 1 \,\&\, x \geq 0 \land v = v_0 - tg \land x = x_0 + v_0 t - \frac{g}{2} t^2]B(x,v)}$$

$$\begin{array}{ll}
 & * \\ \hline
\text{id} & x \geq 0 \land v = v_0 - tg \vdash v = v_0 - 2\frac{g}{2}t \\ \hline
[:=] & x \geq 0 \land v = v_0 - tg \vdash [x' := v][t' := 1]x' = v_0 t' - 2\frac{g}{2}tt' \\ \hline
\text{dI} & A \vdash [x_0 := x][v_0 := v][x'' = -g, t' = 1 \,\&\, x \geq 0 \land v = v_0 - tg]x = x_0 + v_0 t - \frac{g}{2}t^2 \triangleright \\ \hline
\text{dC} & A \vdash [x_0 := x][v_0 := v][x'' = -g, t' = 1 \,\&\, x \geq 0 \land v = v_0 - tg]B(x,v) \\ \hline
\text{iG} & \triangleleft \quad A \vdash [v_0 := v][x'' = -g, t' = 1 \,\&\, x \geq 0 \land v = v_0 - tg]B(x,v) \\ \hline
\text{dC} & A \vdash [v_0 := v][x'' = -g, t' = 1 \,\&\, x \geq 0]B(x,v) \\ \hline
\text{iG} & A \vdash [x'' = -g, t' = 1 \,\&\, x \geq 0]B(x,v)
\end{array}$$

Proving Bouncing Balls with Sneaky Solutions

$$\begin{array}{ll}
 & \overline{x \ge 0 \land v = v_0 - tg \land x = x_0 + v_0 t - \frac{g}{2} t^2 \vdash B(x,v)} \\ \hline
\text{dW} & A \vdash [x_0 := x][v_0 := v][x'' = -g, t' = 1 \,\&\, x \ge 0 \land v = v_0 - tg \land x = x_0 + v_0 t - \frac{g}{2} t^2]B(x,v)
\end{array}$$

$$\begin{array}{ll}
 & \qquad\qquad * \\ \hline
\text{id} & x \ge 0 \land v = v_0 - tg \vdash v = v_0 - 2\frac{g}{2} t \\ \hline
[:=] & x \ge 0 \land v = v_0 - tg \vdash [x' := v][t' := 1]x' = v_0 t' - 2\frac{g}{2} t t' \\ \hline
\text{dI} & A \vdash [x_0 := x][v_0 := v][x'' = -g, t' = 1 \,\&\, x \ge 0 \land v = v_0 - tg]x = x_0 + v_0 t - \frac{g}{2} t^2 \triangleright \\ \hline
\text{dC} & A \vdash [x_0 := x][v_0 := v][x'' = -g, t' = 1 \,\&\, x \ge 0 \land v = v_0 - tg]B(x,v) \\ \hline
\text{iG} & \triangleleft \qquad A \vdash [v_0 := v][x'' = -g, t' = 1 \,\&\, x \ge 0 \land v = v_0 - tg]B(x,v) \\ \hline
\text{dC} & A \vdash [v_0 := v][x'' = -g, t' = 1 \,\&\, x \ge 0]B(x,v) \\ \hline
\text{iG} & A \vdash [x'' = -g, t' = 1 \,\&\, x \ge 0]B(x,v)
\end{array}$$

Proving Bouncing Balls with Sneaky Solutions

$$\begin{array}{rc}
\wedge\mathrm{L} & \\ \hline
 & x \geq 0 \wedge v = v_0 - tg \wedge x = x_0 + v_0 t - \frac{g}{2} t^2 \vdash 2gx = 2gH - v^2 \wedge x \geq 0 \\ \cline{2-2}
 & x \geq 0 \wedge v = v_0 - tg \wedge x = x_0 + v_0 t - \frac{g}{2} t^2 \vdash B(x,v) \\ \hline
\mathrm{dW} & A \vdash [x_0 := x][v_0 := v][x'' = -g, t' = 1 \,\&\, x \geq 0 \wedge v = v_0 - tg \wedge x = x_0 + v_0 t - \frac{g}{2} t^2] B(x,v)
\end{array}$$

$$\begin{array}{rl}
 & \qquad\qquad * \\ \hline
\mathrm{id} & x \geq 0 \wedge v = v_0 - tg \vdash v = v_0 - 2 \frac{g}{2} t \\ \hline
[:=] & x \geq 0 \wedge v = v_0 - tg \vdash [x' := v][t' := 1] x' = v_0 t' - 2 \frac{g}{2} t t' \\ \hline
\mathrm{dI} & \qquad A \vdash [x_0 := x][v_0 := v][x'' = -g, t' = 1 \,\&\, x \geq 0 \wedge v = v_0 - tg] x = x_0 + v_0 t - \frac{g}{2} t^2 \triangleright \\ \hline
\mathrm{dC} & \qquad A \vdash [x_0 := x][v_0 := v][x'' = -g, t' = 1 \,\&\, x \geq 0 \wedge v = v_0 - tg] B(x,v) \\ \hline
\mathrm{iG} & \triangleleft \qquad A \vdash [v_0 := v][x'' = -g, t' = 1 \,\&\, x \geq 0 \wedge v = v_0 - tg] B(x,v) \\ \hline
\mathrm{dC} & \qquad A \vdash [v_0 := v][x'' = -g, t' = 1 \,\&\, x \geq 0] B(x,v) \\ \hline
\mathrm{iG} & \qquad A \vdash [x'' = -g, t' = 1 \,\&\, x \geq 0] B(x,v)
\end{array}$$

$$
\begin{array}{ll}
\hline
\text{=R} & x \geq 0, v = v_0 - tg, x = x_0 + v_0 t - \frac{g}{2} t^2 \vdash 2gx = 2gH - v^2 \land x \geq 0 \\
\hline
\land\text{L} & x \geq 0 \land v = v_0 - tg \land x = x_0 + v_0 t - \frac{g}{2} t^2 \vdash 2gx = 2gH - v^2 \land x \geq 0 \\
\hline
 & x \geq 0 \land v = v_0 - tg \land x = x_0 + v_0 t - \frac{g}{2} t^2 \vdash B(x,v) \\
\hline
\text{dW} & A \vdash [x_0 := x][v_0 := v][x'' = -g, t' = 1 \,\&\, x \geq 0 \land v = v_0 - tg \land x = x_0 + v_0 t - \frac{g}{2} t^2] B(x,v)
\end{array}
$$

$$
\begin{array}{ll}
 & * \\
\hline
\text{id} & x \geq 0 \land v = v_0 - tg \vdash v = v_0 - 2\frac{g}{2} t \\
\hline
[:=] & x \geq 0 \land v = v_0 - tg \vdash [x' := v][t' := 1] x' = v_0 t' - 2\frac{g}{2} t t' \\
\hline
\text{dI} & A \vdash [x_0 := x][v_0 := v][x'' = -g, t' = 1 \,\&\, x \geq 0 \land v = v_0 - tg] x = x_0 + v_0 t - \frac{g}{2} t^2 \rhd \\
\hline
\text{dC} & A \vdash [x_0 := x][v_0 := v][x'' = -g, t' = 1 \,\&\, x \geq 0 \land v = v_0 - tg] B(x,v) \\
\hline
\text{iG} & \lhd \quad A \vdash [v_0 := v][x'' = -g, t' = 1 \,\&\, x \geq 0 \land v = v_0 - tg] B(x,v) \\
\hline
\text{dC} & A \vdash [v_0 := v][x'' = -g, t' = 1 \,\&\, x \geq 0] B(x,v) \\
\hline
\text{iG} & A \vdash [x'' = -g, t' = 1 \,\&\, x \geq 0] B(x,v)
\end{array}
$$

Proving Bouncing Balls with Sneaky Solutions

$$\begin{array}{ll}
\hline
{=}\mathrm{R} & x \geq 0, v = v_0 - tg, x = x_0 + v_0 t - \frac{g}{2} t^2 \vdash 2gx = 2gH - (v_0 - tg)^2 \land x \geq 0 \\
\hline
{=}\mathrm{R} & x \geq 0, v = v_0 - tg, x = x_0 + v_0 t - \frac{g}{2} t^2 \vdash 2gx = 2gH - v^2 \land x \geq 0 \\
\hline
\land\mathrm{L} & x \geq 0 \land v = v_0 - tg \land x = x_0 + v_0 t - \frac{g}{2} t^2 \vdash 2gx = 2gH - v^2 \land x \geq 0 \\
\hline
 & x \geq 0 \land v = v_0 - tg \land x = x_0 + v_0 t - \frac{g}{2} t^2 \vdash B(x,v) \\
\hline
\mathrm{dW} & A \vdash [x_0 := x][v_0 := v][x'' = -g, t' = 1 \,\&\, x \geq 0 \land v = v_0 - tg \land x = x_0 + v_0 t - \frac{g}{2} t^2] B(x,v)
\end{array}$$

$$\begin{array}{ll}
 & * \\
\hline
\mathrm{id} & x \geq 0 \land v = v_0 - tg \vdash v = v_0 - 2 \frac{g}{2} t \\
\hline
[:=] & x \geq 0 \land v = v_0 - tg \vdash [x' := v][t' := 1] x' = v_0 t' - 2 \frac{g}{2} t t' \\
\hline
\mathrm{dI} & A \vdash [x_0 := x][v_0 := v][x'' = -g, t' = 1 \,\&\, x \geq 0 \land v = v_0 - tg] x = x_0 + v_0 t - \frac{g}{2} t^2 \triangleright \\
\hline
\mathrm{dC} & A \vdash [x_0 := x][v_0 := v][x'' = -g, t' = 1 \,\&\, x \geq 0 \land v = v_0 - tg] B(x,v) \\
\hline
\mathrm{iG} & \triangleleft \quad A \vdash [v_0 := v][x'' = -g, t' = 1 \,\&\, x \geq 0 \land v = v_0 - tg] B(x,v) \\
\hline
\mathrm{dC} & A \vdash [v_0 := v][x'' = -g, t' = 1 \,\&\, x \geq 0] B(x,v) \\
\hline
\mathrm{iG} & A \vdash [x'' = -g, t' = 1 \,\&\, x \geq 0] B(x,v)
\end{array}$$

Proving Bouncing Balls with Sneaky Solutions

$$
\begin{array}{ll}
\text{WL} & \hline
x\geq 0, v=v_0-tg, x=x_0+v_0t-\frac{g}{2}t^2 \vdash 2g(x_0+v_0t-\frac{g}{2}t^2)=2gH-(v_0-tg)^2\land x\geq 0 \\
\text{=R} & \hline
x\geq 0, v=v_0-tg, x=x_0+v_0t-\frac{g}{2}t^2 \vdash 2gx=2gH-(v_0-tg)^2\land x\geq 0 \\
\text{=R} & \hline
x\geq 0, v=v_0-tg, x=x_0+v_0t-\frac{g}{2}t^2 \vdash 2gx=2gH-v^2\land x\geq 0 \\
\land\text{L} & \hline
x\geq 0\land v=v_0-tg\land x=x_0+v_0t-\frac{g}{2}t^2 \vdash 2gx=2gH-v^2\land x\geq 0 \\
 & \hline
x\geq 0\land v=v_0-tg\land x=x_0+v_0t-\frac{g}{2}t^2 \vdash B(x,v) \\
\text{dW} & \hline
A\vdash [x_0:=x][v_0:=v][x''=-g,t'=1\,\&\,x\geq 0\land v=v_0-tg\land x=x_0+v_0t-\frac{g}{2}t^2]B(x,v)
\end{array}
$$

$$
\begin{array}{ll}
 & * \\
\text{id} & \hline
x\geq 0\land v=v_0-tg\vdash v=v_0-2\frac{g}{2}t \\
[:=] & \hline
x\geq 0\land v=v_0-tg\vdash [x':=v][t':=1]x'=v_0t'-2\frac{g}{2}tt' \\
\text{dI} & \hline
A\vdash [x_0:=x][v_0:=v][x''=-g,t'=1\,\&\,x\geq 0\land v=v_0-tg]x=x_0+v_0t-\frac{g}{2}t^2 \triangleright \\
\text{dC} & \hline
A\vdash [x_0:=x][v_0:=v][x''=-g,t'=1\,\&\,x\geq 0\land v=v_0-tg]B(x,v) \\
\text{iG} & \hline
\triangleleft \quad A\vdash [v_0:=v][x''=-g,t'=1\,\&\,x\geq 0\land v=v_0-tg]B(x,v) \\
\text{dC} & \hline
A\vdash [v_0:=v][x''=-g,t'=1\,\&\,x\geq 0]B(x,v) \\
\text{iG} & \hline
A\vdash [x''=-g,t'=1\,\&\,x\geq 0]B(x,v)
\end{array}
$$

Proving Bouncing Balls with Sneaky Solutions

$$
\begin{array}{ll}
\wedge\mathrm{R} & \overline{\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad} \\
 & x \ge 0 \vdash 2g(x_0+v_0t-\frac{g}{2}t^2)=2gH-(v_0-tg)^2\wedge x\ge 0 \\
\mathrm{WL} & \overline{x \ge 0, v = v_0-tg, x = x_0+v_0t-\frac{g}{2}t^2 \vdash 2g({\color{blue}x_0+v_0t-\frac{g}{2}t^2})=2gH-(v_0-tg)^2\wedge x\ge 0} \\
=\mathrm{R} & \overline{x \ge 0, v = v_0-tg, {\color{blue}x = x_0+v_0t-\frac{g}{2}t^2} \vdash 2g{\color{blue}x} = 2gH-({\color{red}v_0-tg})^2 \wedge x \ge 0} \\
=\mathrm{R} & \overline{x \ge 0, {\color{red}v = v_0-tg}, x = x_0+v_0t-\frac{g}{2}t^2 \vdash 2gx = 2gH-{\color{red}v}^2 \wedge x \ge 0} \\
\wedge\mathrm{L} & \overline{x \ge 0 \wedge v = v_0-tg \wedge x = x_0+v_0t-\frac{g}{2}t^2 \vdash 2gx = 2gH-v^2 \wedge x \ge 0} \\
 & \overline{x \ge 0 \wedge v = v_0-tg \wedge x = x_0+v_0t-\frac{g}{2}t^2 \vdash B(x,v)} \\
\mathrm{dW} & \overline{A \vdash [x_0 := x][v_0 := v][x'' = -g, t' = 1 \,\&\, x \ge 0 \wedge v = v_0-tg \wedge x = x_0+v_0t-\frac{g}{2}t^2]B(x,v)}
\end{array}
$$

$$
\begin{array}{ll}
 & * \\
\mathrm{id} & \overline{x\ge 0\wedge v=v_0-tg \vdash v = v_0-2\frac{g}{2}t} \\
[:=] & \overline{x\ge 0\wedge v=v_0-tg \vdash [x':=v][t':=1]x' = v_0t'-2\frac{g}{2}tt'} \\
\mathrm{dI} & \overline{A \vdash [x_0:=x][v_0:=v][x''=-g,t'=1\,\&\,x\ge 0\wedge v=v_0-tg]x=x_0+v_0t-\frac{g}{2}t^2 \triangleright} \\
\mathrm{dC} & \overline{A \vdash [x_0:=x][v_0:=v][x''=-g,t'=1\,\&\,x\ge 0\wedge v=v_0-tg]B(x,v)} \\
\mathrm{iG} & \overline{\triangleleft \qquad A \vdash [v_0:=v][x''=-g,t'=1\,\&\,x\ge 0\wedge v=v_0-tg]B(x,v)} \\
\mathrm{dC} & \overline{A \vdash [v_0:=v][x''=-g,t'=1\,\&\,x\ge 0]B(x,v)} \\
\mathrm{iG} & \overline{A \vdash [x''=-g,t'=1\,\&\,x\ge 0]B(x,v)}
\end{array}
$$

Proving Bouncing Balls with Sneaky Solutions

$$
\begin{array}{ll}
 & \\ \hline
\text{WL} & x\geq0 \vdash 2g(x_0+v_0t-\frac{g}{2}t^2)=2gH-(v_0-tg)^2 \qquad \text{id}\ \overline{x\geq 0\vdash x\geq 0} \\ \hline
\land\text{R} & x\geq 0 \vdash 2g(x_0+v_0t-\frac{g}{2}t^2)=2gH-(v_0-tg)^2\land x\geq0 \\ \hline
\text{WL} & x\geq 0, v=v_0-tg, x=x_0+v_0t-\frac{g}{2}t^2 \vdash 2g(x_0+v_0t-\frac{g}{2}t^2)=2gH-(v_0-tg)^2\land x\geq0 \\ \hline
=\text{R} & x\geq 0, v=v_0-tg, x=x_0+v_0t-\frac{g}{2}t^2 \vdash 2gx=2gH-(v_0-tg)^2\land x\geq 0 \\ \hline
=\text{R} & x\geq 0, v=v_0-tg, x=x_0+v_0t-\frac{g}{2}t^2 \vdash 2gx=2gH-v^2\land x\geq 0 \\ \hline
\land\text{L} & x\geq 0\land v=v_0-tg\land x=x_0+v_0t-\frac{g}{2}t^2 \vdash 2gx=2gH-v^2\land x\geq 0 \\ \hline
 & x\geq 0\land v=v_0-tg\land x=x_0+v_0t-\frac{g}{2}t^2 \vdash B(x,v) \\ \hline
\text{dW} & A\vdash [x_0:=x][v_0:=v][x''=-g,t'=1\,\&\,x\geq0\land v=v_0-tg\land x=x_0+v_0t-\frac{g}{2}t^2]B(x,v)
\end{array}
$$

$$
\begin{array}{ll}
 & * \\ \hline
\text{id} & x\geq0\land v=v_0-tg \vdash v=v_0-2\frac{g}{2}t \\ \hline
[:=] & x\geq0\land v=v_0-tg \vdash [x':=v][t':=1]x'=v_0t'-2\frac{g}{2}tt' \\ \hline
\text{dI} & A\vdash [x_0:=x][v_0:=v][x''=-g,t'=1\,\&\,x\geq0\land v=v_0-tg]x=x_0+v_0t-\frac{g}{2}t^2 \triangleright \\ \hline
\text{dC} & A\vdash [x_0:=x][v_0:=v][x''=-g,t'=1\,\&\,x\geq0\land v=v_0-tg]B(x,v) \\ \hline
\text{iG} & \triangleleft \qquad A\vdash [v_0:=v][x''=-g,t'=1\,\&\,x\geq0\land v=v_0-tg]B(x,v) \\ \hline
\text{dC} & A\vdash [v_0:=v][x''=-g,t'=1\,\&\,x\geq 0]B(x,v) \\ \hline
\text{iG} & A\vdash [x''=-g,t'=1\,\&\,x\geq 0]B(x,v)
\end{array}
$$

Proving Bouncing Balls with Sneaky Solutions

$$
\begin{array}{lr}
 & \dfrac{\dfrac{\vdash 2g(x_0+v_0t-\frac{g}{2}t^2)=2gH-(v_0-tg)^2}{x\geq 0 \vdash 2g(x_0+v_0t-\frac{g}{2}t^2)=2gH-(v_0-tg)^2}\text{WL} \qquad \dfrac{}{x \geq 0 \vdash x \geq 0}\text{id}}{x \geq 0 \vdash 2g(x_0+v_0t-\frac{g}{2}t^2)=2gH-(v_0-tg)^2 \land x\geq 0}\land\text{R} \\
\hline
\text{WL} & x \geq 0, v = v_0-tg, x = x_0+v_0t-\frac{g}{2}t^2 \vdash 2g(x_0+v_0t-\frac{g}{2}t^2)=2gH-(v_0-tg)^2 \land x\geq 0 \\
\hline
=\text{R} & x \geq 0, v = v_0-tg, x = x_0+v_0t-\frac{g}{2}t^2 \vdash 2gx = 2gH-(v_0-tg)^2 \land x \geq 0 \\
\hline
=\text{R} & x \geq 0, v = v_0-tg, x = x_0+v_0t-\frac{g}{2}t^2 \vdash 2gx = 2gH - v^2 \land x \geq 0 \\
\hline
\land\text{L} & x \geq 0 \land v = v_0-tg \land x = x_0+v_0t-\frac{g}{2}t^2 \vdash 2gx = 2gH - v^2 \land x \geq 0 \\
\hline
 & x \geq 0 \land v = v_0 - tg \land x = x_0 + v_0t - \frac{g}{2}t^2 \vdash B(x,v) \\
\hline
\text{dW} & A \vdash [x_0 := x][v_0 := v][x'' = -g, t' = 1 \,\&\, x \geq 0 \land v = v_0-tg \land x = x_0+v_0t-\frac{g}{2}t^2]B(x,v)
\end{array}
$$

$$
\begin{array}{ll}
 & \qquad\qquad * \\
\hline
\text{id} & x\geq 0\land v=v_0-tg \vdash v = v_0 - 2\frac{g}{2}t \\
\hline
[:=] & x\geq 0\land v=v_0-tg \vdash [x':=v][t':=1]x' = v_0t' - 2\frac{g}{2}tt' \\
\hline
\text{dI} & \qquad A \vdash [x_0 := x][v_0 := v][x''=-g, t'=1 \,\&\, x\geq 0\land v=v_0-tg]x=x_0+v_0t-\frac{g}{2}t^2 \triangleright \\
\hline
\text{dC} & \qquad A \vdash [x_0 := x][v_0 := v][x'' = -g, t' = 1 \,\&\, x\geq 0\land v=v_0-tg]B(x,v) \\
\hline
\text{iG} & \triangleleft \qquad A \vdash [v_0 := v][x'' = -g, t' = 1 \,\&\, x \geq 0 \land v = v_0 - tg]B(x,v) \\
\hline
\text{dC} & \qquad A \vdash [v_0 := v][x''=-g, t'=1 \,\&\, x \geq 0]B(x,v) \\
\hline
\text{iG} & \qquad A \vdash [x'' = -g, t' = 1 \,\&\, x \geq 0]B(x,v)
\end{array}
$$

Proving Bouncing Balls with Sneaky Solutions

$$\begin{array}{ll}
 & \vdash 2g(x_0+v_0t-\frac{g}{2}t^2)=2gH-v_0^2+2v_0tg-t^2g^2 \\ \hline
 & \vdash 2g(x_0+v_0t-\frac{g}{2}t^2)=2gH-(v_0-tg)^2 \\ \hline
\text{WL} & x\geq0 \vdash 2g(x_0+v_0t-\frac{g}{2}t^2)=2gH-(v_0-tg)^2 \qquad \text{id}\ \overline{x\geq0\vdash x\geq0} \\ \hline
\wedge\text{R} & x\geq0 \vdash 2g(x_0+v_0t-\frac{g}{2}t^2)=2gH-(v_0-tg)^2\wedge x\geq0 \\ \hline
\text{WL} & x\geq0, v=v_0-tg, x=x_0+v_0t-\frac{g}{2}t^2 \vdash 2g(x_0+v_0t-\frac{g}{2}t^2)=2gH-(v_0-tg)^2\wedge x\geq0 \\ \hline
=\text{R} & x\geq0, v=v_0-tg, x=x_0+v_0t-\frac{g}{2}t^2 \vdash 2gx=2gH-(v_0-tg)^2\wedge x\geq0 \\ \hline
=\text{R} & x\geq0, v=v_0-tg, x=x_0+v_0t-\frac{g}{2}t^2 \vdash 2gx=2gH-v^2\wedge x\geq0 \\ \hline
\wedge\text{L} & x\geq0\wedge v=v_0-tg\wedge x=x_0+v_0t-\frac{g}{2}t^2 \vdash 2gx=2gH-v^2\wedge x\geq0 \\ \hline
 & x\geq0\wedge v=v_0-tg\wedge x=x_0+v_0t-\frac{g}{2}t^2 \vdash B(x,v) \\ \hline
\text{dW} & A\vdash [x_0:=x][v_0:=v][x''=-g,t'=1\,\&\,x\geq0\wedge v=v_0-tg\wedge x=x_0+v_0t-\frac{g}{2}t^2]B(x,v)
\end{array}$$

$$\begin{array}{ll}
 & * \\ \hline
\text{id} & x\geq0\wedge v=v_0-tg\vdash v=v_0-2\frac{g}{2}t \\ \hline
[:=] & x\geq0\wedge v=v_0-tg\vdash [x':=v][t':=1]x'=v_0t'-2\frac{g}{2}tt' \\ \hline
\text{dI} & A\vdash [x_0:=x][v_0:=v][x''=-g,t'=1\,\&\,x\geq0\wedge v=v_0-tg]x=x_0+v_0t-\frac{g}{2}t^2 \rhd \\ \hline
\text{dC} & A\vdash [x_0:=x][v_0:=v][x''=-g,t'=1\,\&\,x\geq0\wedge v=v_0-tg]B(x,v) \\ \hline
\text{iG} & \lhd \quad A\vdash [v_0:=v][x''=-g,t'=1\,\&\,x\geq0\wedge v=v_0-tg]B(x,v) \\ \hline
\text{dC} & A\vdash [v_0:=v][x''=-g,t'=1\,\&\,x\geq0]B(x,v) \\ \hline
\text{iG} & A\vdash [x''=-g,t'=1\,\&\,x\geq0]B(x,v)
\end{array}$$

Proving Bouncing Balls with Sneaky Solutions

$$
\begin{array}{ll}
 & \vdash 2g(x_0+\cancel{v_0t}-\cancel{\tfrac{g}{2}t^2})=2gH-v_0^2+\cancel{2v_0tg}-\cancel{t^2g^2} \\ \cline{2-2}
 & \vdash 2g(x_0+v_0t-\tfrac{g}{2}t^2)=2gH-(v_0-tg)^2 \qquad\qquad\qquad * \\ \cline{2-2}
\text{WL} & x\geq 0 \vdash 2g(x_0+v_0t-\tfrac{g}{2}t^2)=2gH-(v_0-tg)^2 \qquad \text{id}\ x\geq 0\vdash x\geq 0 \\ \cline{2-2}
\wedge\text{R} & x\geq 0\vdash 2g(x_0+v_0t-\tfrac{g}{2}t^2)=2gH-(v_0-tg)^2\wedge x\geq 0 \\ \cline{2-2}
\text{WL} & x\geq 0, v=v_0-tg, x=x_0+v_0t-\tfrac{g}{2}t^2 \vdash 2g(x_0+v_0t-\tfrac{g}{2}t^2)=2gH-(v_0-tg)^2\wedge x\geq 0 \\ \cline{2-2}
=\text{R} & x\geq 0, v=v_0-tg, x=x_0+v_0t-\tfrac{g}{2}t^2 \vdash 2gx=2gH-(v_0-tg)^2\wedge x\geq 0 \\ \cline{2-2}
=\text{R} & x\geq 0, v=v_0-tg, x=x_0+v_0t-\tfrac{g}{2}t^2 \vdash 2gx=2gH-v^2\wedge x\geq 0 \\ \cline{2-2}
\wedge\text{L} & x\geq 0\wedge v=v_0-tg\wedge x=x_0+v_0t-\tfrac{g}{2}t^2 \vdash 2gx=2gH-v^2\wedge x\geq 0 \\ \cline{2-2}
 & x\geq 0\wedge v=v_0-tg\wedge x=x_0+v_0t-\tfrac{g}{2}t^2 \vdash B(x,v) \\ \cline{2-2}
\text{dW} & A\vdash [x_0:=x][v_0:=v][x''=-g,t'=1\,\&\,x\geq 0\wedge v=v_0-tg\wedge x=x_0+v_0t-\tfrac{g}{2}t^2]B(x,v)
\end{array}
$$

$$
\begin{array}{ll}
 & * \\ \cline{2-2}
\text{id} & x\geq 0\wedge v=v_0-tg\vdash v=v_0-2\tfrac{g}{2}t \\ \cline{2-2}
[:=] & x\geq 0\wedge v=v_0-tg\vdash [x':=v][t':=1]x'=v_0t'-2\tfrac{g}{2}tt' \\ \cline{2-2}
\text{dI} & A\vdash [x_0:=x][v_0:=v][x''=-g,t'=1\,\&\,x\geq 0\wedge v=v_0-tg]x=x_0+v_0t-\tfrac{g}{2}t^2 \rhd \\ \cline{2-2}
\text{dC} & A\vdash [x_0:=x][v_0:=v][x''=-g,t'=1\,\&\,x\geq 0\wedge v=v_0-tg]B(x,v) \\ \cline{2-2}
\text{iG} & \lhd \qquad A\vdash [v_0:=v][x''=-g,t'=1\,\&\,x\geq 0\wedge v=v_0-tg]B(x,v) \\ \cline{2-2}
\text{dC} & A\vdash [v_0:=v][x''=-g,t'=1\,\&\,x\geq 0]B(x,v) \\ \cline{2-2}
\text{iG} & A\vdash [x''=-g,t'=1\,\&\,x\geq 0]B(x,v)
\end{array}
$$

Proving Bouncing Balls with Sneaky Solutions

$$\vdash 2g(x_0+\cancel{v_0}t-\cancel{\tfrac{g}{2}t^2})=2gH-v_0^2+\cancel{2v_0tg}-\cancel{t^2g^2}$$

$$\vdash 2g(x_0+v_0t-\tfrac{g}{2}t^2)=2gH-(v_0-tg)^2$$

$$\text{WL}\quad x\geq 0\vdash 2g(x_0+v_0t-\tfrac{g}{2}t^2)=2gH-(v_0-tg)^2 \qquad\qquad \text{id}\ \frac{*}{x\geq 0\vdash x\geq 0}$$

$$\wedge\text{R}\quad x\geq 0\vdash 2g(x_0+v_0t-\tfrac{g}{2}t^2)=2gH-(v_0-tg)^2\land x\geq 0$$

$$\text{WL}\quad x\geq 0, v=v_0-tg, x=x_0+v_0t-\tfrac{g}{2}t^2\vdash 2g(x_0+v_0t-\tfrac{g}{2}t^2)=2gH-(v_0-tg)^2\land x\geq 0$$

$$\text{=R}\quad x\geq 0, v=v_0-tg, x=x_0+v_0t-\tfrac{g}{2}t^2\vdash 2gx=2gH-(v_0-tg)^2\land x\geq 0$$

$$\text{=R}\quad x\geq 0, v=v_0-tg, x=x_0+v_0t-\tfrac{g}{2}t^2\vdash 2gx=2gH-v^2\land x\geq 0$$

$$\wedge\text{L}\quad x\geq 0\land v=v_0-tg\land x=x_0+v_0t-\tfrac{g}{2}t^2\vdash 2gx=2gH-v^2\land x\geq 0$$

$$x\geq 0\land v=v_0-tg\land x=x_0+v_0t-\tfrac{g}{2}t^2\vdash B(x,v)$$

$$\text{dW}\quad A\vdash[x_0:=x][v_0:=v][x''=-g,t'=1\,\&\,x\geq 0\land v=v_0-tg\land x=x_0+v_0t-\tfrac{g}{2}t^2]B(x,v)$$

$$*$$

$$\text{id}\quad x\geq 0\land v=v_0-tg\vdash v=v_0-2\tfrac{g}{2}t$$

$$[:=]\quad x\geq 0\land v=v_0-tg\vdash[x':=v][t':=1]x'=v_0t'-2\tfrac{g}{2}tt'$$

Ghost solutions

$$\text{dI}\quad A\vdash[x_0:=x][v_0:=v][x''=-g,t'=1\,\&\,x\geq 0\land v=v_0-tg]x=x_0+v_0t-\tfrac{g}{2}t^2\ \triangleright$$

$$\text{dC}\quad A\vdash[x_0:=x][v_0:=v][x''=-g,t'=1\,\&\,x\geq 0\land v=v_0-tg]B(x,v)$$

Initial ghost

$$\text{iG}\quad\triangleleft\quad \vdash[v_0:=v][x''=-g,t'=1\,\&\,x\geq 0\land v=v_0-tg]B(x,v)$$

$$\text{dC}\quad A\vdash[v_0:=v][x''=-g,t'=1\,\&\,x\geq 0]B(x,v)$$

$$\text{iG}\quad A\vdash[x''=-g,t'=1\,\&\,x\geq 0]B(x,v)$$

Proving Bouncing Balls with Sneaky Solutions

$$\vdash 2g(x_0+\cancel{v_0t}-\cancel{\tfrac{g}{2}t^2})=2gH-v_0^2+\cancel{2v_0tg}-\cancel{t^2g^2}$$

$$\vdash 2g(x_0+v_0t-\tfrac{g}{2}t^2)=2gH-(v_0-tg)^2$$

WL $$x\geq0\vdash 2g(x_0+v_0t-\tfrac{g}{2}t^2)=2gH-(v_0-tg)^2 \qquad \text{id}\ \frac{*}{x\geq0\vdash x\geq0}$$

∧R $$x\geq0\vdash 2g(x_0+v_0t-\tfrac{g}{2}t^2)=2gH-(v_0-tg)^2\land x\geq0$$

WL $$x\geq0, v=v_0-tg, x=x_0+v_0t-\tfrac{g}{2}t^2\vdash 2g(x_0+v_0t-\tfrac{g}{2}t^2)=2gH-(v_0-tg)^2\land x\geq0$$

=R $$x\geq0, v=v_0-tg, x=x_0+v_0t-\tfrac{g}{2}t^2\vdash 2gx=2gH-(v_0-tg)^2\land x\geq0$$

=R $$x\geq0, v=v_0-tg, x=x_0+v_0t-\tfrac{g}{2}t^2\vdash 2gx=2gH-v^2\land x\geq0$$

∧L $$x\geq0\land v=v_0-tg\land x=x_0+v_0t-\tfrac{g}{2}t^2\vdash 2gx=2gH-v^2\land x\geq0$$

$$x\geq0\land v=v_0-tg\land x=x_0+v_0t-\tfrac{g}{2}t^2\vdash B(x,v)$$

dW $$A\vdash[x_0:=x][v_0:=v][x''=-g,t'=1\,\&\,x\geq0\land v=v_0-tg\land x=x_0+v_0t-\tfrac{g}{2}t^2]B(x,v)$$

$$*$$

id $$x\geq0\land v=v_0-tg\vdash v=v_0-2\tfrac{g}{2}t$$

[:=] $$x\geq0\land v=v_0-tg\vdash[x':=v][t':=1]x'=v_0t'-2\tfrac{g}{2}tt'$$

Not initially true

dI $$A\vdash[x_0:=x][v_0:=v][x''=-g,t'=1\,\&\,x\geq0\land v=v_0-tg]x=x_0+v_0t-\tfrac{g}{2}t^2 \ \triangleright$$

dC $$A\vdash[x_0:=x][v_0:=v][x''=-g,t'=1\,\&\,x\geq0\land v=v_0-tg]B(x,v)$$

iG $$\triangleleft\qquad A\vdash[v_0:=v][x''=-g,t'=1\,\&\,x\geq0\land v=v_0-tg]B(x,v)$$

dC $$A\vdash[v_0:=v][x''=-g,t'=1\,\&\,x\geq0]B(x,v)$$

iG $$A\vdash[x''=-g,t'=1\,\&\,x\geq0]B(x,v)$$

Proving Bouncing Balls with Sneaky Solutions

$$
\begin{array}{ll}
 & \vdash 2g(x_0+\cancel{v_0t}-\cancel{\tfrac{g}{2}t^2})=2gH-v_0^2+\cancel{2v_0tg}-\cancel{t^2g^2} \\ \hline
 & \vdash 2g(x_0+v_0t-\tfrac{g}{2}t^2)=2gH-(v_0-tg)^2 \\ \hline
\mathrm{WL} & x\geq 0 \vdash 2g(x_0+v_0t-\tfrac{g}{2}t^2)=2gH-(v_0-tg)^2 \qquad \quad {*} \\
 & \phantom{x\geq 0 \vdash 2g(x_0+v_0t-\tfrac{g}{2}t^2)=2gH-(v_0-tg)^2} \qquad \overline{x\geq 0 \vdash x\geq 0}\ \mathrm{id} \\ \hline
\wedge\mathrm{R} & x\geq 0 \vdash 2g(x_0+v_0t-\tfrac{g}{2}t^2)=2gH-(v_0-tg)^2\wedge x\geq 0 \\ \hline
\mathrm{WL} & x\geq 0, v=v_0-tg, x=x_0+v_0t-\tfrac{g}{2}t^2 \vdash 2g(x_0+v_0t-\tfrac{g}{2}t^2)=2gH-(v_0-tg)^2\wedge x\geq 0 \\ \hline
{=}\mathrm{R} & x\geq 0, v=v_0-tg, x=x_0+v_0t-\tfrac{g}{2}t^2 \vdash 2gx=2gH-(v_0-tg)^2\wedge x\geq 0 \\ \hline
{=}\mathrm{R} & x\geq 0, v=v_0-tg, x=x_0+v_0t-\tfrac{g}{2}t^2 \vdash 2gx=2gH-v^2\wedge x\geq 0 \\ \hline
\wedge\mathrm{L} & x\geq 0\wedge v=v_0-tg\wedge x=x_0+v_0t-\tfrac{g}{2}t^2 \vdash 2gx=2gH-v^2\wedge x\geq 0 \\ \hline
 & x\geq 0\wedge v=v_0-tg\wedge x=x_0+v_0t-\tfrac{g}{2}t^2 \vdash B(x,v) \\ \hline
\mathrm{dW} & A \vdash [x_0:=x][v_0:=v][x''=-g,t'=1\,\&\,x\geq 0\wedge v=v_0-tg\wedge x=x_0+v_0t-\tfrac{g}{2}t^2]B(x,v)
\end{array}
$$

$$
\begin{array}{ll}
 & {*} \\ \hline
\mathrm{id} & x\geq 0\wedge v=v_0-tg \vdash v=v_0-2\tfrac{g}{2}t \\ \hline
[:=] & x\geq 0\wedge v=v_0-tg \vdash [x':=v][t':=1]x'=v_0t'-2\tfrac{g}{2}tt' \\ \hline
\mathrm{dI} & A \vdash [x_0:=x][v_0:=v][x''=-g,t'=1\,\&\,x\geq 0\wedge v=v_0-tg]x=x_0+v_0t-\tfrac{g}{2}t^2 \ \triangleright \\ \hline
\mathrm{dC} & A \vdash [x_0:=x][v_0:=v][x''=-g,t'=1\,\&\,x\geq 0\wedge v=v_0-tg]B(x,v) \\ \hline
\mathrm{iG} & \triangleleft \quad A \vdash [v_0:=v][x''=-g,t'=1\,\&\,x\geq 0\wedge v=v_0-tg]B(x,v) \\ \hline
\mathrm{dC} & A \vdash [v_0:=v][x''=-g,t'=1\,\&\,x\geq 0]B(x,v) \\ \hline
\mathrm{iG} & A \vdash [x''=-g,t'=1\,\&\,x\geq 0]B(x,v)
\end{array}
$$

$v=v_0-(t-t_0)g$

Proving Bouncing Balls with Sneaky Solutions

$$
\begin{array}{ll}
 & \vdash 2g(x_0+\cancel{v_0t}-\cancel{\tfrac{g}{2}t^2})=2gH-v_0^2+\cancel{2v_0tg}-\cancel{t^2g^2} \\ \hline
 & \vdash 2g(x_0+v_0t-\tfrac{g}{2}t^2)=2gH-(v_0-tg)^2 \\ \hline
\text{WL} & x\geq0 \vdash 2g(x_0+v_0t-\tfrac{g}{2}t^2)=2gH-(v_0-tg)^2 \qquad \text{id}\,\dfrac{*}{x\geq0\vdash x\geq0} \\ \hline
\land\text{R} & x\geq0 \vdash 2g(x_0+v_0t-\tfrac{g}{2}t^2)=2gH-(v_0-tg)^2\land x\geq0 \\ \hline
\text{WL} & x\geq0, v=v_0-tg, x=x_0+v_0t-\tfrac{g}{2}t^2 \vdash 2g(x_0+v_0t-\tfrac{g}{2}t^2)=2gH-(v_0-tg)^2\land x\geq0 \\ \hline
=\text{R} & x\geq0, v=v_0-tg, x=x_0+v_0t-\tfrac{g}{2}t^2 \vdash 2gx=2gH-(v_0-tg)^2\land x\geq0 \\ \hline
=\text{R} & x\geq0, v=v_0-tg, x=x_0+v_0t-\tfrac{g}{2}t^2 \vdash 2gx=2gH-v^2\land x\geq0 \\ \hline
\land\text{L} & x\geq0\land v=v_0-tg\land x=x_0+v_0t-\tfrac{g}{2}t^2 \vdash 2gx=2gH-v^2\land x\geq0 \\ \hline
 & x\geq0\land v=v_0-tg\land x=x_0+v_0t-\tfrac{g}{2}t^2 \vdash B(x,v) \\ \hline
\text{dW} & A\vdash [x_0:=x][v_0:=v][x''=-g,t'=1\,\&\,x\geq0\land v=v_0-tg\land x=x_0+v_0t-\tfrac{g}{2}t^2]B(x,v)
\end{array}
$$

$$
\begin{array}{lll}
 & & * \\ \hline
\text{id} & & x\geq0\land v=v_0-tg\vdash v=v_0-2\tfrac{g}{2}t \\ \hline
[:=] & & x\geq0\land v=v_0-tg\vdash [x':=v][t':=1]x'=v_0t'-2\tfrac{g}{2}tt' \\ \hline
\text{dI} & & A\vdash [x_0:=x][v_0:=v][x''=-g,t'=1\,\&\,x\geq0\land v=v_0-tg]x=x_0+v_0t-\tfrac{g}{2}t^2 \triangleright \\ \hline
\text{dC} & & A\vdash [x_0:=x][v_0:=v][x''=-g,t'=1\,\&\,x\geq0\land v=v_0-tg]B(x,v) \\ \hline
\text{iG} & \triangleleft & A\vdash [v_0:=v][x''=-g,t'=1\,\&\,x\geq0\land v=v_0-tg]B(x,v) \\ \hline
\text{dC} & & A\vdash [v_0:=v][x''=-g,t'=1\,\&\,x\geq0]B(x,v) \\ \hline
\text{iG} & & A\vdash [x''=-g,t'=1\,\&\,x\geq0]B(x,v)
\end{array}
$$

ghost $[t_0 := t]$ by iG

Proving Bouncing Balls with Sneaky Solutions

$$\vdash 2g(x_0+\cancel{v_0t}-\cancel{\tfrac{g}{2}t^2})=2gH-v_0^2+\cancel{2v_0tg}-\cancel{t^2g^2}$$

$$\vdash 2g(x_0+v_0t-\tfrac{g}{2}t^2)=2gH-(v_0-tg)^2$$

$$\text{WL}\quad x\geq 0\vdash 2g(x_0+v_0t-\tfrac{g}{2}t^2)=2gH-(v_0-tg)^2 \qquad \text{id}\ \dfrac{*}{x\geq 0\vdash x\geq 0}$$

$$\text{∧R}\quad x\geq 0\vdash 2g(x_0+v_0t-\tfrac{g}{2}t^2)=2gH-(v_0-tg)^2\land x\geq 0$$

$$\text{WL}\quad x\geq 0, v=v_0-tg, x=x_0+v_0t-\tfrac{g}{2}t^2\vdash 2g(x_0+v_0t-\tfrac{g}{2}t^2)=2gH-(v_0-tg)^2\land x\geq 0$$

$$\text{=R}\quad x\geq 0, v=v_0-tg, x=x_0+v_0t-\tfrac{g}{2}t^2\vdash 2gx=2gH-(v_0-tg)^2\land x\geq 0$$

$$\text{=R}\quad x\geq 0, v=v_0-tg, x=x_0+v_0t-\tfrac{g}{2}t^2\vdash 2gx=2gH-v^2\land x\geq 0$$

$$\text{∧L}\quad x\geq 0\land v=v_0-tg\land x=x_0+v_0t-\tfrac{g}{2}t^2\vdash 2gx=2gH-v^2\land x\geq 0$$

$$x\geq 0\land v=v_0-tg\land x=x_0+v_0t-\tfrac{g}{2}t^2\vdash B(x,v)$$

$$\text{dW}\quad A\vdash [x_0:=x][v_0:=v][x''=-g,t'=1\ \&\ x\geq 0\land v=v_0-tg\land x=x_0+v_0t-\tfrac{g}{2}t^2]B(x,v)$$

$$*$$

$$\text{id}\quad x\geq 0\land v=v_0-tg\vdash v=v_0-2\tfrac{g}{2}t$$

$$[:=]\quad x\geq 0\land v= \quad =1]x'=v_0t'-2\tfrac{g}{2}tt'$$

What about time?

$$\text{dI}\quad A\vdash [x_0:=x][v_0:=v][x''=-g,t'=1\ \&\ x\geq 0\land v=v_0-tg]x=x_0+v_0t-\tfrac{g}{2}t^2 \rhd$$

$$\text{dC}\quad A\vdash [x_0:=x][v_0:=v][x''=-g,t'=1\ \&\ x\geq 0\land v=v_0-tg]B(x,v)$$

$$\text{iG}\quad \lhd\quad A\vdash [v_0:=v][x''=-g,t'=1\ \&\ x\geq 0\land v=v_0-tg]B(x,v)$$

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On the Significance of Time

Why does the proof with ghost solutions need $t' = 1$ in the model?

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Could we just add in $t' = 1$ if we need it?

Differential Ghosts of Time

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$$\frac{\Gamma \vdash [x' = f(x), t' = 1 \,\&\, Q]P, \Delta}{\Gamma \vdash [x' = f(x) \,\&\, Q]P, \Delta}$$

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Differential Ghosts of Time

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$$\frac{\Gamma \vdash [x' = f(x), t' = 1 \,\&\, Q]P, \Delta}{\Gamma \vdash [x' = f(x) \,\&\, Q]P, \Delta} \quad (t \text{ fresh})$$

What could possibly go wrong?

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This is a perfectly harmless proof rule with fresh t.

Differential Ghosts of Time

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This is a perfectly harmless proof rule with fresh t.
But it's too specific and cannot add any other ODEs.

Constructing Differential Ghosts

Differential Ghost

$$[x' = f(x) \,\&\, Q]P \leftrightarrow \exists y\, [x' = f(x), y' = g(x,y) \,\&\, Q]P$$

Differential Ghosts of Time

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Differential Ghost $[x' = f(x) \,\&\, Q]P \leftrightarrow \exists y\, [x' = f(x), y' = g(x,y) \,\&\, Q]P$

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Get differential ghosts of time by axiom DG, even with clever initial $t = 0$:

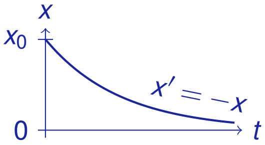
$$\text{DG}\,\dfrac{\exists\text{R}\,\dfrac{\Gamma, t = 0 \vdash [x' = f(x), t' = 1 \,\&\, Q]P, \Delta}{\Gamma \vdash \exists t\, [x' = f(x), t' = 1 \,\&\, Q]P, \Delta}}{\Gamma \vdash [x' = f(x) \,\&\, Q]P, \Delta}$$

Differential Ghost $[x' = f(x) \,\&\, Q]P \leftrightarrow \exists y\, [x' = f(x), y' = g(x, y) \,\&\, Q]P$

Exponentials Make Matters Worse

Example ()

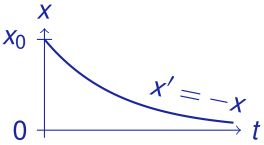
$$\text{dI}\ \frac{}{x > 0 \vdash [x' = -x]x > 0}$$



Exponentials Make Matters Worse

Example ()

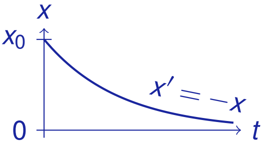
$$\dfrac{\overline{\vdash [x':=-x]x' > 0}^{\,[:=]}}{x > 0 \vdash [x' = -x]x > 0}\,\mathrm{dI}$$



Exponentials Make Matters Worse

Example ()

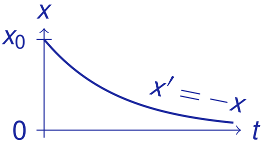
$$\dfrac{\dfrac{\dfrac{}{\vdash {\color{red}-x} > 0}\ \mathbb{R}}{\vdash [x' := {\color{red}-x}]x' > 0}\ [:=]}{x > 0 \vdash [x' = -x]x > 0}\ \text{dI}$$



Exponentials Make Matters Worse

Example (Cannot prove like this)

$$\text{dI}\;\dfrac{[:=]\;\dfrac{\mathbb{R}\;\dfrac{\text{not valid}}{\vdash -x > 0}}{\vdash [x':=-x]x' > 0}}{x > 0 \vdash [x' = -x]x > 0}$$

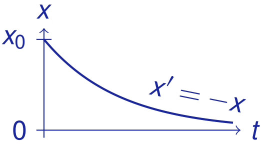


Exponentials Make Matters Worse

Example (Cannot prove like this)

$$\dfrac{\dfrac{\dfrac{\text{not valid}}{\vdash -x > 0}\ \mathbb{R}}{\vdash [x' := -x] x' > 0}\ [:=]}{x > 0 \vdash [x' = -x] x > 0}\ \text{dI}$$

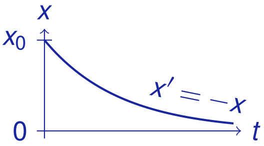
Matters get worse over time in this dynamics



Differential Ghosts: Proofs in Extra Dimensions

Example (▸ Spooky proof)

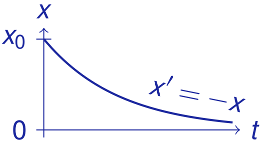
$$\text{DG}\ \frac{}{x > 0 \vdash [x' = -x]x > 0}$$



Differential Ghosts: Proofs in Extra Dimensions

Example (▸ Spooky proof)

$$\text{DG}\ \dfrac{\exists\text{R,cut}\ \dfrac{}{x>0 \vdash \exists y\,[x'=-x, y'=\frac{y}{2}]x>0}}{x>0 \vdash [x'=-x]x>0}$$

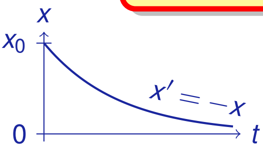


Differential Ghosts: Proofs in Extra Dimensions

Example (▶ Spooky proof)

$$\text{DG}\;\dfrac{\exists\text{R,cut}\;\dfrac{\text{MR}\;\dfrac{}{xy^2 = 1 \vdash [x' = -x, y' = \frac{y}{2}]x > 0}}{x > 0 \vdash \exists y\, [x' = -x, y' = \frac{y}{2}]x > 0}}{x > 0 \vdash [x' = -x]x > 0}$$

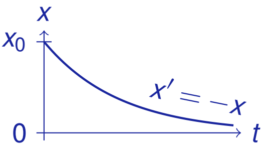
differential ghost: dream me up



Differential Ghosts: Proofs in Extra Dimensions

Example (▸ Spooky proof)

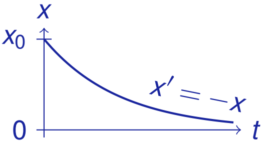
$$\mathrm{DG}\ \dfrac{\exists\mathrm{R,cut}\ \dfrac{\mathrm{MR}\ \dfrac{\mathbb{R}\ \overline{xy^2{=}1 \vdash x{>}0} \qquad \mathrm{dI}\ \overline{xy^2{=}1 \vdash [x' = -x, y' = \frac{y}{2}]xy^2 = 1}}{xy^2 = 1 \vdash [x' = -x, y' = \frac{y}{2}]x > 0}}{x > 0 \vdash \exists y\,[x' = -x, y' = \frac{y}{2}]x > 0}}{x > 0 \vdash [x' = -x]x > 0}$$



Differential Ghosts: Proofs in Extra Dimensions

Example (▶ Spooky proof)

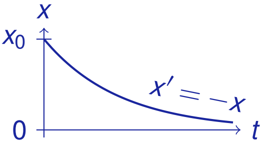
$$\text{DG}\;\dfrac{\exists\text{R,cut}\;\dfrac{\text{MR}\;\dfrac{\mathbb{R}\,\dfrac{*}{xy^2{=}1 \vdash x{>}0} \qquad \text{dI}\,\dfrac{}{xy^2{=}1 \vdash [x' = -x, y' = \frac{y}{2}]xy^2 = 1}}{xy^2 = 1 \vdash [x' = -x, y' = \frac{y}{2}]x > 0}}{x > 0 \vdash \exists y\,[x' = -x, y' = \frac{y}{2}]x > 0}}{x > 0 \vdash [x' = -x]x > 0}$$



Differential Ghosts: Proofs in Extra Dimensions

Example (▶ Spooky proof)

$$
\text{DG}\;\dfrac{\exists\text{R,cut}\;\dfrac{\text{MR}\;\dfrac{\mathbb{R}\,\dfrac{*}{xy^2=1 \vdash x>0} \qquad \text{dI}\;\dfrac{[:=]\;\dfrac{}{\vdash [x':=-x][y':=\frac{y}{2}]x'y^2+x2yy'=0}}{xy^2=1 \vdash [x'=-x, y'=\frac{y}{2}]xy^2=1}}{xy^2=1 \vdash [x'=-x, y'=\frac{y}{2}]x>0}}{x>0 \vdash \exists y\,[x'=-x, y'=\frac{y}{2}]x>0}}{x>0 \vdash [x'=-x]x>0}
$$



Differential Ghosts: Proofs in Extra Dimensions

Example (▶ Spooky proof)

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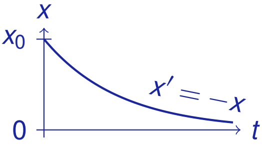
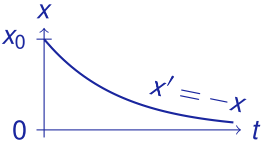


Differential Ghosts: Proofs in Extra Dimensions

Example (▶ Spooky proof)

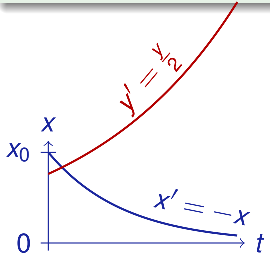
$$\text{DG}\;\dfrac{\exists\text{R,cut}\;\dfrac{\text{MR}\;\dfrac{\mathbb{R}\;\dfrac{*}{xy^2{=}1 \vdash x{>}0} \qquad \text{dI}\;\dfrac{[:=]\;\dfrac{\mathbb{R}\;\dfrac{*}{\vdash -xy^2+2xy\frac{y}{2}=0}}{\vdash [x'{:=}{-x}][y'{:=}\frac{y}{2}]x'y^2{+}x2yy'{=}0}}{xy^2{=}1 \vdash [x'=-x,y'=\frac{y}{2}]xy^2=1}}{xy^2=1 \vdash [x'=-x,y'=\frac{y}{2}]x>0}}{x>0 \vdash \exists y\,[x'=-x,y'=\frac{y}{2}]x>0}}{x>0 \vdash [x'=-x]x>0}$$



Differential Ghosts: Proofs in Extra Dimensions

Example (▶ Spooky proof with counterweight ghost)

$$\text{DG}\,\dfrac{\exists\text{R,cut}\,\dfrac{\text{MR}\,\dfrac{\mathbb{R}\,\dfrac{*}{xy^2{=}1 \vdash x{>}0} \qquad \text{dI}\,\dfrac{[:=]\,\dfrac{\mathbb{R}\,\dfrac{*}{\vdash -xy^2+2xy\frac{y}{2}=0}}{\vdash [x'{:=}{-}x][y'{:=}\frac{y}{2}]x'y^2{+}x2yy'{=}0}}{xy^2{=}1 \vdash [x'=-x,y'=\frac{y}{2}]xy^2=1}}{xy^2=1 \vdash [x'=-x,y'=\frac{y}{2}]x>0}}{x>0 \vdash \exists y\,[x'=-x,y'=\frac{y}{2}]x>0}}{x>0 \vdash [x'=-x]x>0}$$



Differential Ghosts: Proofs in Extra Dimensions

Example (Spooky proof with counterweight ghost)

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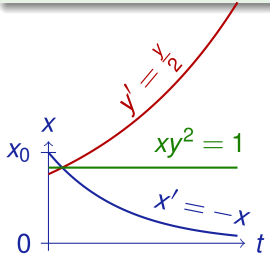

Differential Ghosts: Proofs in Extra Dimensions

Example (▶ Spooky proof with counterweight ghost)

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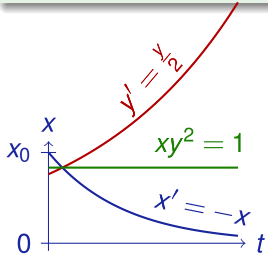
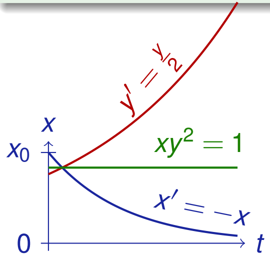


Creative proofs with differential ghosts prove what we otherwise couldn't!

Differential Ghosts: Proofs in Extra Dimensions

Example (▶ Spooky proof with counterweight ghost)

$$\text{DG}\,\dfrac{\exists\text{R,cut}\,\dfrac{\text{MR}\,\dfrac{\mathbb{R}\,\dfrac{*}{xy^2{=}1 \vdash x{>}0} \qquad \text{dI}\,\dfrac{[:=]\,\dfrac{\mathbb{R}\,\dfrac{*}{\vdash -xy^2+2xy\frac{y}{2}=0}}{\vdash [x'{:=}{-x}][y'{:=}\frac{y}{2}]x'y^2{+}x2yy'{=}0}}{xy^2{=}1 \vdash [x'=-x,y'=\frac{y}{2}]xy^2=1}}{xy^2=1 \vdash [x'=-x,y'=\frac{y}{2}]x>0}}{x>0 \vdash \exists y\,[x'=-x,y'=\frac{y}{2}]x>0}}{x>0 \vdash [x'=-x]x>0}$$



Creative proofs with differential ghosts prove what we otherwise couldn't!

Wait, are differential ghosts actually sound?

Outline

1. Learning Objectives
2. Recap: Proofs for Differential Equations
3. A Gradual Introduction to Ghost Variables
   - Discrete Ghosts
   - Proving Bouncing Balls with Sneaky Solutions
   - Differential Ghosts of Time
   - Constructing Differential Ghosts
4. Differential Ghosts
   - Substitute Ghosts
   - Solvable Ghosts
   - Limit Velocity of an Aerodynamic Ball
5. Summary

Constructing Differential Ghosts

Differential Ghost

$$[x' = f(x) \,\&\, Q]P \leftrightarrow \exists y\, [x' = f(x), y' = g(x,y) \,\&\, Q]P$$

Constructing Differential Ghosts

Differential Ghost

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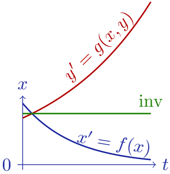

Constructing Differential Ghosts

Differential Ghost

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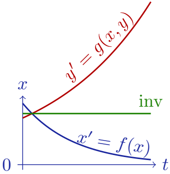
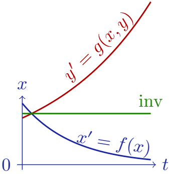


if new $y' = g(x,y)$ has a global solution $y : [0,\infty) \to \mathbb{R}^n$

Differential Ghosts

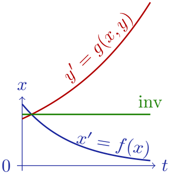
Differential Ghost

$$[x' = f(x)\,\&\,Q]P \leftrightarrow \exists y\,[x' = f(x), y' = a(x)y + b(x)\,\&\,Q]P$$



since new $y' = a(x)y + b(x)$ has a long enough solution

Differential Ghosts



Differential Ghost

$$[x' = f(x) \,\&\, Q]P \leftrightarrow \exists y\, [x' = f(x), y' = a(x)y + b(x) \,\&\, Q]P$$

Differential Ghost

$$\text{dG}\ \frac{\Gamma \vdash \exists y\, [x' = f(x), y' = a(x)y + b(x) \,\&\, Q]P, \Delta}{\Gamma \vdash [x' = f(x) \,\&\, Q]P, \Delta}$$

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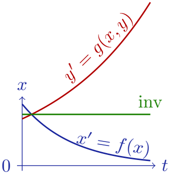
Differential Ghosts



Differential Ghost

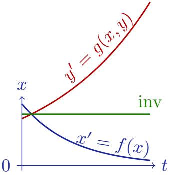
$$[x' = f(x)\,\&\,Q]P \leftrightarrow \exists y\,[x' = f(x), y' = a(x)y + b(x)\,\&\,Q]P$$

Differential Ghost

$$\text{dG}\ \frac{\Gamma \vdash \exists y\,[x' = f(x), y' = a(x)y + b(x)\,\&\,Q]P, \Delta}{\Gamma \vdash [x' = f(x)\,\&\,Q]P, \Delta}$$

since new $y' = a(x)y + b(x)$ has a long enough solution

Differential Ghosts



Differential Ghost

$$[x' = f(x) \,\&\, Q]P \leftrightarrow \exists y\,[x' = f(x), y' = a(x)y + b(x) \,\&\, Q]P$$

Differential Ghost

$$\text{dG}\ \frac{\Gamma \vdash \exists y\,[x' = f(x), y' = a(x)y + b(x) \,\&\, Q]P, \Delta}{\Gamma \vdash [x' = f(x) \,\&\, Q]P, \Delta}$$

Differential Auxiliary

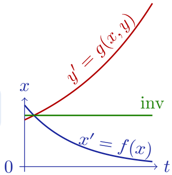
$$\text{dA}\ \frac{\vdash F \leftrightarrow \exists y\, G \quad G \vdash [x' = f(x), y' = a(x)y + b(x) \,\&\, Q]G}{F \vdash [x' = f(x) \,\&\, Q]F}$$

since new $y' = a(x)y + b(x)$ has a long enough solution

Differential Ghosts

Differential Ghost

$$[x' = f(x) \,\&\, Q]P \leftrightarrow \exists y\,[x' = f(x), y' = a(x)y + b(x) \,\&\, Q]P$$



Differential Ghost

$$\text{dG}\ \frac{\Gamma \vdash \exists y\,[x'=f(x), y'=a(x)y+b(x)\,\&\,Q]P, \Delta}{\Gamma \vdash [x'=f(x)\,\&\,Q]P, \Delta}$$

Differential Auxiliary

$$\text{dA}\ \frac{\vdash F\leftrightarrow\exists y\,G \quad G\vdash [x'=f(x), y'=a(x)y+b(x)\,\&\,Q]G}{F\vdash [x'=f(x)\,\&\,Q]F}$$

$$\text{DG}\ \dfrac{\text{MR}\ \dfrac{\dfrac{\exists y\,G\vdash F}{G\vdash F} \qquad \exists\text{R,cut}\ \dfrac{F\vdash\exists y\,G \quad G\vdash[x'=f(x),y'=a(x)y+b(x)]G}{F\vdash \exists y\,[x'=f(x),y'=a(x)y+b(x)]G}}{F\vdash\exists y\,[x'=f(x),y'=a(x)y+b(x)]F}}{F\vdash[x'=f(x)]F}$$

Nonexistent Differential Ghosts

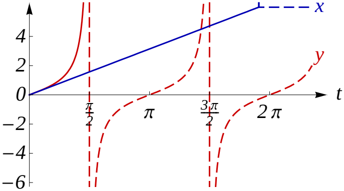
What could possibly go wrong?

$$\lightning\ \cfrac{\exists\mathrm{R}\ \cfrac{x{=}0, y{=}0 \vdash [x' = 1, y' = y^2 + 1]\, x \le 6}{x = 0 \vdash \exists y\, [x' = 1, y' = y^2 + 1]\, x \le 6}}{x = 0 \vdash [x' = 1]\, x \le 6}$$

Nonexistent Differential Ghosts

What could possibly go wrong?

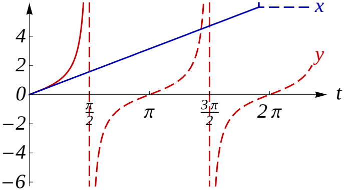
$$\text{↯}\;\dfrac{\exists\mathrm{R}\;\dfrac{x=0, y=0 \vdash [x'=1, y'=y^2+1]\,x \le 6}{x=0 \vdash \exists y\,[x'=1, y'=y^2+1]\,x \le 6}}{x=0 \vdash [x'=1]\,x \le 6}$$



Nonexistent Differential Ghosts

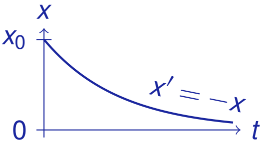
What could possibly go wrong? Explosive ghosts stop the world. Don't!

$$
\lightning\;\frac{\exists R\;\dfrac{x=0, y=0 \vdash [x'=1, y'=y^2+1]\, x \le 6}{x = 0 \vdash \exists y\, [x'=1, y'=y^2+1]\, x \le 6}}{x = 0 \vdash [x'=1]\, x \le 6}
$$



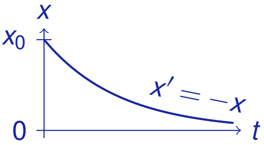
Substitute Ghosts

$$\frac{}{x > 0 \vdash [x' = -x]x > 0}\text{dA}$$



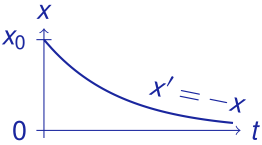
Substitute Ghosts

$$\text{dA}\;\dfrac{\mathbb{R}\;\dfrac{}{\vdash x>0 \leftrightarrow \exists y\, xy^2=1} \qquad \text{dI}\;\dfrac{}{xy^2=1 \vdash [x' = -x, y' = \qquad]xy^2 = 1}}{x > 0 \vdash [x' = -x]x > 0}$$



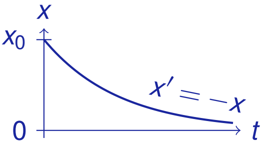
Substitute Ghosts

$$\text{dA}\;\dfrac{\mathbb{R}\,\dfrac{*}{\vdash x>0 \leftrightarrow \exists y\, xy^2=1} \qquad \text{dI}\,\dfrac{}{xy^2=1 \vdash [x' = -x, y' = \text{☁}]xy^2 = 1}}{x > 0 \vdash [x' = -x]x > 0}$$



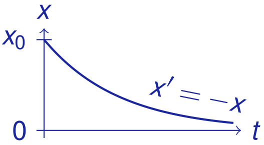
Substitute Ghosts

$$\text{dA}\ \cfrac{\mathbb{R}\ \cfrac{*}{\vdash x>0 \leftrightarrow \exists y\, xy^2=1} \qquad \text{dI}\ \cfrac{[:=]\ \cfrac{}{\vdash [x':=-x][y':=\text{☁}]x'y^2+x2yy'=0}}{xy^2=1 \vdash [x'=-x, y'=\text{☁}]xy^2=1}}{x>0 \vdash [x'=-x]x>0}$$



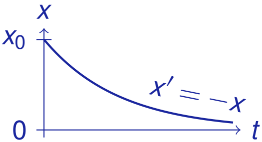
Substitute Ghosts

$$\text{dA}\;\dfrac{\mathbb{R}\;\dfrac{*}{\vdash x>0 \leftrightarrow \exists y\, xy^2=1} \qquad \text{dI}\;\dfrac{[:=]\;\dfrac{\dfrac{}{\vdash -xy^2+2xy\text{☁}=0}}{\vdash [x':=-x][y':=\text{☁}]x'y^2+x2yy'=0}}{xy^2=1 \vdash [x'=-x, y'=\text{☁}]xy^2=1}}{x>0 \vdash [x'=-x]x>0}$$



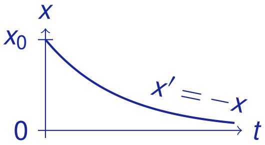
Substitute Ghosts

$$\dfrac{\mathbb{R}\,\dfrac{*}{\vdash x>0 \leftrightarrow \exists y\, xy^2=1} \qquad \text{dI}\,\dfrac{[:=]\,\dfrac{\dfrac{\text{could prove if } ☁ = \frac{y}{2}}{\vdash -xy^2+2xy☁=0}}{\vdash [x':=-x][y':=☁]x'y^2+x2yy'=0}}{xy^2=1 \vdash [x'=-x, y'=☁]xy^2=1}}{x>0 \vdash [x'=-x]x>0}\,\text{dA}$$



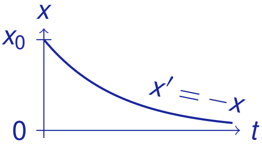
Substitute Ghosts

$$\text{dA}\;\dfrac{\mathbb{R}\;\dfrac{*}{\vdash x>0 \leftrightarrow \exists y\, xy^2=1} \qquad \text{dI}\;\dfrac{[:=]\;\dfrac{\dfrac{\text{could prove if } ☁ = \frac{y}{2}}{\vdash -xy^2+2xy☁=0}}{\vdash [x':=-x][y':=☁]x'y^2+x2yy'=0}}{xy^2=1 \vdash [x'=-x, y'=☁]xy^2=1}}{x>0 \vdash [x'=-x]x>0}$$



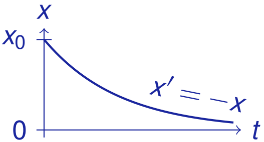
Substitute Ghosts

$$
\dfrac{\mathbb{R}\,\dfrac{*}{\vdash x>0 \leftrightarrow \exists y\, xy^2=1} \qquad [:=]\,\text{dI}\,\dfrac{\dfrac{\dfrac{\text{could prove if } \frac{y}{2} \qquad = \frac{y}{2} \text{ proved!}}{\vdash -xy^2+2xy\frac{y}{2} \qquad =0}}{\vdash [x':=-x][y':=\frac{y}{2} \qquad]x'y^2+x2yy'=0}}{xy^2=1 \vdash [x'=-x, y'=\frac{y}{2} \qquad]xy^2=1}}{x>0 \vdash [x'=-x]x>0}\,\text{dA}
$$



Substitute Ghosts

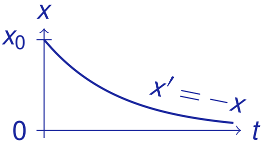
$$\text{dA}\,\dfrac{\mathbb{R}\,\dfrac{*}{\vdash x>0 \leftrightarrow \exists y\, xy^2=1} \qquad \text{dI}\,\dfrac{[:=]\,\dfrac{\dfrac{\text{could prove if } \frac{y}{2} = \frac{y}{2}}{\vdash -xy^2+2xy\frac{y}{2} = 0}}{\vdash [x':=-x][y':=\frac{y}{2}]x'y^2+x2yy'=0}}{xy^2=1 \vdash [x'=-x, y'=\frac{y}{2}]xy^2=1}}{x>0 \vdash [x'=-x]x>0}$$



This is a recipe for brewing suitable differential ghosts!

Substitute Ghosts

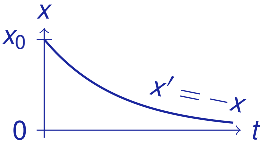
$$\text{dA}\,\dfrac{\mathbb{R}\,\dfrac{*}{\vdash x>0 \leftrightarrow \exists y\, xy^2=1} \qquad \text{dI}\,\dfrac{[:=]\,\dfrac{\dfrac{\text{could prove if } j(y) \;=\frac{y}{2} \text{ proved!}}{\vdash -xy^2+2xy\,j(y) \;=0}}{\vdash [x':=-x][y':=j(y)\;]x'y^2+x2yy'=0}}{xy^2=1 \vdash [x'=-x, y'=j(y)\;]xy^2=1}}{x>0 \vdash [x'=-x]x>0}$$



Function symbol $j(y)$ can play the rôle of a substitute ghost

Substitute Ghosts

$$\text{dA}\ \dfrac{\mathbb{R}\ \dfrac{*}{\vdash x>0 \leftrightarrow \exists y\, xy^2=1} \qquad \text{dI}\ \dfrac{[:=]\ \dfrac{\dfrac{\text{could prove if } \frac{y}{2} \quad = \frac{y}{2}}{\vdash -xy^2+2xy\frac{y}{2} \quad = 0}}{\vdash [x':=-x][y':=\frac{y}{2} \quad]x'y^2+x2yy'=0}}{xy^2=1 \vdash [x'=-x, y'=\frac{y}{2} \quad]xy^2=1}}{x>0 \vdash [x'=-x]x>0}$$

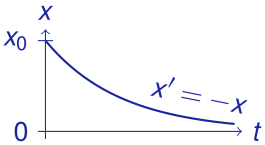


Function symbol $j(y)$ can be substituted uniformly

▸ Chapter 18

Substitute Ghosts

$$
\text{dA}\;\dfrac{\mathbb{R}\;\dfrac{*}{\vdash x>0 \leftrightarrow \exists y\, xy^2=1} \qquad \text{dI}\;\dfrac{[:=]\;\dfrac{\dfrac{\text{could prove if } j(y) \;=\frac{y}{2}}{\vdash \;-xy^2+2xyj(y) \;=0}}{\vdash [x':=-x][y':=j(y)\;]x'y^2+x2yy'=0}}{xy^2=1 \vdash [x'=-x, y'=j(y)\;]xy^2=1}}{x>0 \vdash [x'=-x]x>0}
$$



Function symbol $j(y)$ needs to be instantiated linearly in y

Solvable Ghosts: Axiomatic Differential Equation Solver

① DG introduces time t, DC cuts solution in, that DI proves and

② DW exports to postcondition

③ inverse DC removes evolution domain constraints

④ inverse DG removes original ODE

⑤ DS solves remaining ODE for time $[x'=c()]P \leftrightarrow \forall t{\geq}0\,[x:=x+c()t]P$

$$\begin{array}{lll}
 & * & \\ \hline
\mathbb{R} & \Gamma \vdash \forall s{\geq}0\,(x_0 + \frac{a}{2}s^2 + v_0 s \geq 0) & \\ \hline
[:=] & \Gamma \vdash \forall s{\geq}0\,[t:=0+1s]x_0 + \frac{a}{2}t^2 + v_0 t \geq 0 & \\ \hline
\text{DS} & \Gamma \vdash [t'=1]x_0 + \frac{a}{2}t^2 + v_0 t \geq 0 & \\ \hline
\text{DG} & \Gamma \vdash [v'=a, t'=1]x_0 + \frac{a}{2}t^2 + v_0 t \geq 0 & \\ \hline
\text{DG} & \Gamma \vdash [x'=v, v'=a, t'=1]x_0 + \frac{a}{2}t^2 + v_0 t \geq 0 & \triangleright \\ \hline
\text{DC} & \Gamma \vdash [x'=v, v'=a, t'=1\,\&\, v=v_0{+}at]x_0 + \frac{a}{2}t^2 + v_0 t \geq 0 & \triangleright \\ \hline
\text{DC} & \Gamma \vdash [x'=v, v'=a, t'=1\,\&\, v=v_0{+}at \wedge x=x_0{+}\frac{a}{2}t^2{+}v_0 t]x_0 + \frac{a}{2}t^2 + v_0 t \geq 0 & \\ \hline
\text{MR} & \Gamma \vdash [x'=v, v'=a, t'=1\,\&\, v=v_0{+}at \wedge x=x_0{+}\frac{a}{2}t^2{+}v_0 t](x{=}x_0{+}\frac{a}{2}t^2{+}v_0 t \rightarrow x{\geq}0) & \\ \hline
\text{DW} & \Gamma \vdash [x'=v, v'=a, t'=1\,\&\, v=v_0{+}at \wedge x=x_0{+}\frac{a}{2}t^2{+}v_0 t]x \geq 0 & \triangleright \\ \hline
\text{DC} & \Gamma \vdash [x'=v, v'=a, t'=1\,\&\, v=v_0{+}at]x \geq 0 & \triangleright \\ \hline
\text{DC} & \Gamma \vdash [x'=v, v'=a, t'=1]x \geq 0 & t:=0 \\ \hline
 & \Gamma \vdash \exists t\,[x'=v, v'=a, t'=1]x \geq 0 & \\ \hline
\text{DG} & \Gamma \vdash [x'=v, v'=a]x \geq 0 &
\end{array}$$

▷ Proving Solutions of Differential Equations

These are the side branches elided above by ▷

$$\text{dI}\ \frac{}{\phi \vdash [x' = v, v' = a, t' = 1]v = v_0 + at}$$

$$\text{dI}\ \frac{}{\phi \vdash [x' = v, v' = a, t' = 1 \,\&\, v = v_0 + at]x = x_0 + \frac{a}{2}t^2 + v_0 t}$$

▷ Proving Solutions of Differential Equations

These are the side branches elided above by ▷

$$\dfrac{\dfrac{}{\vdash [v' := a][t' := 1]v' = at'}{\scriptstyle [:=]}}{\phi \vdash [x' = v, v' = a, t' = 1]v = v_0 + at}{\scriptstyle \text{dI}}$$

$$\dfrac{}{\phi \vdash [x' = v, v' = a, t' = 1 \,\&\, v = v_0 + at]x = x_0 + \frac{a}{2}t^2 + v_0 t}{\scriptstyle \text{dI}}$$

▷ Proving Solutions of Differential Equations

These are the side branches elided above by ▷

$$\text{dI}\ \dfrac{[:=]\ \dfrac{\mathbb{R}\ \dfrac{}{\vdash a = a \cdot 1}}{\vdash [v' := a][t' := 1]v' = at'}}{\phi \vdash [x' = v, v' = a, t' = 1]v = v_0 + at}$$

$$\text{dI}\ \dfrac{}{\phi \vdash [x' = v, v' = a, t' = 1 \,\&\, v = v_0 + at]x = x_0 + \frac{a}{2}t^2 + v_0 t}$$

▷ Proving Solutions of Differential Equations

These are the side branches elided above by ▷

$$\mathrm{dI}\;\dfrac{[:=]\;\dfrac{\mathbb{R}\;\dfrac{*}{\vdash a = a \cdot 1}}{\vdash [v' := a][t' := 1]v' = at'}}{\phi \vdash [x' = v, v' = a, t' = 1]\color{red}{v = v_0 + at}}$$

$$\mathrm{dI}\;\dfrac{}{\phi \vdash [x' = v, v' = a, t' = 1 \,\&\, \color{red}{v = v_0 + at}]x = x_0 + \frac{a}{2}t^2 + v_0 t}$$

⊳ Proving Solutions of Differential Equations

These are the side branches elided above by ⊳

$$\text{dI}\;\dfrac{[:=]\;\dfrac{\mathbb{R}\;\dfrac{*}{\vdash a = a\cdot 1}}{\vdash [v' := a][t' := 1]v' = at'}}{\phi \vdash [x' = v, v' = a, t' = 1]v = v_0 + at}$$

$$\text{dI}\;\dfrac{[:=]\;\dfrac{}{\vdash v = v_0 + at \to [x' := v][t' := 1]x' = at\,t' + v_0\,t'}}{\phi \vdash [x' = v, v' = a, t' = 1 \,\&\, v = v_0 + at]x = x_0 + \frac{a}{2}t^2 + v_0\,t}$$

▷ Proving Solutions of Differential Equations

These are the side branches elided above by ▷

$$\text{dI}\;\dfrac{[:=]\;\dfrac{\mathbb{R}\;\dfrac{*}{\vdash a = a\cdot 1}}{\vdash [v' := a][t' := 1]v' = at'}}{\phi \vdash [x' = v, v' = a, t' = 1]v = v_0 + at}$$

$$\text{dI}\;\dfrac{[:=]\;\dfrac{\mathbb{R}\;\dfrac{}{\vdash v = v_0 + at \rightarrow {\color{red}v} = at\cdot {\color{red}1} + v_0\cdot {\color{red}1}}}{\vdash v = v_0 + at \rightarrow [x' := {\color{red}v}][t' := {\color{red}1}]x' = att' + v_0 t'}}{\phi \vdash [x' = v, v' = a, t' = 1 \,\&\, v = v_0 + at]x = x_0 + \frac{a}{2}t^2 + v_0 t}$$

▷ Proving Solutions of Differential Equations

These are the side branches elided above by ▷

$$\text{dI}\,\dfrac{[:=]\,\dfrac{\mathbb{R}\,\dfrac{*}{\vdash a = a\cdot 1}}{\vdash [v' := a][t' := 1]v' = at'}}{\phi \vdash [x' = v, v' = a, t' = 1]v = v_0 + at}$$

$$\text{dI}\,\dfrac{[:=]\,\dfrac{\mathbb{R}\,\dfrac{*}{\vdash v = v_0 + at \rightarrow v = at\cdot 1 + v_0\cdot 1}}{\vdash v = v_0 + at \rightarrow [x' := v][t' := 1]x' = att' + v_0t'}}{\phi \vdash [x' = v, v' = a, t' = 1\,\&\, v = v_0 + at]x = x_0 + \frac{a}{2}t^2 + v_0t}$$

▷ Proving Solutions of Differential Equations

These are the side branches elided above by ▷

$$\text{dI}\;\dfrac{[:=]\;\dfrac{\mathbb{R}\;\dfrac{*}{\vdash a = a\cdot 1}}{\vdash [v' := a][t' := 1]v' = at'}}{\phi \vdash [x' = v, v' = a, t' = 1]v = v_0 + at}$$

$$\text{dI}\;\dfrac{[:=]\;\dfrac{\mathbb{R}\;\dfrac{*}{\vdash v = v_0 + at \rightarrow v = at\cdot 1 + v_0\cdot 1}}{\vdash v = v_0 + at \rightarrow [x' := v][t' := 1]x' = att' + v_0 t'}}{\phi \vdash [x' = v, v' = a, t' = 1 \,\&\, v = v_0 + at]x = x_0 + \frac{a}{2}t^2 + v_0 t}$$

But ϕ needs $v = v_0 \wedge x = x_0$ initially for dI

▷ Proving Solutions of Differential Equations

These are the side branches elided above by ▷

$$\mathrm{dI}\;\dfrac{[:=]\;\dfrac{\mathbb{R}\;\dfrac{*}{\vdash a = a\cdot 1}}{\vdash [v' := a][t' := 1]v' = at'}}{\phi \vdash [x' = v, v' = a, t' = 1]v = v_0 + at}$$

$$\mathrm{dI}\;\dfrac{[:=]\;\dfrac{\mathbb{R}\;\dfrac{*}{\vdash v = v_0 + at \rightarrow v = at\cdot 1 + v_0\cdot 1}}{\vdash v = v_0 + at \rightarrow [x' := v][t' := 1]x' = att' + v_0 t'}}{\phi \vdash [x' = v, v' = a, t' = 1 \,\&\, v = v_0 + at]x = x_0 + \frac{a}{2}t^2 + v_0 t}$$

But ϕ needs $v = v_0 \land x = x_0$ initially for dI

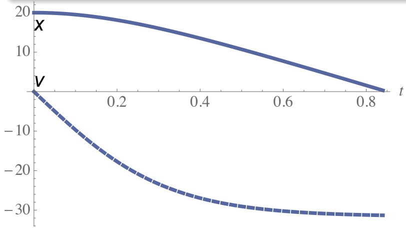
Discrete ghosts to the rescue: $[x_0 := x][v_0 := v]\ldots$

who can remember initial value on demand.

Limit Velocity of an Aerodynamic Ball

Proposition (Aerodynamic velocity limits)

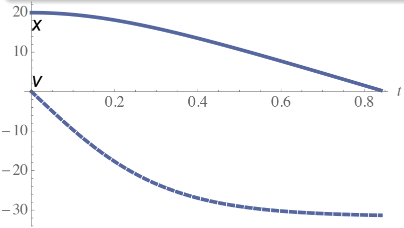
$$g>0 \land r>0 \qquad \rightarrow [x'=v, v'=-g+rv^2 \;\&\; x\geq 0 \land v\leq 0]$$



Limit Velocity of an Aerodynamic Ball

Proposition (Aerodynamic velocity limits)

$$g>0 \land r>0 \qquad \rightarrow [x'=v, v'=-g+rv^2 \,\&\, x\geq 0 \land v\leq 0]$$



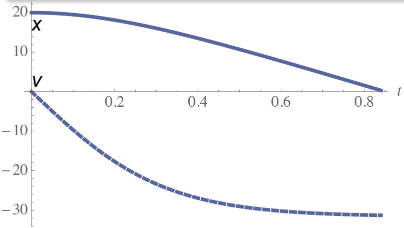
Equilibrium

$v' = 0$ iff $-g + rv^2 = 0$

Limit Velocity of an Aerodynamic Ball

Proposition (Aerodynamic velocity limits)

$$g>0 \land r>0 \quad \rightarrow [x'=v, v'=-g+rv^2 \,\&\, x\geq 0 \land v\leq 0]$$



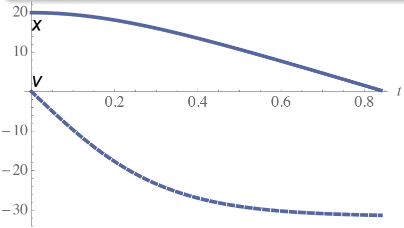
Equilibrium

$v' = 0$ iff $-g + rv^2 = 0$ iff $v = \pm\sqrt{\frac{g}{r}}$

Limit Velocity of an Aerodynamic Ball

Proposition (Aerodynamic velocity limits)

$$g>0 \land r>0 \land v>-\sqrt{\frac{g}{r}} \rightarrow [x'=v, v'=-g+rv^2 \,\&\, x\geq 0 \land v\leq 0]\, v>-\sqrt{\frac{g}{r}}$$



Equilibrium

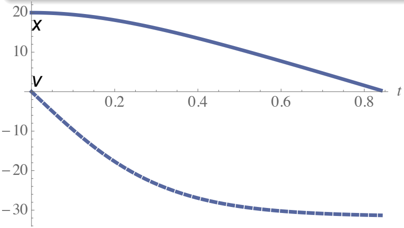
$v' = 0$ iff $-g+rv^2 = 0$ iff $v = \pm\sqrt{\frac{g}{r}}$

Limit Velocity of an Aerodynamic Ball

$$\text{dA}\ \frac{}{v>-\sqrt{g/r} \vdash [x' = v, v' = -g + rv^2]\, v > -\sqrt{g/r}}$$

Proposition (Aerodynamic velocity limits)

$$g>0 \land r>0 \land v>-\sqrt{\frac{g}{r}} \rightarrow [x'=v, v'=-g+rv^2\, \&\, x\geq 0 \land v\leq 0]\, v>-\sqrt{\frac{g}{r}}$$



Equilibrium

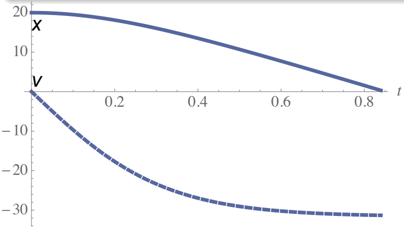
$v' = 0$ iff $-g + rv^2 = 0$ iff $v = \pm\sqrt{\frac{g}{r}}$

Limit Velocity of an Aerodynamic Ball

$$\cfrac{\cfrac{}{y^2(v+\sqrt{g/r})=1 \vdash [x'=v, v'=-g+rv^2, y'=j(x,v,y)\quad]y^2(v+\sqrt{g/r})=1}\,\text{dI} \quad \triangleright}{v>-\sqrt{g/r} \vdash [x'=v, v'=-g+rv^2]\,v>-\sqrt{g/r}}\,\text{dA}$$

Proposition (Aerodynamic velocity limits)

$$g>0 \land r>0 \land v>-\sqrt{\frac{g}{r}} \rightarrow [x'=v, v'=-g+rv^2 \,\&\, x\geq 0 \land v\leq 0]\, v>-\sqrt{\frac{g}{r}}$$



Equilibrium

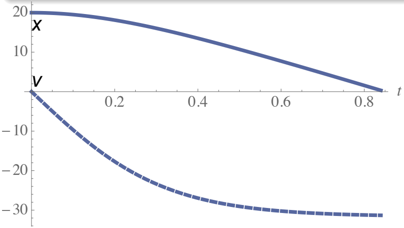
$v'=0$ iff $-g+rv^2=0$ iff $v=\pm\sqrt{\frac{g}{r}}$

Limit Velocity of an Aerodynamic Ball

$$\begin{array}{ll}
\text{[:=]} & \overline{\vdash [x':=v][v':=-g+rv^2][{\color{red}y':=j(x,v,y)}]2y{\color{red}y'}(v+\sqrt{g/r})+y^2v'=0} \\
\text{dI} & \overline{y^2(v+\sqrt{g/r})=1 \vdash [x'=v,v'=-g+rv^2,{\color{red}y'=j(x,v,y)}]y^2(v+\sqrt{g/r})=1} \quad \triangleright \\
\text{dA} & \overline{v>-\sqrt{g/r} \vdash [x'=v,v'=-g+rv^2]\,v>-\sqrt{g/r}}
\end{array}$$

Proposition (Aerodynamic velocity limits)

$$g>0 \land r>0 \land v>-\sqrt{\frac{g}{r}} \rightarrow [x'=v,v'=-g+rv^2\,\&\,x\geq 0 \land v\leq 0]\,v>-\sqrt{\frac{g}{r}}$$



Equilibrium

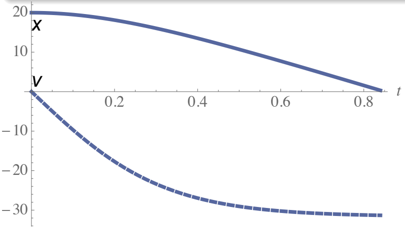
$v' = 0$ iff $-g+rv^2=0$ iff $v=\pm\sqrt{\frac{g}{r}}$

Limit Velocity of an Aerodynamic Ball

$$\begin{array}{l|l}
 & \hline \vdash 2y(j(x,v,y) \qquad)(v+\sqrt{g/r})+y^2(-g+rv^2)=0 \\
[:=] & \hline \vdash [x':=v][v':=-g+rv^2][y':=j(x,v,y) \qquad]2yy'(v+\sqrt{g/r})+y^2v'=0 \\
\text{dI} & \hline y^2(v+\sqrt{g/r})=1 \vdash [x'=v, v'=-g+rv^2, y'=j(x,v,y) \qquad]y^2(v+\sqrt{g/r})=1 \qquad \triangleright \\
\text{dA} & \hline v>-\sqrt{g/r} \vdash [x'=v, v'=-g+rv^2]\, v>-\sqrt{g/r}
\end{array}$$

Proposition (Aerodynamic velocity limits)

$$g>0 \land r>0 \land v>-\sqrt{\frac{g}{r}} \rightarrow [x'=v, v'=-g+rv^2 \,\&\, x\geq 0 \land v\leq 0]\, v>-\sqrt{\frac{g}{r}}$$



Equilibrium

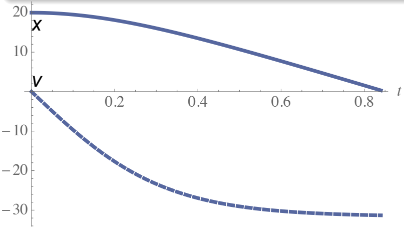
$v' = 0$ iff $-g+rv^2=0$ iff $v=\pm\sqrt{\frac{g}{r}}$

Limit Velocity of an Aerodynamic Ball

$$
\begin{array}{lr}
\mathbb{R} & \\ \hline
 & \vdash -ry^2(v^2-{}^g/_r)+y^2(-g+rv^2)=0 \\ \hline
 & \vdash 2y(-{}^r/_2(v-\sqrt{{}^g/_r})y)(v+\sqrt{{}^g/_r})+y^2(-g+rv^2)=0 \\
[:=] & \\ \hline
 & \vdash [x':=v][v':=-g+rv^2][y':=-{}^r/_2(v-\sqrt{{}^g/_r})y]2yy'(v+\sqrt{{}^g/_r})+y^2v'=0 \\
\text{dI} & \\ \hline
 & y^2(v+\sqrt{{}^g/_r})=1 \vdash [x'=v,v'=-g+rv^2,y'=-{}^r/_2(v-\sqrt{{}^g/_r})y]y^2(v+\sqrt{{}^g/_r})=1 \quad \triangleright \\
\text{dA} & \\ \hline
 & v>-\sqrt{{}^g/_r} \vdash [x'=v,v'=-g+rv^2]\,v>-\sqrt{{}^g/_r}
\end{array}
$$

Proposition (Aerodynamic velocity limits)

$$g>0 \land r>0 \land v>-\sqrt{\frac{g}{r}} \rightarrow [x'=v,v'=-g+rv^2\,\&\,x\geq 0 \land v\leq 0]\,v>-\sqrt{\frac{g}{r}}$$



Equilibrium

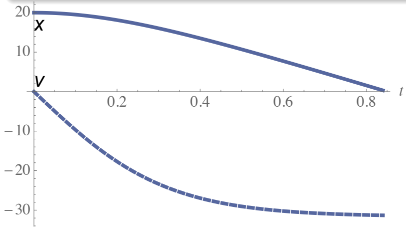
$v' = 0$ iff $-g + rv^2 = 0$ iff $v = \pm\sqrt{\frac{g}{r}}$

Limit Velocity of an Aerodynamic Ball

$$\begin{array}{lrl}
 & & * \\ \hline
\mathbb{R} & & \vdash -ry^2(v^2 - {}^g\!/_r) + y^2(-g + rv^2) = 0 \\ \hline
 & & \vdash 2y(-{}^r\!/_2(v-\sqrt{{}^g\!/_r})y)(v+\sqrt{{}^g\!/_r}) + y^2(-g+rv^2) = 0 \\ \hline
[:=] & & \vdash [x':=v][v':=-g+rv^2][y':=-{}^r\!/_2(v-\sqrt{{}^g\!/_r})y]2yy'(v+\sqrt{{}^g\!/_r})+y^2v'=0 \\ \hline
\mathrm{dI} & y^2(v+\sqrt{{}^g\!/_r})=1 & \vdash [x'=v, v'=-g+rv^2, y'=-{}^r\!/_2(v-\sqrt{{}^g\!/_r})y]y^2(v+\sqrt{{}^g\!/_r})=1 \quad \triangleright \\ \hline
\mathrm{dA} & v>-\sqrt{{}^g\!/_r} & \vdash [x' = v, v' = -g + rv^2]\, v > -\sqrt{{}^g\!/_r}
\end{array}$$

Proposition (Aerodynamic velocity limits)

$$g>0 \land r>0 \land v>-\sqrt{\frac{g}{r}} \rightarrow [x'=v, v'=-g+rv^2 \,\&\, x\geq 0 \land v\leq 0]\, v>-\sqrt{\frac{g}{r}}$$



Equilibrium

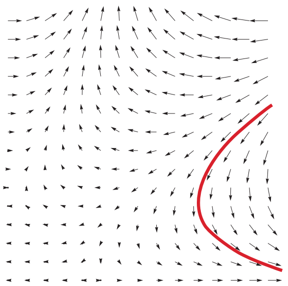
$v' = 0$ iff $-g + rv^2 = 0$ iff $v = \pm\sqrt{\frac{g}{r}}$

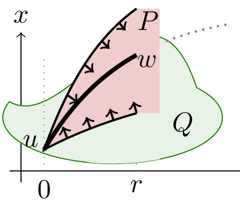
Outline

1. Learning Objectives
2. Recap: Proofs for Differential Equations
3. A Gradual Introduction to Ghost Variables
    - Discrete Ghosts
    - Proving Bouncing Balls with Sneaky Solutions
    - Differential Ghosts of Time
    - Constructing Differential Ghosts
4. Differential Ghosts
    - Substitute Ghosts
    - Solvable Ghosts
    - Limit Velocity of an Aerodynamic Ball
5. Summary

Differential Invariants for Differential Equations

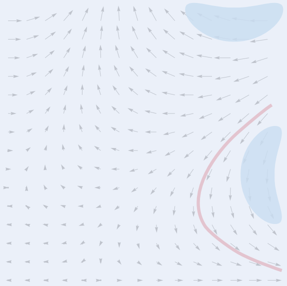
Differential Invariant

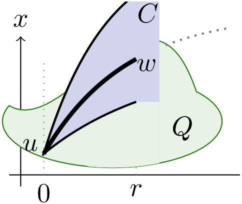




$x' = f(x) \,\&\, Q$

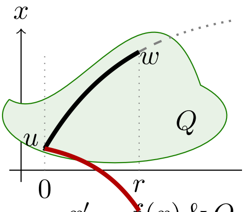
Differential Cut





$x' = f(x) \,\&\, Q$

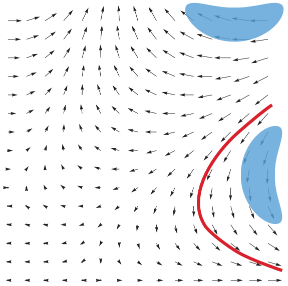
Differential Ghost

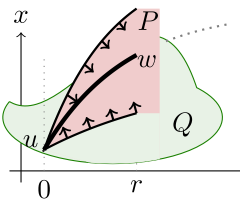


$x' = f(x) \,\&\, Q$

Differential Invariants for Differential Equations

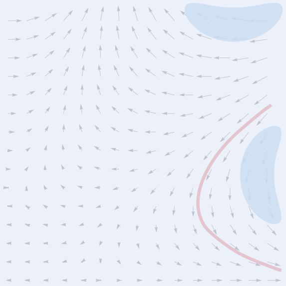
Differential Invariant

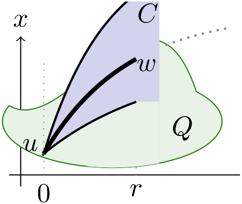




$$x' = f(x)\,\&\,Q$$

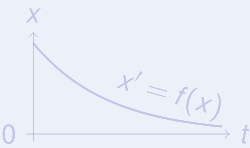
Differential Cut

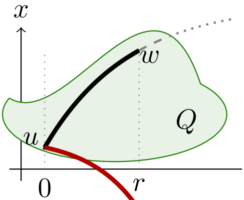




$$x' = f(x)\,\&\,Q$$

Differential Ghost

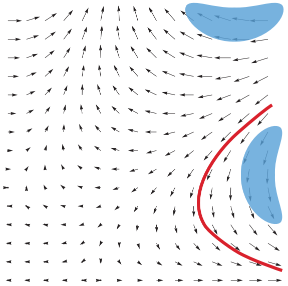


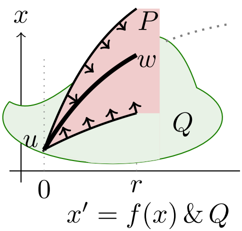


$$x' = f(x)\,\&\,Q$$

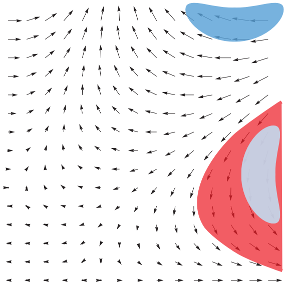
Differential Invariants for Differential Equations

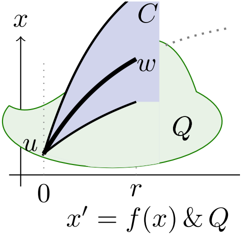
Differential Invariant





Differential Cut





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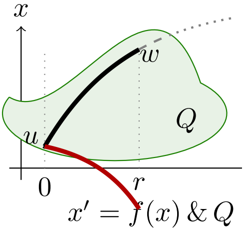
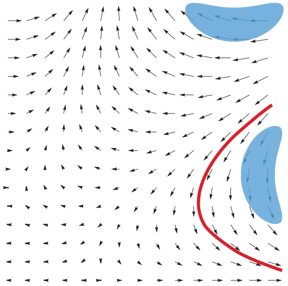


Differential Invariants for Differential Equations

Differential Invariant



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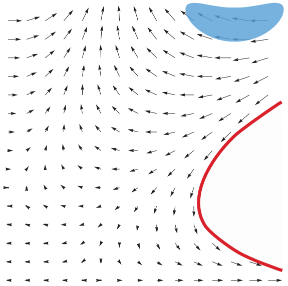
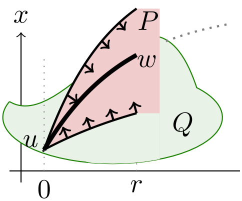
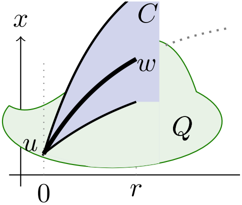


Differential Ghost



$x' = f(x) \,\&\, Q$



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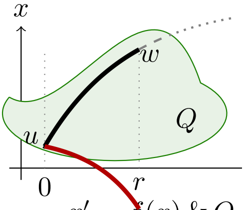

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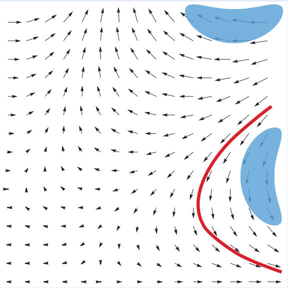
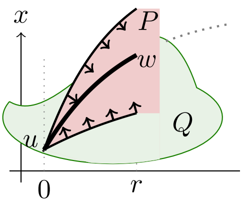
Differential Invariants for Differential Equations

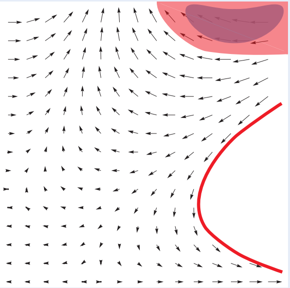
Differential Invariant

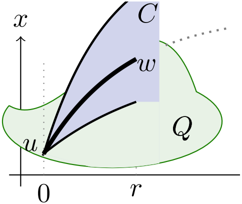




$$x' = f(x) \,\&\, Q$$

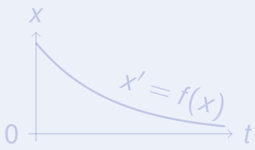
Differential Cut

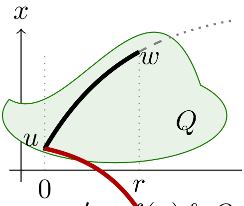




$$x' = f(x) \,\&\, Q$$

Differential Ghost

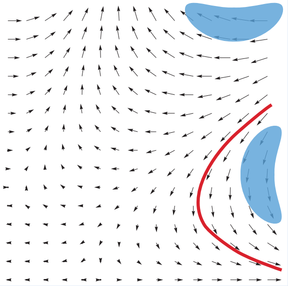


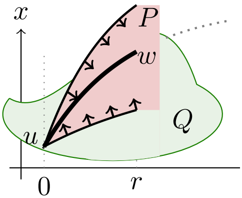


$$x' = f(x) \,\&\, Q$$

Differential Invariants for Differential Equations

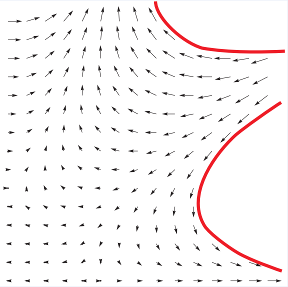
Differential Invariant

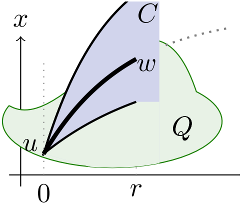




$x' = f(x) \,\&\, Q$

Differential Cut





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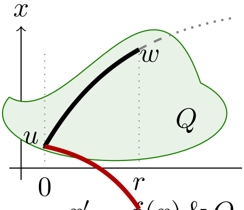
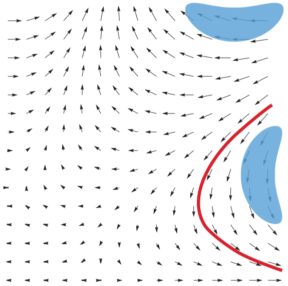
Differential Ghost

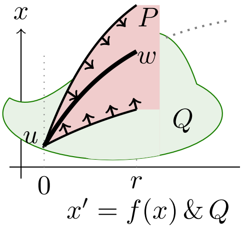


$x' = f(x) \,\&\, Q$

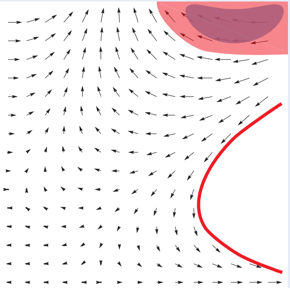
Differential Invariants for Differential Equations

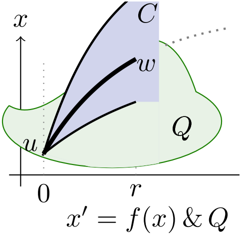
Differential Invariant



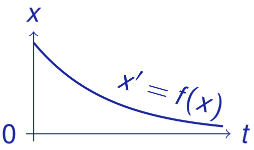


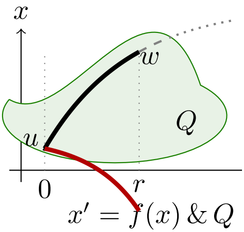
Differential Cut





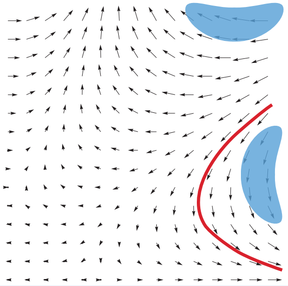
Differential Ghost

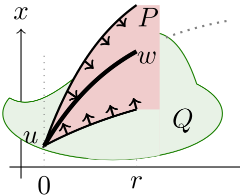




Differential Invariants for Differential Equations

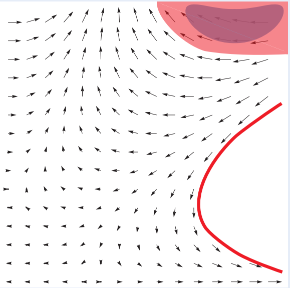
Differential Invariant

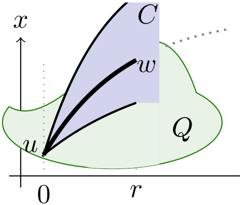




$x' = f(x) \,\&\, Q$

Differential Cut





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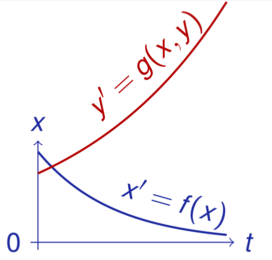
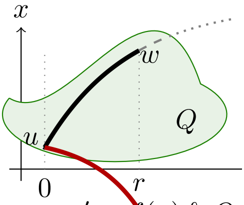
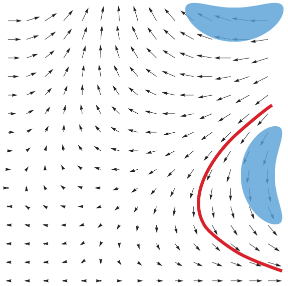




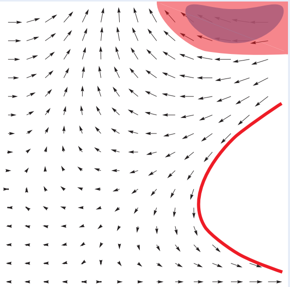
$x' = f(x) \,\&\, Q$

Differential Invariants for Differential Equations

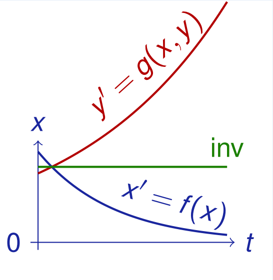
Differential Invariant

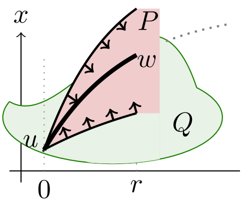


Differential Cut

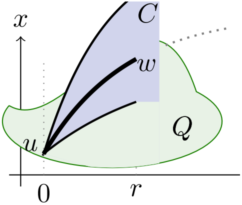


Differential Ghost

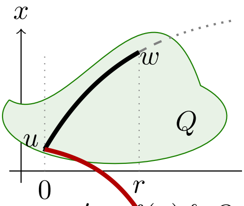




$x' = f(x) \,\&\, Q$



$x' = f(x) \,\&\, Q$

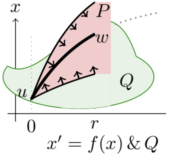


$x' = f(x) \,\&\, Q$

Differential Invariants for Differential Equations

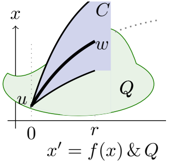
Differential Invariant

$$\frac{Q \vdash [x' := f(x)](P)'}{P \vdash [x' = f(x) \,\&\, Q]P}$$



Differential Cut

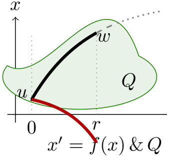
$$\frac{P \vdash [x' = f(x) \,\&\, Q]C \quad P \vdash [x' = f(x) \,\&\, Q \land C]P}{P \vdash [x' = f(x) \,\&\, Q]P}$$



Differential Ghost

$$\frac{P \leftrightarrow \exists y\, G \quad G \vdash [x' = f(x), y' = g(x,y) \,\&\, Q]G}{P \vdash [x' = f(x) \,\&\, Q]P}$$

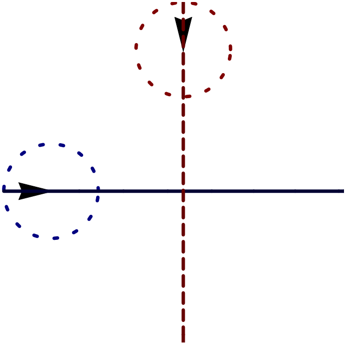
if new $y' = g(x,y)$ has long enough solution



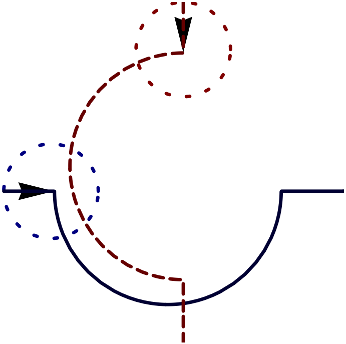
Outline

6 Appendix
- Axiomatic Ghosts
- Arithmetic Ghosts
- Nondeterministic Assignments & Ghosts of Choice
- Differential-Algebraic Ghosts

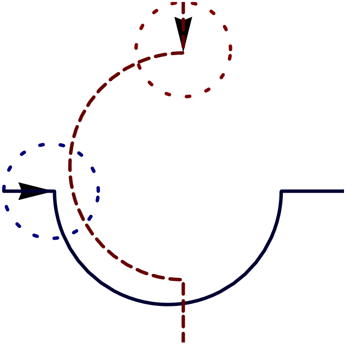
Air Traffic Control



Air Traffic Control



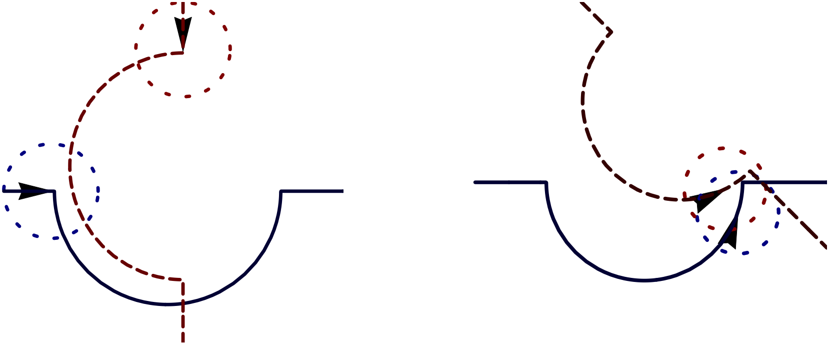
Air Traffic Control



Verification?

looks correct

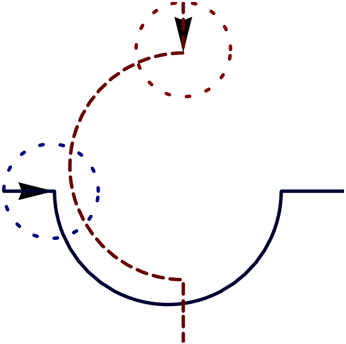
Air Traffic Control

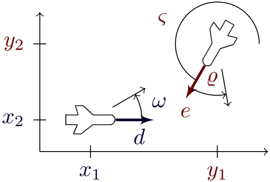


Verification?

looks correct NO!

Air Traffic Control



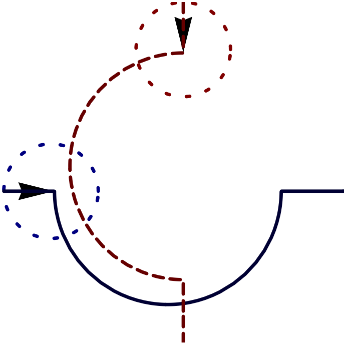


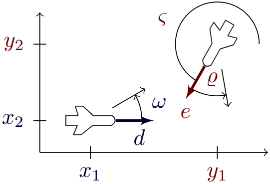
$$\begin{bmatrix} x_1' = -v_1 + v_2 \cos\vartheta + \omega x_2 \\ x_2' = v_2 \sin\vartheta - \omega x_1 \\ \vartheta' = \varpi - \omega \end{bmatrix}$$

Verification?

looks correct NO!

Air Traffic Control



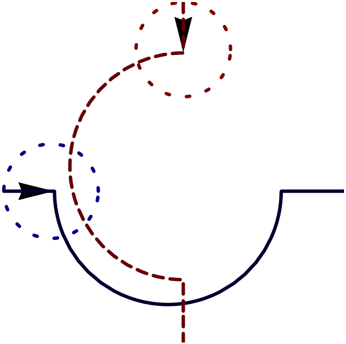


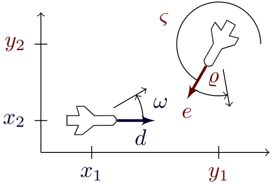
$$\left[\begin{aligned} x_1' &= -v_1 + v_2 \textcolor{red}{\cos\vartheta} + \omega x_2 \\ x_2' &= \qquad\quad v_2 \textcolor{red}{\sin\vartheta} - \omega x_1 \\ \vartheta' &= \qquad\qquad\quad \varpi - \omega \end{aligned}\right]$$

Example (“Solving” differential equations)

$$\begin{aligned} x_1(t) = \frac{1}{\omega\varpi}\big(& x_1\omega\varpi\cos t\omega - v_2\omega\cos t\omega\sin\vartheta + v_2\omega\cos t\omega\cos t\varpi\sin\vartheta - v_1\varpi\sin t\omega \\ &+ x_2\omega\varpi\sin t\omega - v_2\omega\cos\vartheta\cos t\varpi\sin t\omega - v_2\omega\sqrt{1-\sin\vartheta^2}\sin t\omega \\ &+ v_2\omega\cos\vartheta\cos t\omega\sin t\varpi + v_2\omega\sin\vartheta\sin t\omega\sin t\varpi\big)\ldots \end{aligned}$$

Air Traffic Control





$$\left[\begin{aligned} x_1' &= -v_1 + v_2 \cos\vartheta + \omega x_2 \\ x_2' &= v_2 \sin\vartheta - \omega x_1 \\ \vartheta' &= \varpi - \omega \end{aligned}\right]$$

Example ("Solving" differential equations)

$$\forall t{\geq}0 \quad \frac{1}{\omega\varpi}\big(x_1\omega\varpi\cos t\omega - v_2\omega\cos t\omega\sin\vartheta + v_2\omega\cos t\omega\cos t\varpi\sin\vartheta - v_1\varpi\sin t\omega$$
$$+ x_2\omega\varpi\sin t\omega - v_2\omega\cos\vartheta\cos t\varpi\sin t\omega - v_2\omega\sqrt{1-\sin\vartheta^2}\sin t\omega$$
$$+ v_2\omega\cos\vartheta\cos t\omega\sin t\varpi + v_2\omega\sin\vartheta\sin t\omega\sin t\varpi\big)\ldots$$

```
 \forall R ts2.
   (   0 <= ts2 & ts2 <= t2_0
    ->       (  (om_1)^-1
              * (omb_1)^-1
              * (  om_1 * omb_1 * x1 * cos(om_1 * ts2)
                 + om_1 * v2 * cos(om_1 * ts2) * (1 + -1 * (cos(u))^2)^(1 / 2)
                 + -1 * omb_1 * v1 * sin(om_1 * ts2)
                 + om_1 * omb_1 * x2 * sin(om_1 * ts2)
                 + om_1 * v2 * cos(u) * sin(om_1 * ts2)
                 + -1 * om_1 * v2 * cos(omb_1 * ts2) * cos(u) * sin(om_1 * ts2)
                 + om_1 * v2 * cos(om_1 * ts2) * cos(u) * sin(omb_1 * ts2)
                 + om_1 * v2 * cos(om_1 * ts2) * cos(omb_1 * ts2) * sin(u)
                 + om_1 * v2 * sin(om_1 * ts2) * sin(omb_1 * ts2) * sin(u)))
         ^2
        +    (  (om_1)^-1
              * (omb_1)^-1
              * (  -1 * omb_1 * v1 * cos(om_1 * ts2)
                 + om_1 * omb_1 * x2 * cos(om_1 * ts2)
                 + omb_1 * v1 * (cos(om_1 * ts2))^2
                 + om_1 * v2 * cos(om_1 * ts2) * cos(u)
                 + -1 * om_1 * v2 * cos(om_1 * ts2) * cos(omb_1 * ts2) * cos(u)
                 + -1 * om_1 * omb_1 * x1 * sin(om_1 * ts2)
                 +   -1
                   * om_1
                   * v2
                   * (1 + -1 * (cos(u))^2)^(1 / 2)
                   * sin(om_1 * ts2)
                 + omb_1 * v1 * (sin(om_1 * ts2))^2
                 + -1 * om_1 * v2 * cos(u) * sin(om_1 * ts2) * sin(omb_1 * ts2)
                 + -1 * om_1 * v2 * cos(omb_1 * ts2) * sin(om_1 * ts2) * sin(u)
                 + om_1 * v2 * cos(om_1 * ts2) * sin(omb_1 * ts2) * sin(u)))
         ^2
       >= (p)^2),
 t2_0 >= 0,
 x1^2 + x2^2 >= (p)^2
==>
```

```
\forall R t7.
  (   t7 >= 0
   ->        (  (om_3)^-1
            * (    om_3
                 * (  (om_1)^-1
                    * (omb_1)^-1
                    * (  om_1 * omb_1 * x1 * cos(om_1 * t2_0)
                       +   om_1
                         * v2
                         * cos(om_1 * t2_0)
                         * (1 + -1 * (cos(u))^2)^(1 / 2)
                       + -1 * omb_1 * v1 * sin(om_1 * t2_0)
                       + om_1 * omb_1 * x2 * sin(om_1 * t2_0)
                       + om_1 * v2 * cos(u) * sin(om_1 * t2_0)
                       +   -1
                         * om_1
                         * v2
                         * cos(omb_1 * t2_0)
                         * cos(u)
                         * sin(om_1 * t2_0)
                       +   om_1
                         * v2
                         * cos(om_1 * t2_0)
                         * cos(u)
                         * sin(omb_1 * t2_0)
                       +   om_1
                         * v2
                         * cos(om_1 * t2_0)
                         * cos(omb_1 * t2_0)
                         * sin(u)
                       +   om_1
                         * v2
                         * sin(om_1 * t2_0)
                         * sin(omb_1 * t2_0)
                         * sin(u)))
```

```
  * cos(om_3 * t5)
+   v2
  * cos(om_3 * t5)
  *   (  1
       +   -1
         * (cos(-1 * om_1 * t2_0 + omb_1 * t2_0 + u + Pi / 4))^2)
   ^(1 / 2)
+ -1 * v1 * sin(om_3 * t5)
+   om_3
  * (  (om_1)^-1
     * (omb_1)^-1
     * (  -1 * omb_1 * v1 * cos(om_1 * t2_0)
        + om_1 * omb_1 * x2 * cos(om_1 * t2_0)
        + omb_1 * v1 * (cos(om_1 * t2_0))^2
        + om_1 * v2 * cos(om_1 * t2_0) * cos(u)
        +   -1
          * om_1
          * v2
          * cos(om_1 * t2_0)
          * cos(omb_1 * t2_0)
          * cos(u)
        + -1 * om_1 * omb_1 * x1 * sin(om_1 * t2_0)
        +   -1
          * om_1
          * v2
          * (1 + -1 * (cos(u))^2)^(1 / 2)
          * sin(om_1 * t2_0)
        + omb_1 * v1 * (sin(om_1 * t2_0))^2
        +   -1
          * om_1
          * v2
          * cos(u)
          * sin(om_1 * t2_0)
          * sin(omb_1 * t2_0)
```

```
                           +   -1
                             * om_1
                             * v2
                             * cos(omb_1 * t2_0)
                             * sin(om_1 * t2_0)
                             * sin(u)
                           +   om_1
                             * v2
                             * cos(om_1 * t2_0)
                             * sin(omb_1 * t2_0)
                             * sin(u)))
                     * sin(om_3 * t5)
                   +   v2
                     * cos(-1 * om_1 * t2_0 + omb_1 * t2_0 + u + Pi / 4)
                     * sin(om_3 * t5)
                   +   v2
                     * (cos(om_3 * t5))^2
                     * sin(-1 * om_1 * t2_0 + omb_1 * t2_0 + u + Pi / 4)
                   +   v2
                     * (sin(om_3 * t5))^2
                     * sin(-1 * om_1 * t2_0 + omb_1 * t2_0 + u + Pi / 4)))
            ^2
           +   (  (om_3)^-1
                * (  -1 * v1 * cos(om_3 * t5)
                   +   om_3
                     * (  (om_1)^-1
                        * (omb_1)^-1
                        * (  -1 * omb_1 * v1 * cos(om_1 * t2_0)
                           + om_1 * omb_1 * x2 * cos(om_1 * t2_0)
                           + omb_1 * v1 * (cos(om_1 * t2_0))^2
                           + om_1 * v2 * cos(om_1 * t2_0) * cos(u)
                           +   -1
                             * om_1
                             * v2
                             * cos(om_1 * t2_0)
                             * cos(omb_1 * t2_0)
```

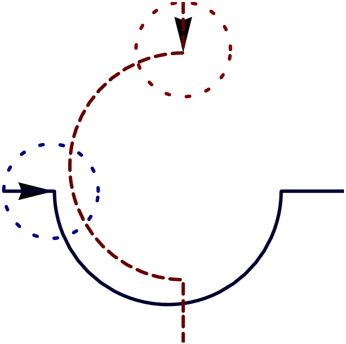
```
        + -1 * om_1 * omb_1 * x1 * sin(om_1 * t2_0)
        +   -1
          * om_1
          * v2
          * (1 + -1 * (cos(u))^2)^(1 / 2)
          * sin(om_1 * t2_0)
        + omb_1 * v1 * (sin(om_1 * t2_0))^2
        +   -1
          * om_1
          * v2
          * cos(u)
          * sin(om_1 * t2_0)
          * sin(omb_1 * t2_0)
        +   -1
          * om_1
          * v2
          * cos(omb_1 * t2_0)
          * sin(om_1 * t2_0)
          * sin(u)
        +   om_1
          * v2
          * cos(om_1 * t2_0)
          * sin(omb_1 * t2_0)
          * sin(u)))
  * cos(om_3 * t5)
+ v1 * (cos(om_3 * t5))^2
+   v2
  * cos(om_3 * t5)
  * cos(-1 * om_1 * t2_0 + omb_1 * t2_0 + u + Pi / 4)
+   -1
  * v2
  * (cos(om_3 * t5))^2
  * cos(-1 * om_1 * t2_0 + omb_1 * t2_0 + u + Pi / 4)
```

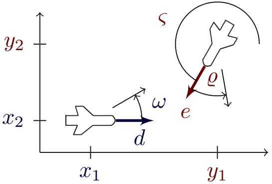
```
+   -1
  * om_3
  * (  (om_1)^-1
     * (omb_1)^-1
     * (  om_1 * omb_1 * x1 * cos(om_1 * t2_0)
        +   om_1
          * v2
          * cos(om_1 * t2_0)
          * (1 + -1 * (cos(u))^2)^(1 / 2)
        + -1 * omb_1 * v1 * sin(om_1 * t2_0)
        + om_1 * omb_1 * x2 * sin(om_1 * t2_0)
        + om_1 * v2 * cos(u) * sin(om_1 * t2_0)
        +   -1
          * om_1
          * v2
          * cos(omb_1 * t2_0)
          * cos(u)
          * sin(om_1 * t2_0)
        +   om_1
          * v2
          * cos(om_1 * t2_0)
          * cos(u)
          * sin(omb_1 * t2_0)
        +   om_1
          * v2
          * cos(om_1 * t2_0)
          * cos(omb_1 * t2_0)
          * sin(u)
        +   om_1
          * v2
          * sin(om_1 * t2_0)
          * sin(omb_1 * t2_0)
          * sin(u)))
  * sin(om_3 * t5)
```

```
         +   -1
           * v2
           *   (  1
                +   -1
                  * (cos(-1 * om_1 * t2_0 + omb_1 * t2_0 + u + Pi / 4))^2)
            ^(1 / 2)
           * sin(om_3 * t5)
         + v1 * (sin(om_3 * t5))^2
         +   -1
           * v2
           * cos(-1 * om_1 * t2_0 + omb_1 * t2_0 + u + Pi / 4)
           * (sin(om_3 * t5))^2))
  ^2
>= (p)^2)
```

This is just one branch to prove for aircraft . . .

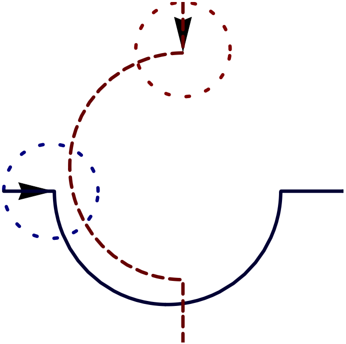
Air Traffic Control

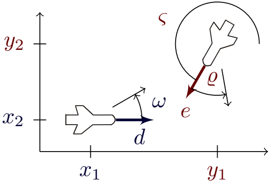




$$\begin{bmatrix} x_1' = v\cos\vartheta & & y_1' = u\cos\varsigma \\ x_2' = v\sin\vartheta & & y_2' = u\sin\varsigma \\ \vartheta' = \omega & & \varsigma' = \rho \end{bmatrix}$$

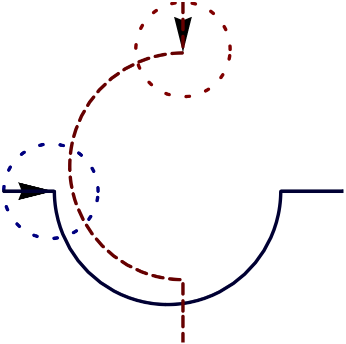
Air Traffic Control

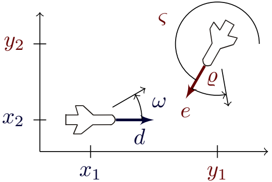




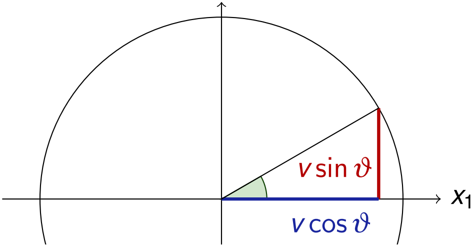
$$\begin{bmatrix} x_1' = v\cos\vartheta & y_1' = u\cos\varsigma \\ x_2' = v\sin\vartheta & y_2' = u\sin\varsigma \\ \vartheta' = \omega & \varsigma' = \rho \end{bmatrix}$$

Air Traffic Control

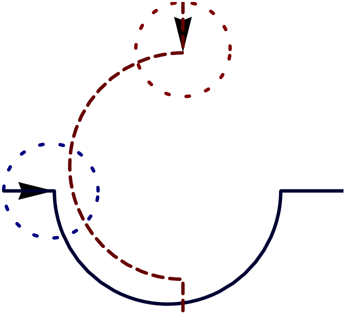


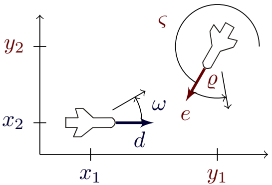


$$\begin{bmatrix} x_1' = v\cos\vartheta & & y_1' = u\cos\varsigma \\ x_2' = v\sin\vartheta & & y_2' = u\sin\varsigma \\ \vartheta' = \omega & & \varsigma' = \rho \end{bmatrix}$$

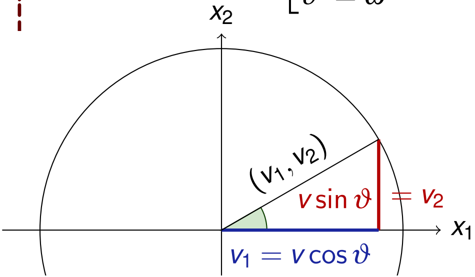


Air Traffic Control

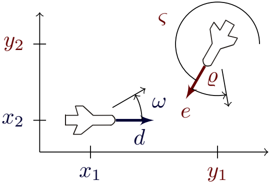




$$\begin{bmatrix} x_1' = v\cos\vartheta = v_1 & y_1' = u\cos\varsigma \\ x_2' = v\sin\vartheta = v_2 & y_2' = u\sin\varsigma \\ \vartheta' = \omega & \varsigma' = \rho \end{bmatrix}$$



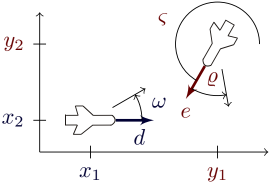
Differential Axiomatization of Flight Dynamics



$$\left[\begin{array}{ll}
x_1' = v\cos\vartheta = v_1 & y_1' = u\cos\varsigma = u_1 \\
x_2' = v\sin\vartheta = v_2 & y_2' = u\sin\varsigma = u_2 \\
v_1' = & u_1' = \\
v_2' = & u_2' = \\
\vartheta' = \omega & \varsigma' = \rho
\end{array}\right]$$

$$v_1' =$$
$$v_2' =$$

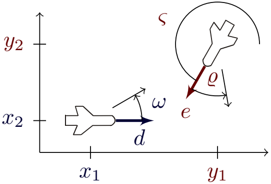
Differential Axiomatization of Flight Dynamics



$$\begin{bmatrix} x_1' = v\cos\vartheta = v_1 & y_1' = u\cos\varsigma = u_1 \\ x_2' = v\sin\vartheta = v_2 & y_2' = u\sin\varsigma = u_2 \\ v_1' = & u_1' = \\ v_2' = & u_2' = \\ \vartheta' = \omega & \varsigma' = \rho \end{bmatrix}$$

$$v_1' = (v\cos\vartheta)'$$
$$v_2' = (v\sin\vartheta)'$$

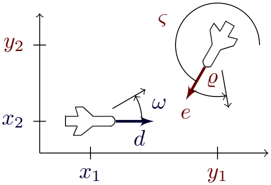
Differential Axiomatization of Flight Dynamics



$$\begin{bmatrix} x_1' = \color{blue}{v\cos\vartheta = v_1} & y_1' = u\cos\varsigma = u_1 \\ x_2' = \color{red}{v\sin\vartheta = v_2} & y_2' = u\sin\varsigma = u_2 \\ v_1' = & u_1' = \\ v_2' = & u_2' = \\ \vartheta' = \omega & \varsigma' = \rho \end{bmatrix}$$

$$v_1' = (v\cos\vartheta)' = v'\cos\vartheta + v(-\sin\vartheta)\vartheta'$$
$$v_2' = (v\sin\vartheta)' = v'\sin\vartheta + v(\cos\vartheta)\vartheta'$$

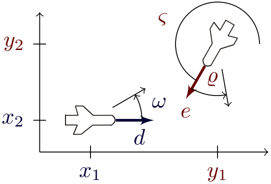
Differential Axiomatization of Flight Dynamics



$$\begin{bmatrix} x_1' = v\cos\vartheta = v_1 & y_1' = u\cos\varsigma = u_1 \\ x_2' = v\sin\vartheta = v_2 & y_2' = u\sin\varsigma = u_2 \\ v_1' = & u_1' = \\ v_2' = & u_2' = \\ \vartheta' = \omega & \varsigma' = \rho \end{bmatrix}$$

$$v_1' = (v\cos\vartheta)' = v'\cos\vartheta + v(-\sin\vartheta)\vartheta' = -(v\sin\vartheta)\omega$$
$$v_2' = (v\sin\vartheta)' = v'\sin\vartheta + v(\cos\vartheta)\vartheta' = (v\cos\vartheta)\omega$$

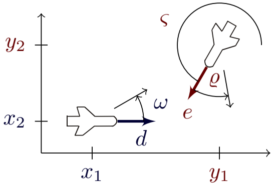
Differential Axiomatization of Flight Dynamics



$$\begin{bmatrix} x_1' = v\cos\vartheta = v_1 & y_1' = u\cos\varsigma = u_1 \\ x_2' = v\sin\vartheta = v_2 & y_2' = u\sin\varsigma = u_2 \\ v_1' = -\omega v_2 & u_1' = \\ v_2' = \quad \omega v_1 & u_2' = \\ \vartheta' = \omega & \varsigma' = \rho \end{bmatrix}$$

$$v_1' = (v\cos\vartheta)' = v'\cos\vartheta + v(-\sin\vartheta)\vartheta' = -(v\sin\vartheta)\omega = -\omega v_2$$
$$v_2' = (v\sin\vartheta)' = v'\sin\vartheta + v(\cos\vartheta)\vartheta' = (v\cos\vartheta)\omega = \omega v_1$$

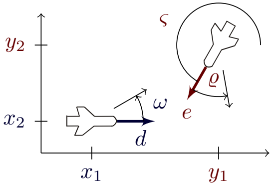
Differential Axiomatization of Flight Dynamics



$$\begin{bmatrix} x_1' = v\cos\vartheta = v_1 & y_1' = u\cos\varsigma = u_1 \\ x_2' = v\sin\vartheta = v_2 & y_2' = u\sin\varsigma = u_2 \\ v_1' = -\omega v_2 & u_1' = -\rho u_2 \\ v_2' = \omega v_1 & u_2' = \rho u_1 \\ \vartheta' = \omega & \varsigma' = \rho \end{bmatrix}$$

$$v_1' = (v\cos\vartheta)' = v'\cos\vartheta + v(-\sin\vartheta)\vartheta' = -(v\sin\vartheta)\omega = -\omega v_2$$
$$v_2' = (v\sin\vartheta)' = v'\sin\vartheta + v(\cos\vartheta)\vartheta' = (v\cos\vartheta)\omega = \omega v_1$$

Differential Axiomatization of Flight Dynamics

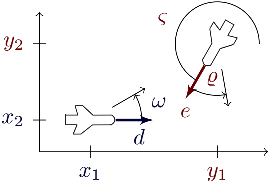


$$\begin{bmatrix} x_1' = v\cos\vartheta = v_1 & y_1' = u\cos\varsigma = u_1 \\ x_2' = v\sin\vartheta = v_2 & y_2' = u\sin\varsigma = u_2 \\ v_1' = -\omega v_2 & u_1' = -\rho u_2 \\ v_2' = \omega v_1 & u_2' = \rho u_1 \\ & \end{bmatrix}$$

$$\begin{aligned} v_1' &= (v\cos\vartheta)' = v'\cos\vartheta + v(-\sin\vartheta)\vartheta' = -(v\sin\vartheta)\omega = -\omega v_2 \\ v_2' &= (v\sin\vartheta)' = v'\sin\vartheta + v(\cos\vartheta)\vartheta' = (v\cos\vartheta)\omega = \omega v_1 \end{aligned}$$

$$v = \|(v_1, v_2)\| = \sqrt{v_1^2 + v_2^2}$$

Differential Axiomatization of Flight Dynamics



$$\begin{bmatrix} x_1' = v_1 & y_1' = u_1 \\ x_2' = v_2 & y_2' = u_2 \\ v_1' = -\omega v_2 & u_1' = -\rho u_2 \\ v_2' = \omega v_1 & u_2' = \rho u_1 \\ & \end{bmatrix}$$

$$v_1' = (v\cos\vartheta)' = v'\cos\vartheta + v(-\sin\vartheta)\vartheta' = -(v\sin\vartheta)\omega = -\omega v_2$$
$$v_2' = (v\sin\vartheta)' = v'\sin\vartheta + v(\cos\vartheta)\vartheta' = (v\cos\vartheta)\omega = \omega v_1$$

$$v = \|(v_1, v_2)\| = \sqrt{v_1^2 + v_2^2}$$

Arithmetic Ghosts

Syntax $e ::= x \mid x' \mid f(e) \mid e+k \mid e \cdot k \mid (e)'$

Syntax $\alpha ::= x := e \mid ?Q \mid x' = f(x) \,\&\, Q \mid \alpha \cup \beta \mid \alpha;\beta \mid \alpha^*$

Syntax $P ::= e \geq k \mid p(e) \mid \neg P \mid P \land Q \mid \forall x\, P \mid \exists x\, Q \mid [\alpha]P \mid \langle\alpha\rangle P$

Wait, what about . . .

Arithmetic Ghosts

Syntax	$e ::= x \mid x' \mid f(e) \mid e+k \mid e\cdot k \mid (e)'$
Syntax	$\alpha ::= x := e \mid ?Q \mid x' = f(x)\,\&\,Q \mid \alpha\cup\beta \mid \alpha;\beta \mid \alpha^*$
Syntax	$P ::= e\geq k \mid p(e) \mid \neg P \mid P\wedge Q \mid \forall x\, P \mid \exists x\, Q \mid [\alpha]P \mid \langle\alpha\rangle P$

1. $e-k$

Arithmetic Ghosts

Syntax $e ::= x \mid x' \mid f(e) \mid e+k \mid e\cdot k \mid (e)'$

Syntax $\alpha ::= x := e \mid ?Q \mid x' = f(x) \,\&\, Q \mid \alpha \cup \beta \mid \alpha;\beta \mid \alpha^*$

Syntax $P ::= e \geq k \mid p(e) \mid \neg P \mid P \wedge Q \mid \forall x\, P \mid \exists x\, Q \mid [\alpha]P \mid \langle\alpha\rangle P$

1. $e - k \overset{\text{def}}{=} e + (-1)\cdot k$

Arithmetic Ghosts

Syntax	$e ::= x \mid x' \mid f(e) \mid e+k \mid e\cdot k \mid (e)'$
Syntax	$\alpha ::= x := e \mid ?Q \mid x' = f(x)\,\&\,Q \mid \alpha\cup\beta \mid \alpha;\beta \mid \alpha^*$
Syntax	$P ::= e{\geq}k \mid p(e) \mid \neg P \mid P\wedge Q \mid \forall x\, P \mid \exists x\, Q \mid [\alpha]P \mid \langle\alpha\rangle P$

1. $e-k \overset{\text{def}}{=} e+(-1)\cdot k$
2. $-e$

Arithmetic Ghosts

Syntax $e ::= x \mid x' \mid f(e) \mid e+k \mid e\cdot k \mid (e)'$

Syntax $\alpha ::= x := e \mid ?Q \mid x' = f(x) \,\&\, Q \mid \alpha\cup\beta \mid \alpha;\beta \mid \alpha^*$

Syntax $P ::= e\geq k \mid p(e) \mid \neg P \mid P\wedge Q \mid \forall x\, P \mid \exists x\, Q \mid [\alpha]P \mid \langle\alpha\rangle P$

1. $e-k \overset{\text{def}}{=} e+(-1)\cdot k$
2. $-e \overset{\text{def}}{=} 0-e$

Arithmetic Ghosts

Syntax $e ::= x \mid x' \mid f(e) \mid e+k \mid e \cdot k \mid (e)'$

Syntax $\alpha ::= x := e \mid ?Q \mid x' = f(x) \,\&\, Q \mid \alpha \cup \beta \mid \alpha;\beta \mid \alpha^*$

Syntax $P ::= e \geq k \mid p(e) \mid \neg P \mid P \wedge Q \mid \forall x\, P \mid \exists x\, Q \mid [\alpha]P \mid \langle\alpha\rangle P$

1. $e - k \overset{\text{def}}{=} e + (-1) \cdot k$
2. $-e \overset{\text{def}}{=} 0 - e$
3. e^n

Arithmetic Ghosts

Syntax $e ::= x \mid x' \mid f(e) \mid e+k \mid e\cdot k \mid (e)'$

Syntax $\alpha ::= x := e \mid ?Q \mid x' = f(x) \,\&\, Q \mid \alpha \cup \beta \mid \alpha;\beta \mid \alpha^*$

Syntax $P ::= e \geq k \mid p(e) \mid \neg P \mid P \wedge Q \mid \forall x\, P \mid \exists x\, Q \mid [\alpha]P \mid \langle\alpha\rangle P$

1. $e - k \overset{\text{def}}{=} e + (-1)\cdot k$
2. $-e \overset{\text{def}}{=} 0 - e$
3. $e^n \overset{\text{def}}{=} e\cdot\ldots\cdot e$

Arithmetic Ghosts

Syntax $e ::= x \mid x' \mid f(e) \mid e+k \mid e\cdot k \mid (e)'$

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2. $-e \overset{\text{def}}{=} 0 - e$
3. $e^n \overset{\text{def}}{=} e\cdot\ldots\cdot e$ $n \in \mathbb{N}$ times, not e^π

Arithmetic Ghosts

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2. $-e \overset{\text{def}}{=} 0 - e$
3. $e^n \overset{\text{def}}{=} e \cdot \ldots \cdot e$ $n \in \mathbb{N}$ times, not e^π
4. e/k

Arithmetic Ghosts

Syntax $e ::= x \mid x' \mid f(e) \mid e+k \mid e\cdot k \mid (e)'$

Syntax $\alpha ::= x := e \mid ?Q \mid x' = f(x) \,\&\, Q \mid \alpha \cup \beta \mid \alpha;\beta \mid \alpha^*$

Syntax $P ::= e \geq k \mid p(e) \mid \neg P \mid P \land Q \mid \forall x\, P \mid \exists x\, Q \mid [\alpha]P \mid \langle\alpha\rangle P$

1. $e - k \overset{\text{def}}{=} e + (-1)\cdot k$
2. $-e \overset{\text{def}}{=} 0 - e$
3. $e^n \overset{\text{def}}{=} e \cdot \ldots \cdot e$ $n \in \mathbb{N}$ times, not e^π
4. $e/k \overset{\text{def}}{=}$ depends

Arithmetic Ghosts

Syntax $e ::= x \mid x' \mid f(e) \mid e+k \mid e\cdot k \mid (e)'$

Syntax $\alpha ::= x := e \mid ?Q \mid x' = f(x)\,\&\,Q \mid \alpha \cup \beta \mid \alpha;\beta \mid \alpha^*$

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3. $e^n \overset{\text{def}}{=} e\cdot\ldots\cdot e$ $n\in\mathbb{N}$ times, not e^π
4. $e/k \overset{\text{def}}{=}$ depends $\quad q=\frac{b}{c} \overset{\text{def}}{\equiv} qc=b$

Arithmetic Ghosts

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4. $e/k \overset{\text{def}}{=}$ depends $\quad q = \frac{b}{c} \overset{\text{def}}{\equiv} qc = b$

$$q := \frac{b}{c}$$

Arithmetic Ghosts / Ghosts of Division

Syntax $e ::= x \mid x' \mid f(e) \mid e+k \mid e\cdot k \mid (e)'$

Syntax $\alpha ::= x:=e \mid ?Q \mid x'=f(x)\,\&\,Q \mid \alpha\cup\beta \mid \alpha;\beta \mid \alpha^*$

Syntax $P ::= e\geq k \mid p(e) \mid \neg P \mid P\wedge Q \mid \forall x\, P \mid \exists x\, Q \mid [\alpha]P \mid \langle\alpha\rangle P$

1. $e-k \overset{\text{def}}{=} e+(-1)\cdot k$
2. $-e \overset{\text{def}}{=} 0-e$
3. $e^n \overset{\text{def}}{=} e\cdot\ldots\cdot e\ n\in\mathbb{N}$ times, not e^π
4. $e/k \overset{\text{def}}{=}$ depends $\quad q=\frac{b}{c} \overset{\text{def}}{\equiv} qc=b$

$$q:=\frac{b}{c} \quad\rightsquigarrow\quad q:=*;\ ?qc=b$$

Arithmetic Ghosts / Ghosts of Division

Syntax $e ::= x \mid x' \mid f(e) \mid e+k \mid e\cdot k \mid (e)'$

Syntax $\alpha ::= x := e \mid ?Q \mid x' = f(x)\,\&\,Q \mid \alpha\cup\beta \mid \alpha;\beta \mid \alpha^*$

Syntax $P ::= e{\geq}k \mid p(e) \mid \neg P \mid P\wedge Q \mid \forall x\, P \mid \exists x\, Q \mid [\alpha]P \mid \langle\alpha\rangle P$

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4. $e/k \overset{\text{def}}{=}$ depends $\quad q = \frac{b}{c} \overset{\text{def}}{\equiv} qc = b$ where $c \neq 0$

$$q := \frac{b}{c} \quad \rightsquigarrow \quad q := *;\ ?qc = b \wedge c \neq 0$$

Arithmetic Ghosts / Ghosts of Division

Syntax	$e ::= x \mid x' \mid f(e) \mid e+k \mid e\cdot k \mid (e)'$
Syntax	$\alpha ::= x := e \mid ?Q \mid x' = f(x) \,\&\, Q \mid \alpha \cup \beta \mid \alpha;\beta \mid \alpha^*$
Syntax	$P ::= e \geq k \mid p(e) \mid \neg P \mid P \land Q \mid \forall x\, P \mid \exists x\, Q \mid [\alpha]P \mid \langle\alpha\rangle P$

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2. $-e \overset{\text{def}}{=} 0 - e$
3. $e^n \overset{\text{def}}{=} e \cdot \ldots \cdot e$ $n \in \mathbb{N}$ times, not e^π
4. $e/k \overset{\text{def}}{=}$ depends $\quad q = \frac{b}{c} \overset{\text{def}}{\equiv} qc = b$ where $c \neq 0$

$$q := \frac{b}{c} \quad \rightsquigarrow \quad q := *;\ ?qc = b \land c \neq 0$$

$$x := 2 + \frac{b}{c} + e$$

Arithmetic Ghosts / Ghosts of Division

Syntax $e ::= x \mid x' \mid f(e) \mid e+k \mid e\cdot k \mid (e)'$

Syntax $\alpha ::= x := e \mid ?Q \mid x' = f(x)\,\&\,Q \mid \alpha\cup\beta \mid \alpha;\beta \mid \alpha^*$

Syntax $P ::= e \geq k \mid p(e) \mid \neg P \mid P\wedge Q \mid \forall x\,P \mid \exists x\,Q \mid [\alpha]P \mid \langle\alpha\rangle P$

1. $e-k \overset{\text{def}}{=} e+(-1)\cdot k$
2. $-e \overset{\text{def}}{=} 0-e$
3. $e^n \overset{\text{def}}{=} e\cdot\ldots\cdot e$ $n\in\mathbb{N}$ times, not e^π
4. $e/k \overset{\text{def}}{=}$ depends $\quad q=\frac{b}{c} \overset{\text{def}}{\equiv} qc=b$ where $c\neq 0$

$$q := \frac{b}{c} \quad\leadsto\quad q := *;\ ?qc=b\wedge c\neq 0$$

$$x := 2+\frac{b}{c}+e \quad\leadsto\quad q := *;\ ?qc=b \qquad ;\ x := 2+q+e$$

Arithmetic Ghosts / Ghosts of Division

Syntax $e ::= x \mid x' \mid f(e) \mid e+k \mid e\cdot k \mid (e)'$

Syntax $\alpha ::= x := e \mid ?Q \mid x' = f(x) \,\&\, Q \mid \alpha \cup \beta \mid \alpha;\beta \mid \alpha^*$

Syntax $P ::= e \geq k \mid p(e) \mid \neg P \mid P \land Q \mid \forall x\, P \mid \exists x\, Q \mid [\alpha]P \mid \langle\alpha\rangle P$

1. $e - k \overset{\text{def}}{=} e + (-1)\cdot k$
2. $-e \overset{\text{def}}{=} 0 - e$
3. $e^n \overset{\text{def}}{=} e\cdot\ldots\cdot e$ $n \in \mathbb{N}$ times, not e^π
4. $e/k \overset{\text{def}}{=}$ depends $\quad q = \frac{b}{c} \overset{\text{def}}{\equiv} qc = b$ where $c \neq 0$

$$q := \frac{b}{c} \quad \rightsquigarrow \quad q := *;\ ?qc = b \land c \neq 0$$

$$x := 2 + \frac{b}{c} + e \quad \rightsquigarrow \quad q := *;\ ?qc = b \land c \neq 0;\ x := 2 + q + e$$

Arithmetic Ghosts / Ghosts of Division

Syntax $e ::= x \mid x' \mid f(e) \mid e+k \mid e\cdot k \mid (e)'$

Syntax $\alpha ::= x := e \mid ?Q \mid x' = f(x)\,\&\,Q \mid \alpha\cup\beta \mid \alpha;\beta \mid \alpha^*$

Syntax $P ::= e\geq k \mid p(e) \mid \neg P \mid P\wedge Q \mid \forall x\, P \mid \exists x\, Q \mid [\alpha]P \mid \langle\alpha\rangle P$

1. $e-k \overset{\text{def}}{=} e+(-1)\cdot k$
2. $-e \overset{\text{def}}{=} 0-e$
3. $e^n \overset{\text{def}}{=} e\cdot\ldots\cdot e$ $n\in\mathbb{N}$ times, not e^π
4. $e/k \overset{\text{def}}{=}$ depends $\quad q=\frac{b}{c} \overset{\text{def}}{\equiv} qc=b$ where $c\neq 0$

$$q := \frac{b}{c} \quad\rightsquigarrow\quad q:=*;\ ?qc=b\wedge c\neq 0$$

$$x := 2+\frac{b}{c}+e \quad\rightsquigarrow\quad q:=*;\ ?qc=b\wedge c\neq 0;\ x:=2+q+e$$

$$x := a+\sqrt{4y}$$

Arithmetic Ghosts / Ghosts of Division

Syntax $e ::= x \mid x' \mid f(e) \mid e+k \mid e\cdot k \mid (e)'$

Syntax $\alpha ::= x:=e \mid ?Q \mid x'=f(x)\,\&\,Q \mid \alpha\cup\beta \mid \alpha;\beta \mid \alpha^*$

Syntax $P ::= e\geq k \mid p(e) \mid \neg P \mid P\land Q \mid \forall x\,P \mid \exists x\,Q \mid [\alpha]P \mid \langle\alpha\rangle P$

1. $e-k \overset{\text{def}}{=} e+(-1)\cdot k$
2. $-e \overset{\text{def}}{=} 0-e$
3. $e^n \overset{\text{def}}{=} e\cdot\ldots\cdot e$ $n\in\mathbb{N}$ times, not e^π
4. $e/k \overset{\text{def}}{=}$ depends $\quad q=\frac{b}{c} \overset{\text{def}}{\equiv} qc=b$ where $c\neq 0$

$$q:=\frac{b}{c} \quad\rightsquigarrow\quad q:=*;\ ?qc=b\land c\neq 0$$

$$x:=2+\frac{b}{c}+e \quad\rightsquigarrow\quad q:=*;\ ?qc=b\land c\neq 0;\ x:=2+q+e$$

$$x:=a+\sqrt{4y} \quad\rightsquigarrow\quad q:=*;\ ?q^2=4y;\ x:=a+q$$

Arithmetic Ghosts / Ghosts of Division

Syntax $e ::= x \mid x' \mid f(e) \mid e+k \mid e\cdot k \mid (e)'$

Syntax $\alpha ::= x := e \mid ?Q \mid x' = f(x) \,\&\, Q \mid \alpha \cup \beta \mid \alpha;\beta \mid \alpha^*$

Syntax $P ::= e \geq k \mid p(e) \mid \neg P \mid P \wedge Q \mid \forall x\, P \mid \exists x\, Q \mid [\alpha]P \mid \langle\alpha\rangle P$

1. $e - k \overset{\text{def}}{=} e + (-1)\cdot k$
2. $-e \overset{\text{def}}{=} 0 - e$
3. $e^n \overset{\text{def}}{=} e\cdot\ldots\cdot e$ $n \in \mathbb{N}$ times, not e^π
4. $e/k \overset{\text{def}}{=}$ depends $\quad q = \frac{b}{c} \overset{\text{def}}{\equiv} qc = b$ where $c \neq 0$

$$\begin{aligned}
q := \frac{b}{c} \quad &\leadsto \quad q := *;\ ?qc = b \wedge c \neq 0\\
x := 2 + \frac{b}{c} + e \quad &\leadsto \quad q := *;\ ?qc = b \wedge c \neq 0;\ x := 2 + q + e\\
x := a + \sqrt{4y} \quad &\leadsto \quad q := *;\ ?q^2 = 4y \wedge 4y \geq 0;\ x := a + q
\end{aligned}$$

Arithmetic Ghosts / Ghosts of Division

Syntax $e ::= x \mid x' \mid f(e) \mid e+k \mid e\cdot k \mid (e)'$

Syntax $\alpha ::= x := e \mid ?Q \mid x' = f(x) \,\&\, Q \mid \alpha \cup \beta \mid \alpha;\beta \mid \alpha^*$

Syntax $P ::= e \geq k \mid p(e) \mid \neg P \mid P \land Q \mid \forall x\, P \mid \exists x\, Q \mid [\alpha]P \mid \langle\alpha\rangle P$

1. $e - k \overset{\text{def}}{=} e + (-1)\cdot k$
2. $-e \overset{\text{def}}{=} 0 - e$
3. $e^n \overset{\text{def}}{=} e\cdot\ldots\cdot e$ $n \in \mathbb{N}$ times, not e^π
4. arithmetic ghost: auxiliary for the model ere $c \neq 0$

$$q := \frac{b}{c} \quad \rightsquigarrow \quad q := *;\ ?qc = b \land c \neq 0$$

$$x := 2 + \frac{b}{c} + e \quad \rightsquigarrow \quad q := *;\ ?qc = b \land c \neq 0;\ x := 2 + q + e$$

$$x := a + \sqrt{4y} \quad \rightsquigarrow \quad q := *;\ ?q^2 = 4y \land 4y \geq 0;\ x := a + q$$

Arithmetic Ghosts / Ghosts of Division

Syntax	$e ::= x \mid x' \mid f(e) \mid e+k \mid e\cdot k \mid (e)'$
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1. $e - k \overset{\text{def}}{=} e + (-1)\cdot k$
2. $-e \overset{\text{def}}{=} 0 - e$
3. $e^n \overset{\text{def}}{=} e\cdot\ldots\cdot e$ $n \in \mathbb{N}$ times, not e^π

nondeterministic assignment $q := *$ not in syntax

4. e/k depends $q = \frac{b}{c} = qc = b$ where $c \neq 0$

$$q := \frac{b}{c} \quad\leadsto\quad q := *;\ ?qc = b \wedge c \neq 0$$

$$x := 2 + \frac{b}{c} + e \quad\leadsto\quad q := *;\ ?qc = b \wedge c \neq 0;\ x := 2 + q + e$$

$$x := a + \sqrt{4y} \quad\leadsto\quad q := *;\ ?q^2 = 4y \wedge 4y \geq 0;\ x := a + q$$

Nondeterministic Assignments & Ghosts of Choice

Nondeterministic assignment $x := *$ not in HP syntax.

Nondeterministic Assignments & Ghosts of Choice

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Syntax $\alpha ::= \ldots \mid x := *$

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Axioms

$\langle{:}*\rangle \;\; \langle x := *\rangle P \leftrightarrow$

$[{:}*] \;\; [x := *]P \leftrightarrow$

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Axioms

$\langle{:}*\rangle \ \ \langle x := *\rangle P \leftrightarrow \exists x \, P$

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---

② Or derived definition

Nondeterministic Assignments & Ghosts of Choice

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Syntax $\alpha ::= \dots \mid x := *$

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Derived $x := * \overset{\text{def}}{\equiv}$

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Derived $x := * \overset{\text{def}}{\equiv} x' = 1 \cup x' = -1$

Derived $x := * \overset{\text{def}}{\equiv} x' = 1; x' = -1$

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discrete time?

continuous time?

---

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Axioms

$\langle :* \rangle$ $\langle x := * \rangle P \leftrightarrow \exists x\, P$

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invisible time! time is relative.

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Derived $x := * \overset{\text{def}}{\equiv} x' = 1 \cup x' = -1$

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Derived $x := * \overset{\text{def}}{\equiv} x' = 1; x' = -1$

$x := * \not\equiv x' = 1, t' = 1 \cup x' = -1, t' = 1$ visible time

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I'm just a ghost of your imagination. I'm definable.

Differential-Algebraic Ghosts

Syntax	$e ::= x \mid x' \mid f(e) \mid e+k \mid e\cdot k \mid (e)'$
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$$\{x'=\frac{2x}{c} \,\&\, c\neq 0 \land \frac{x+1}{c}>0\}$$

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$$\{x' = \frac{c}{2x} \,\&\, 2x \neq 0 \wedge \frac{c}{2x} > 0\} \not\rightsquigarrow q := *; ?q2x = 1; \{x' = cq \,\&\, 2x \neq 0 \wedge cq > 0\}$$

inverse only of initial x

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change rate of q:

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change rate of q: $\quad q' = \left(\frac{1}{2x}\right)' = \frac{-2x'}{4x^2} = \frac{-2\frac{c}{2x}}{4x^2} = -\frac{c}{4x^3}$

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still $1/x$

change rate of q: $\quad q' = \left(\frac{1}{2x}\right)' = \frac{-2x'}{4x^2} = \frac{-2\frac{c}{2x}}{4x^2} = -\frac{c}{4x^3}$

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$$\rightsquigarrow \{x' = cq, q' = * \,\&\, q2x = 1 \wedge 2x \neq 0 \wedge cq > 0\}$$

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$$\{x'=\frac{c}{2x}\,\&\,2 \quad \ldots \quad x\neq0\land cq>0\}$$

continuously changing nondeterministic value

change rate of q: $\quad q' = \left(\frac{1}{2x}\right)' = \frac{-2x'}{4x^2} = \frac{-2\frac{c}{2x}}{4x^2} = -\frac{c}{4x^3}$

$$\rightsquigarrow \{x'=cq, q' = *\,\&\,q2x=1\land 2x\neq0\land cq>0\}$$

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$\{x'=\frac{c}{2x}\,\&$

differential-algebraic ghost: auxiliary for the model

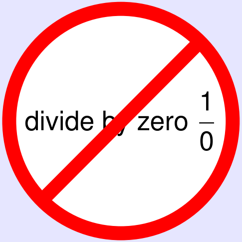
$\neq 0 \land cq>0\}$

change rate of q: $\quad q' = \left(\frac{1}{2x}\right)' = \frac{-2x'}{4x^2} = \frac{-2\frac{c}{2x}}{4x^2} = -\frac{c}{4x^3}$

$$\rightsquigarrow \{x'=cq, q' = *\,\&\,q2x=1 \land 2x\neq 0 \land cq>0\}$$

Divisions by Zero

Divisions

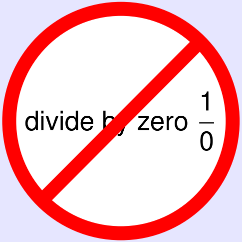


1. Scrutinize every division or possible singularity.
2. Missing requirements in the system.
3. Stopping distance $\frac{v^2}{2b}$ from initial velocity v

Don't divide by zero. It's not worth it.
~~Divide & Conquer~~ Divide & Regret

Divisions by Zero

Divisions



1. Scrutinize every division or possible singularity.
2. Missing requirements in the system.
3. Stopping distance $\frac{v^2}{2b}$ from initial velocity v
4. … needs brakes to work $b \neq 0$ though …

Don't divide by zero. It's not worth it.
~~Divide & Conquer~~ Divide & Regret

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