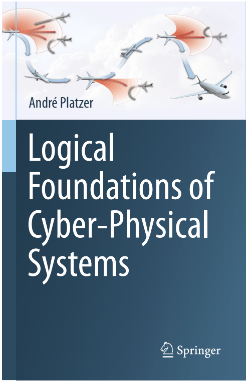
# 11: Differential Equations & Proofs

## Logical Foundations of Cyber-Physical Systems

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# Outline

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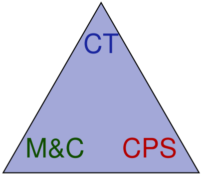
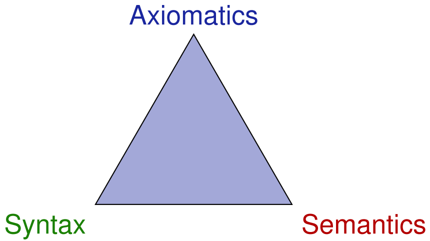
discrete vs. continuous analogy
rigorous reasoning about ODEs
beyond differential invariant terms
differential invariant formulas
cut principles for differential equations
axiomatization of ODEs
differential facet of logical trinity




understanding continuous dynamics
relate discrete+continuous

operational CPS effects
state changes along ODE

# Differential Facet of Logical Trinity




**Syntax** defines the notation
What problems are we allowed to write down?

**Semantics** what carries meaning.
What real or mathematical objects does the syntax stand for?

**Axiomatics** internalizes semantic relations into universal syntactic transformations.
How does the semantics of $e \geq \tilde{e}$ relate to semantics of $e - \tilde{e} \geq 0$, syntactically? What about derivatives?

# Outline

# Differentials

**Syntax** $e ::= x \mid x' \mid c \mid e+k \mid e \cdot k \mid (e)'$

**Semantics** $\omega[\![(e)']\!] = \sum_x \omega(x') \frac{\partial [\![e]\!]}{\partial x}(\omega)$

**Axioms**

$$(e+k)' = (e)' + (k)'$$
$$(e \cdot k)' = (e)' \cdot k + e \cdot (k)'$$
$$(c())' = 0 \qquad \text{for constants/numbers } c()$$
$$(x)' = x' \qquad \text{for variables } x \in \mathscr{V}$$

**ODE**

$$[\![x' = f(x) \,\&\, Q]\!] = \{(\varphi(0)|_{\{x'\}^\complement}, \varphi(r)) : \varphi \models x' = f(x) \land Q$$
$$\text{for some } \varphi : [0,r] \to \mathscr{S}, \text{ some } r \in \mathbb{R}\}$$

$$\varphi(z)(x') = \frac{\mathrm{d}\varphi(t)(x)}{\mathrm{d}t}(z) \qquad \ldots$$

# Differential Substitution Lemmas

**Lemma (Differential lemma)** (Differential value vs. Time-derivative)

*If* $\varphi \models x' = f(x) \land Q$ *for duration* $r>0$*, then for all* $0 \le z \le r$, $FV(e) \subseteq \{x\}$:

$$\varphi(z)[\![(e)']\!] = \frac{d\varphi(t)[\![e]\!]}{dt}(z)$$

**Lemma (Differential assignment)** (Effect on Differentials)

*If* $\varphi \models x' = f(x) \land Q$ *then* $\varphi \models P \leftrightarrow [x' := f(x)]P$

**Lemma (Derivations)** (Equations of Differentials)

$$\begin{aligned}
(e+k)' &= (e)' + (k)' \\
(e \cdot k)' &= (e)' \cdot k + e \cdot (k)' \\
(c())' &= 0 && \text{for constants/numbers } c() \\
(x)' &= x' && \text{for variables } x \in \mathcal{V}
\end{aligned}$$

**Lemma (Differential lemma)** (Differential value vs. Time-derivative)

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**Lemma (Derivations)** (Equations of Differentials)

$$
\begin{array}{ll}
+' & (e+k)' = (e)' + (k)' \\
\cdot' & (e\cdot k)' = (e)'\cdot k + e\cdot (k)' \\
c' & (c())' = 0 \\
x' & (x)' = x'
\end{array}
$$

**Lemma (Derivations)** (Equations of Differentials)

| | |
|---|---|
| $+'$ | $(e+k)' = (e)' + (k)'$ |
| $\cdot'$ | $(e \cdot k)' = (e)' \cdot k + e \cdot (k)'$ |
| $c'$ | $(c())' = 0$ |
| $x'$ | $(x)' = x'$ |

# Soundness: Proof of Derivations Lemma

## Lemma (Derivations) (Equations of Differentials)

$+'$ $\quad (e+k)' = (e)' + (k)'$

## Proof.

$$\omega[\![(e+k)']\!] =$$

$\square$

**Lemma (Derivations)** (Equations of Differentials)

$+'$ $\quad (e+k)' = (e)' + (k)'$

**Proof.**

$$\omega[\![(e+k)']\!] = \sum_x \omega(x') \frac{\partial [\![e+k]\!]}{\partial x}(\omega)$$

□

# Soundness: Proof of Derivations Lemma

**Lemma (Derivations)** (Equations of Differentials)

$+'$ $\quad (e+k)' = (e)' + (k)'$

**Proof.**

$$\omega[\![(e+k)']\!] = \sum_x \omega(x') \frac{\partial [\![e+k]\!]}{\partial x}(\omega) = \sum_x \omega(x') \frac{\partial ([\![e]\!] + [\![k]\!])}{\partial x}(\omega)$$

$\square$

# Soundness: Proof of Derivations Lemma

**Lemma (Derivations)** (Equations of Differentials)

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**Proof.**

$$\omega[\![(e+k)']\!] = \sum_x \omega(x') \frac{\partial [\![e+k]\!]}{\partial x}(\omega) = \sum_x \omega(x') \frac{\partial ([\![e]\!] + [\![k]\!])}{\partial x}(\omega)$$

$$= \sum_x \omega(x') \Big( \frac{\partial [\![e]\!]}{\partial x}(\omega) + \frac{\partial [\![k]\!]}{\partial x}(\omega) \Big)$$

$\square$

**Lemma (Derivations)** (Equations of Differentials)

$+'$ $\quad (e+k)' = (e)' + (k)'$

**Proof.**

$$\omega[\![(e+k)']\!] = \sum_x \omega(x')\frac{\partial[\![e+k]\!]}{\partial x}(\omega) = \sum_x \omega(x')\frac{\partial([\![e]\!]+[\![k]\!])}{\partial x}(\omega)$$
$$= \sum_x \omega(x')\Big(\frac{\partial[\![e]\!]}{\partial x}(\omega) + \frac{\partial[\![k]\!]}{\partial x}(\omega)\Big)$$
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$\square$

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**Proof.**

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&= \sum_x \omega(x') \Big( \frac{\partial [\![e]\!]}{\partial x}(\omega) + \frac{\partial [\![k]\!]}{\partial x}(\omega) \Big) \\
&= \sum_x \omega(x') \frac{\partial [\![e]\!]}{\partial x}(\omega) + \sum_x \omega(x') \frac{\partial [\![k]\!]}{\partial x}(\omega) \\
&= \omega[\![(e)']\!] + \omega[\![(k)']\!]
\end{aligned}
$$

□

# Soundness: Proof of Derivations Lemma

**Lemma (Derivations)** (Equations of Differentials)

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&= \omega[\![(e)']\!] + \omega[\![(k)']\!] = \omega[\![(e)'+(k)']\!]
\end{aligned}$$

$\square$

# Soundness: Proof of Derivations Lemma

## Lemma (Derivations) (Equations of Differentials)

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&= \sum_x \omega(x') \Big( \frac{\partial [\![e]\!]}{\partial x}(\omega) + \frac{\partial [\![k]\!]}{\partial x}(\omega) \Big) \\
&= \sum_x \omega(x') \frac{\partial [\![e]\!]}{\partial x}(\omega) + \sum_x \omega(x') \frac{\partial [\![k]\!]}{\partial x}(\omega) \\
&= \omega[\![(e)']\!] + \omega[\![(k)']\!] = \omega[\![(e)' + (k)']\!] \quad \text{for all } \omega
\end{aligned}$$

□

# Differential Substitution Lemmas $\rightsquigarrow$ Proofs

**Lemma (Differential lemma)** (Differential value vs. Time-derivative)

*If* $\varphi \models x' = f(x) \land Q$ *for duration* $r>0$*, then for all* $0 \le z \le r$*,* $FV(e) \subseteq \{x\}$*:*

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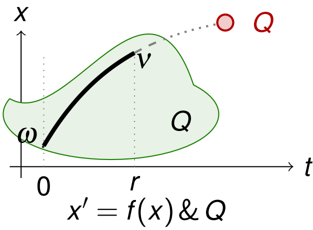


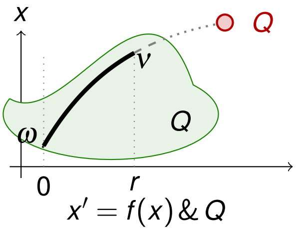

ODE

$$[\![x' = f(x) \,\&\, Q]\!] = \{(\varphi(0)|_{\{x'\}^\complement}, \varphi(r)) : \varphi \models x' = f(x) \land Q$$

$$\text{for some } \varphi : [0,r] \to \mathscr{S}, \text{ some } r \in \mathbb{R}\}$$

$$\varphi(z)(x') = \frac{\mathrm{d}\varphi(t)(x)}{\mathrm{d}t}(z)$$

# Differential Weakening

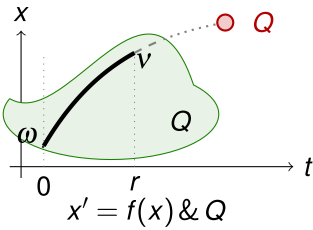



DW $[x' = f(x) \,\&\, Q]P \leftrightarrow [x' = f(x) \,\&\, Q](Q \to P)$

ODE

$$[\![x' = f(x) \,\&\, Q]\!] = \{(\varphi(0)|_{\{x'\}^\complement}, \varphi(r)) : \varphi \models x' = f(x) \wedge Q$$
$$\text{for some } \varphi : [0,r] \to \mathscr{S}, \text{ some } r \in \mathbb{R}\}$$
$$\varphi(z)(x') = \frac{\mathrm{d}\varphi(t)(x)}{\mathrm{d}t}(z)$$

Differential equations cannot leave their domains.

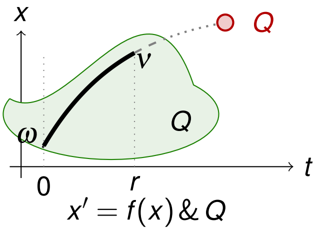



DW $[x' = f(x) \,\&\, Q]P \leftrightarrow [x' = f(x) \,\&\, Q](Q \to P)$

## Example (Bouncing ball)

$$\text{DW}\ \frac{}{\vdash [x' = v, v' = -g \,\&\, x \geq 0]\, 0 \leq x}$$

No need to solve any ODEs to prove that bouncing ball is above ground.




DW $[x' = f(x) \,\&\, Q]P \leftrightarrow [x' = f(x) \,\&\, Q](Q \to P)$

**Example (Bouncing ball)**

$$\dfrac{\text{G}\;\dfrac{}{\vdash [x' = v, v' = -g \,\&\, x \geq 0](x \geq 0 \to 0 \leq x)}}{\vdash [x' = v, v' = -g \,\&\, x \geq 0]0 \leq x}\;\text{DW}$$

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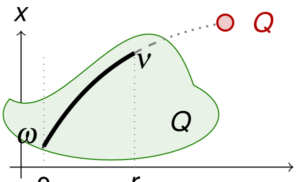
# Differential Weakening




DW $[x' = f(x) \,\&\, Q]P \leftrightarrow [x' = f(x) \,\&\, Q](Q \rightarrow P)$

## Example (Bouncing ball)

$$
\dfrac{\dfrac{\dfrac{}{\vdash x \geq 0 \rightarrow 0 \leq x}\;\mathbb{R}}{\vdash [x' = v, v' = -g \,\&\, x \geq 0](x \geq 0 \rightarrow 0 \leq x)}\;\text{G}}{\vdash [x' = v, v' = -g \,\&\, x \geq 0]\,0 \leq x}\;\text{DW}
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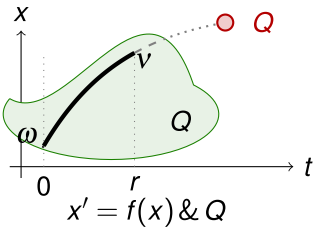
# Differential Weakening




DW $[x' = f(x) \,\&\, Q]P \leftrightarrow [x' = f(x) \,\&\, Q](Q \to P)$

## Example (Bouncing ball)

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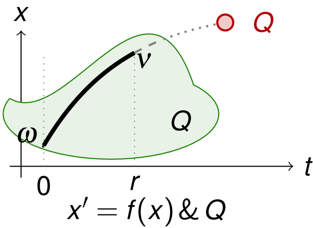
No need to solve any ODEs to prove that bouncing ball is above ground.

# Differential Weakening

## Differential Weakening

$$\text{dW}\ \frac{}{\Gamma \vdash [x' = f(x) \,\&\, Q]P, \Delta}$$




DW $[x' = f(x) \,\&\, Q]P \leftrightarrow [x' = f(x) \,\&\, Q](Q \rightarrow P)$

### Example (Bouncing ball)

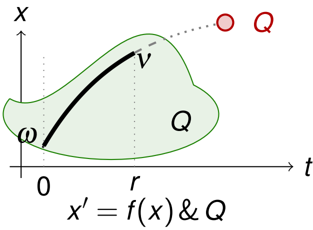
$$\frac{\frac{\frac{*}{\vdash x \geq 0 \rightarrow 0 \leq x}\ \mathbb{R}}{\vdash [x' = v, v' = -g \,\&\, x \geq 0](x \geq 0 \rightarrow 0 \leq x)}\ \text{G}}{\vdash [x' = v, v' = -g \,\&\, x \geq 0]\, 0 \leq x}\ \text{DW}$$

No need to solve any ODEs to prove that bouncing ball is above ground.

# Differential Weakening

## Differential Weakening

$$\text{dW}\ \frac{Q \vdash P}{\Gamma \vdash [x' = f(x) \,\&\, Q]P, \Delta}$$

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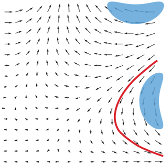
No need to solve any ODEs to prove that bouncing ball is above ground.



**Differential Invariant**

$$\text{dI}\ \frac{\vdash [x' := f(x)](e)' = 0}{e = 0 \vdash [x' = f(x) \,\&\, Q]e = 0}$$

$$\text{DI}\ \big([x' = f(x)]\, e = 0 \leftrightarrow e = 0\big) \leftarrow [x' = f(x)]\,(e)' = 0$$

$$\text{DE}\ [x' = f(x) \,\&\, Q]P \leftrightarrow [x' = f(x) \,\&\, Q][x' := f(x)]P$$

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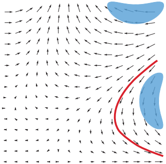

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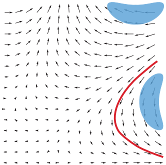

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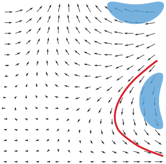
**Proof (dI is a derived rule).**

$\square$

$$\text{DI}\ \frac{}{e = 0 \vdash [x' = f(x) \,\&\, Q]e = 0}$$

# Differential Invariant Terms for Differential Equations



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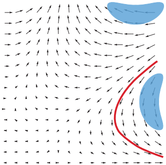

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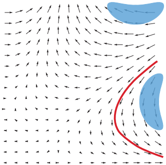

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**Proof (dI is a derived rule).**

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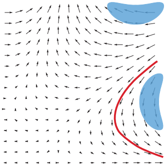
□



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DW $[x' = f(x)\,\&\,Q]P \leftrightarrow [x' = f(x)\,\&\,Q](Q \to P)$

**Proof (dI is a derived rule).**

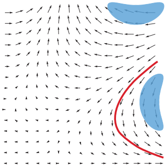
$$\cfrac{\cfrac{\cfrac{\cfrac{Q \vdash [x' := f(x)](e)' = 0}{\vdash [x' = f(x)\,\&\,Q](Q \to [x' := f(x)](e)' = 0)}\ \text{G},\!\to\!\text{R}}{\vdash [x' = f(x)\,\&\,Q][x' := f(x)](e)' = 0}\ \text{DW}}{\vdash [x' = f(x)\,\&\,Q](e)' = 0}\ \text{DE}}{e = 0 \vdash [x' = f(x)\,\&\,Q]e = 0}\ \text{DI}$$

$$\text{G}\ \frac{P}{[\alpha]P}$$

$\square$

**Lemma (Differential lemma) (Differential value vs. Time-derivative)**

$$\varphi \models x' = f(x) \land Q \text{ for } r > 0 \;\Rightarrow\; \forall 0 \le z \le r \;\; \varphi(z)[\![(e)']\!] = \frac{\mathrm{d}\varphi(t)[\![e]\!]}{\mathrm{d}t}(z)$$

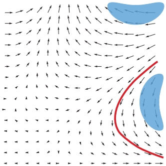


**Differential Invariant**

$$\text{dI}\;\frac{}{e = k \vdash [x' = f(x)]e = k}$$

**Lemma (Differential lemma)** (Differential value vs. Time-derivative)

$$\varphi \models x' = f(x) \wedge Q \text{ for } r > 0 \ \Rightarrow\ \forall 0 \le z \le r \ \ \varphi(z)[\![(e)']\!] = \frac{d\varphi(t)[\![e]\!]}{dt}(z)$$



**Differential Invariant**

$$\text{dI}\ \frac{\vdash [x' := f(x)](e)' = (k)'}{e = k \vdash [x' = f(x)]e = k}$$

$$\text{DI}\ \big([x' = f(x)]\, e = k \leftrightarrow e = k\big) \leftarrow [x' = f(x)]\,(e)' = (k)'$$

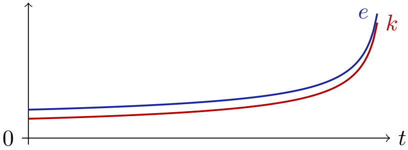
# Differential Invariant Equations

**Lemma (Differential lemma)** (Differential value vs. Time-derivative)

$$\varphi \models x' = f(x) \land Q \text{ for } r > 0 \;\Rightarrow\; \forall 0 \le z \le r \;\; \varphi(z)[\![(e)']\!] = \frac{d\varphi(t)[\![e]\!]}{dt}(z)$$

**Differential Invariant**

$$\text{dI}\ \frac{\vdash [x' := f(x)](e)' = (k)'}{e = k \vdash [x' = f(x)]e = k}$$




$$\text{DI}\ \big([x' = f(x)]\, e = k \leftrightarrow e = k\big) \leftarrow [x' = f(x)]\,(e)' = (k)'$$

**Proof** (= rate of change from = initial value. Mean-value theorem).

$$\frac{d\varphi(t)[\![e]\!]}{dt}(z) = \varphi(z)[\![(e)']\!] = \varphi(z)[\![(k)']\!] = \frac{d\varphi(t)[\![k]\!]}{dt}(z)$$ $\square$

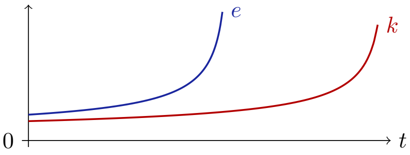
# Differential Invariant Inequalities

**Lemma (Differential lemma)** (Differential value vs. Time-derivative)

$$\varphi \models x' = f(x) \land Q \text{ for } r > 0 \;\Rightarrow\; \forall 0 \le z \le r \;\; \varphi(z)[\![ (e)' ]\!] = \frac{d\varphi(t)[\![e]\!]}{dt}(z)$$

**Differential Invariant**

$$\text{dI}\; \frac{\vdash [x' := f(x)](e)' \ge (k)'}{e \ge k \vdash [x' = f(x)] e \ge k}$$




$$\text{DI}\; \big([x' = f(x)]\, e \ge k \leftrightarrow e \ge k\big) \leftarrow [x' = f(x)]\, (e)' \ge (k)'$$

**Proof** ($\ge$ rate of change from $\ge$ initial value. Mean-value theorem).

$$\frac{d\varphi(t)[\![e]\!]}{dt}(z) = \varphi(z)[\![(e)']\!] \ge \varphi(z)[\![(k)']\!] = \frac{d\varphi(t)[\![k]\!]}{dt}(z)$$ □

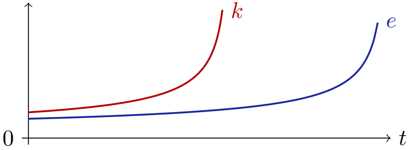
# Differential Invariant Inequalities

**Lemma (Differential lemma)** (Differential value vs. Time-derivative)

$$\varphi \models x' = f(x) \land Q \text{ for } r > 0 \;\Rightarrow\; \forall 0 \le z \le r \;\; \varphi(z)[\![(e)']\!] = \frac{d\varphi(t)[\![e]\!]}{dt}(z)$$

**Differential Invariant**

$$\text{dI}\; \frac{\vdash [x' := f(x)](e)' \le (k)'}{e \le k \vdash [x' = f(x)]e \le k}$$




$$\text{DI}\;\; \big([x' = f(x)]\, e \le k \leftrightarrow e \le k\big) \leftarrow [x' = f(x)]\,(e)' \le (k)'$$

**Proof** ($\le$ rate of change from $\le$ initial value. Mean-value theorem).

$$\frac{d\varphi(t)[\![e]\!]}{dt}(z) = \varphi(z)[\![(e)']\!] \le \varphi(z)[\![(k)']\!] = \frac{d\varphi(t)[\![k]\!]}{dt}(z)$$ □

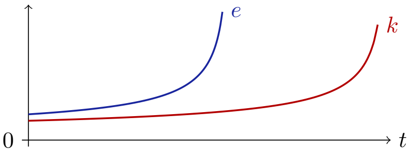
# Differential Invariant Inequalities

**Lemma (Differential lemma)** (Differential value vs. Time-derivative)

$$\varphi \models x' = f(x) \wedge Q \textit{ for } r > 0 \;\Rightarrow\; \forall 0 \le z \le r \;\; \varphi(z)[\![(e)']\!] = \frac{\mathrm{d}\varphi(t)[\![e]\!]}{\mathrm{d}t}(z)$$

**Differential Invariant**

$$\text{dI}\ \frac{\vdash [x' := f(x)](e)' > (k)'}{e > k \vdash [x' = f(x)]e > k}$$




$$\text{DI}\ \big([x' = f(x)]\,e > k \leftrightarrow e > k\big) \leftarrow [x' = f(x)]\,(e)' > (k)'$$

**Proof** ($>$ rate of change from $>$ initial value. Mean-value theorem).

$$\frac{\mathrm{d}\varphi(t)[\![e]\!]}{\mathrm{d}t}(z) = \varphi(z)[\![(e)']\!] > \varphi(z)[\![(k)']\!] = \frac{\mathrm{d}\varphi(t)[\![k]\!]}{\mathrm{d}t}(z)$$

□

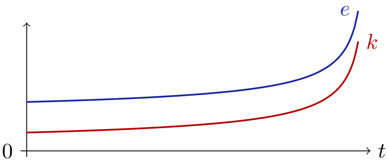
# Differential Invariant Inequalities

**Lemma (Differential lemma)** (Differential value vs. Time-derivative)

$$\varphi \models x' = f(x) \land Q \text{ for } r > 0 \;\Rightarrow\; \forall 0 \le z \le r \;\; \varphi(z)[\![(e)']\!] = \frac{d\varphi(t)[\![e]\!]}{dt}(z)$$

**Differential Invariant**

$$\text{dI}\;\frac{\vdash [x' := f(x)](e)' \ge (k)'}{e > k \vdash [x' = f(x)]e > k}$$




$$\text{DI}\;\big([x' = f(x)]\,e > k \leftrightarrow e > k\big) \leftarrow [x' = f(x)]\,(e)' \ge (k)'$$

**Proof ($\ge$ rate of change from $>$ initial value. Mean-value theorem).**

$$\frac{d\varphi(t)[\![e]\!]}{dt}(z) = \varphi(z)[\![(e)']\!] \ge \varphi(z)[\![(k)']\!] = \frac{d\varphi(t)[\![k]\!]}{dt}(z)$$

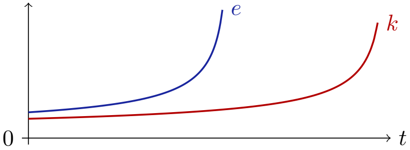
□

**Lemma (Differential lemma)** (Differential value vs. Time-derivative)

$$\varphi \models x' = f(x) \land Q \text{ for } r > 0 \;\Rightarrow\; \forall 0 \le z \le r \;\; \varphi(z)[\![(e)']\!] = \frac{d\varphi(t)[\![e]\!]}{dt}(z)$$

**Differential Invariant**

$$\text{dI}\ \frac{\vdash [x' := f(x)](e)' \neq (k)'}{e \neq k \vdash [x' = f(x)]e \neq k}$$




$$\text{DI}\ \big([x' = f(x)]\, e \neq k \leftrightarrow e \neq k\big) \leftarrow [x' = f(x)]\, (e)' \neq (k)'$$

**Proof** ($\neq$ rate of change from $\neq$ initial value. Mean-value theorem).

$$\frac{d\varphi(t)[\![e]\!]}{dt}(z) = \varphi(z)[\![(e)']\!] \neq \varphi(z)[\![(k)']\!] = \frac{d\varphi(t)[\![k]\!]}{dt}(z) \qquad \square$$

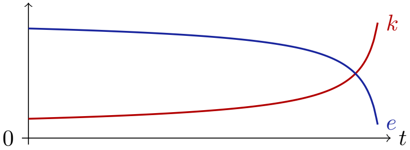
# Differential Invariant Inequalities

**Lemma (Differential lemma)** (Differential value vs. Time-derivative)

$$\varphi \models x' = f(x) \land Q \text{ for } r > 0 \;\Rightarrow\; \forall 0 \le z \le r \;\; \varphi(z)[\![(e)']\!] = \frac{d\varphi(t)[\![e]\!]}{dt}(z)$$

**Differential Invariant**

$$\text{dI}\;\frac{\vdash [x' := f(x)](e)' \neq (k)'}{e \neq k \vdash [x' = f(x)]e \neq k}$$




$$\text{DI}\;\big([x' = f(x)]\,e \neq k \leftrightarrow e \neq k\big) \leftarrow [x' = f(x)]\,(e)' \neq (k)'$$

**Proof** ($\neq$ rate of change from $\neq$ initial value. Mean-value theorem).

$$\frac{d\varphi(t)[\![e]\!]}{dt}(z) = \varphi(z)[\![(e)']\!] \neq \varphi(z)[\![(k)']\!] = \frac{d\varphi(t)[\![k]\!]}{dt}(z)$$

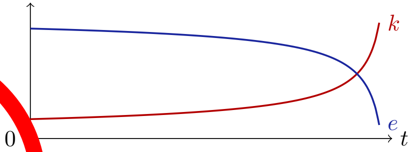
$\square$

**Lemma (Differential lemma)** (Differential value vs. Time-derivative)

$$\varphi \models x' = f(x) \land Q \text{ for } r > 0 \;\Rightarrow\; \forall 0 \le z \le r \;\; \varphi(z)[\![(e)']\!] = \frac{d\varphi(t)[\![e]\!]}{dt}(z)$$

**Differential Invariant**

$$\text{dI}\;\frac{\vdash [x' := f(x)](e)' \neq (k)'}{e \neq k \vdash [x' = f(x)]e \neq k}$$




$$\text{DI}\;\big([x' = f(x)]\,e \neq k \leftrightarrow e \neq k\big) \leftarrow [x' := f(x)](e)' \neq (k)'$$

**Proof** ($\neq$ rate of change from $\neq$ initial value. Mean-value theorem).

$$\frac{d\varphi(t)[\![e]\!]}{dt}(z) = \varphi(z)[\![(e)']\!] \neq \varphi(z)[\![(k)']\!] = \frac{d\varphi(t)[\![k]\!]}{dt}(z)$$ □

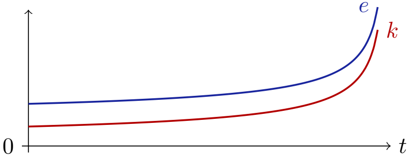
# Differential Invariant Inequalities

**Lemma (Differential lemma) (Differential value vs. Time-derivative)**

$$\varphi \models x' = f(x) \land Q \text{ for } r > 0 \;\Rightarrow\; \forall 0 \le z \le r \;\; \varphi(z)[\![(e)']\!] = \frac{d\varphi(t)[\![e]\!]}{dt}(z)$$

**Differential Invariant**

$$\text{dI}\ \frac{\vdash [x' := f(x)](e)' = (k)'}{e \neq k \vdash [x' = f(x)]e \neq k}$$




$$\text{DI}\ \big([x' = f(x)]\, e \neq k \leftrightarrow e \neq k\big) \leftarrow [x' = f(x)]\,(e)' = (k)'$$

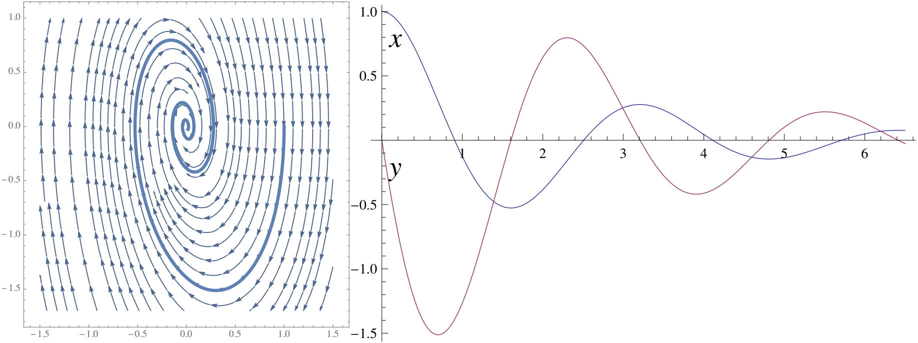
**Proof (= rate of change from ≠ initial value. Mean-value theorem).**

$$\frac{d\varphi(t)[\![e]\!]}{dt}(z) = \varphi(z)[\![(e)']\!] = \varphi(z)[\![(k)']\!] = \frac{d\varphi(t)[\![k]\!]}{dt}(z)$$

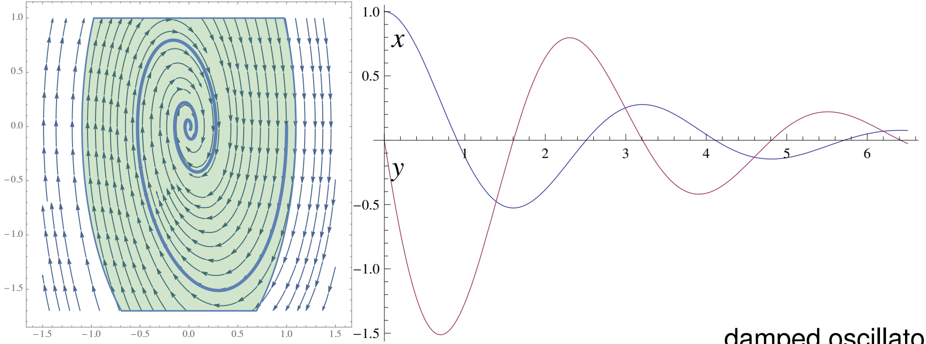
□

# Example: Differential Invariant Inequalities

$$\frac{}{\omega^2x^2+y^2\leq c^2 \vdash [x'=y, y'=-\omega^2x-2d\omega y \,\&\, \omega\geq 0 \land d\geq 0]\,\omega^2x^2+y^2\leq c^2}$$

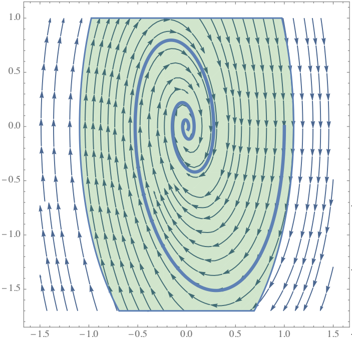


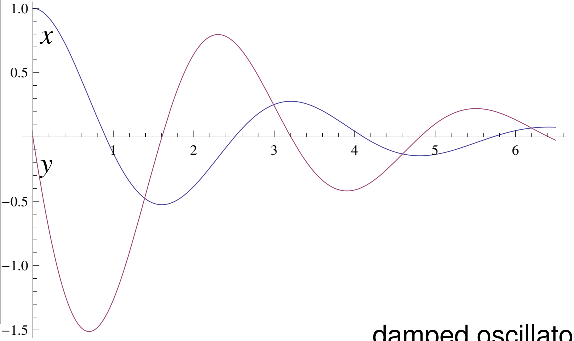
$$\frac{}{\omega^2x^2+y^2\le c^2 \vdash [x'=y, y'=-\omega^2x-2d\omega y \,\&\, \omega\ge 0 \land d\ge 0]\, \omega^2x^2+y^2\le c^2}$$




damped oscillator

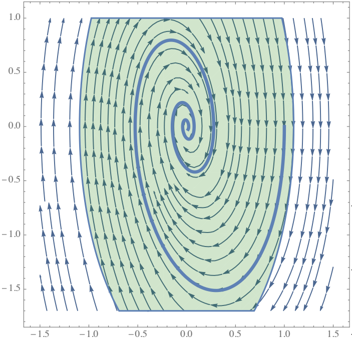
$$\dfrac{\omega\geq 0 \land d\geq 0 \vdash [x':=y][y':=-\omega^2 x - 2d\omega y]\, 2\omega^2 x x' + 2 y y' \leq 0}{\omega^2x^2+y^2\leq c^2 \vdash [x' = y, y' = -\omega^2 x - 2d\omega y \,\&\, \omega\geq 0 \land d\geq 0]\, \omega^2x^2+y^2\leq c^2}$$

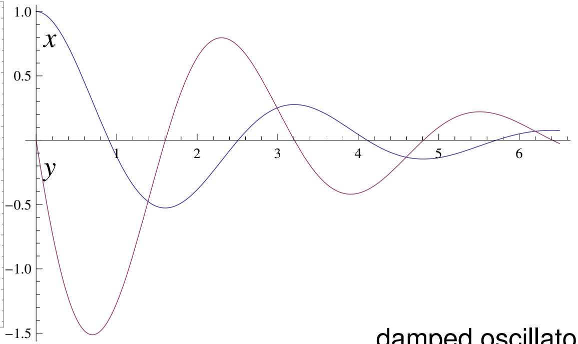



damped oscillator

$$\cfrac{\cfrac{\overline{\omega{\geq}0 \land d{\geq}0 \vdash 2\omega^2 x y + 2y(-\omega^2 x - 2d\omega y) \leq 0}}{\omega{\geq}0 \land d{\geq}0 \vdash [x'{:=}y][y'{:=}-\omega^2 x - 2d\omega y]\, 2\omega^2 x x' + 2yy' \leq 0}}{\omega^2 x^2 {+} y^2 {\leq} c^2 \vdash [x' = y, y' = -\omega^2 x - 2d\omega y \,\&\, \omega{\geq}0 \land d{\geq}0]\, \omega^2 x^2 {+} y^2 {\leq} c^2}$$

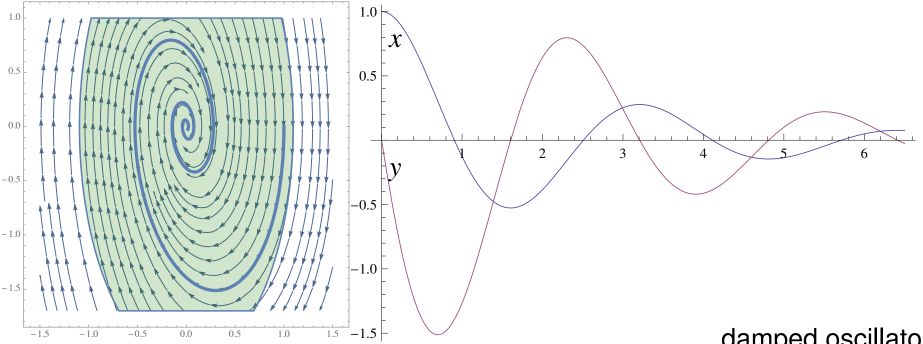



damped oscillator

# Example: Differential Invariant Inequalities: Oscillator

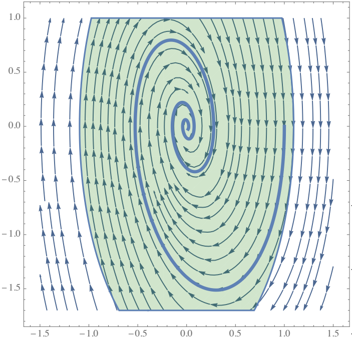
$$\dfrac{\dfrac{\dfrac{*}{\omega\geq0\land d\geq0 \vdash 2\omega^2xy+2y(-\omega^2x-2d\omega y)\leq 0}}{\omega\geq0\land d\geq0 \vdash [x':=y][y':=-\omega^2x-2d\omega y]\,2\omega^2xx'+2yy'\leq 0}}{\omega^2x^2+y^2\leq c^2 \vdash [x'=y,y'=-\omega^2x-2d\omega y\,\&\,\omega\geq0\land d\geq0]\,\omega^2x^2+y^2\leq c^2}$$

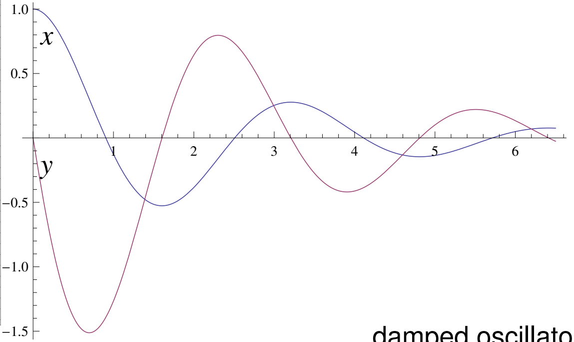



damped oscillator

# Example: Differential Invariant Inequalities: Oscillator

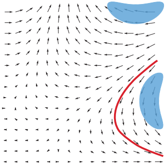
$$\dfrac{\dfrac{\dfrac{*}{\omega\geq 0 \land d\geq 0 \vdash 2\omega^2 xy + 2y(-\omega^2 x - 2d\omega y) \leq 0}}{\omega\geq 0 \land d\geq 0 \vdash [x':=y][y':=-\omega^2 x - 2d\omega y]\, 2\omega^2 x x' + 2y y' \leq 0}}{\omega^2 x^2 + y^2 \leq c^2 \vdash [x' = y, y' = -\omega^2 x - 2d\omega y \,\&\, \omega\geq 0 \land d\geq 0]\, \omega^2 x^2 + y^2 \leq c^2}$$




damped oscillator

# Differential Invariant Conjunctions

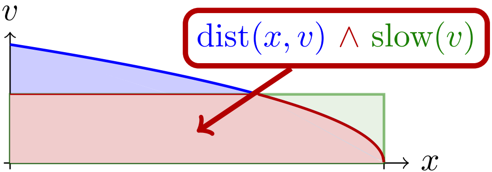


**Differential Invariant**

$$\text{dI}\ \frac{}{A \wedge B \vdash [x' = f(x)](A \wedge B)}$$

**Differential Invariant**

$$\text{dI}\ \frac{\vdash [x' := f(x)]((A)' \land (B)')}{A \land B \vdash [x' = f(x)](A \land B)}$$

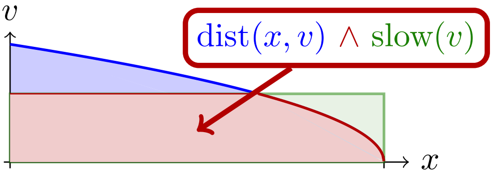



$$\text{DI}\ \big([x' = f(x)](A \land B) \leftrightarrow (A \land B)\big) \leftarrow [x' = f(x)]((A)' \land (B)')$$

# Differential Invariant Conjunctions

**Differential Invariant**

$$\text{dI}\ \frac{\vdash [x' := f(x)]((A)' \land (B)')}{A \land B \vdash [x' = f(x)](A \land B)}$$




$$\text{DI}\ \big([x' = f(x)](A \land B) \leftrightarrow (A \land B)\big) \leftarrow [x' = f(x)]((A)' \land (B)')$$

**Proof (separately).**

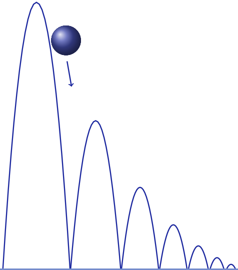
$$\text{[]}\land,\text{WL}\ \frac{\text{DI}\ \dfrac{\vdash [x' = f(x)](A)'}{A \vdash [x' = f(x)]A} \qquad \text{DI}\ \dfrac{\vdash [x' = f(x)](B)'}{B \vdash [x' = f(x)]B}}{A \land B \vdash [x' = f(x)](A \land B)}$$

$\square$

$[]\land\ \ [\alpha](P \land Q) \leftrightarrow [\alpha]P \land [\alpha]Q$

$$\overline{2gx=2gH-v^2 \vdash [x''=-g \,\&\, x\geq 0](2gx=2gH-v^2 \land x\geq 0)}$$

No solutions but still a proof.
Simple proof with simple arithmetic.



# Quantum's Back for a Differential Invariant Proof

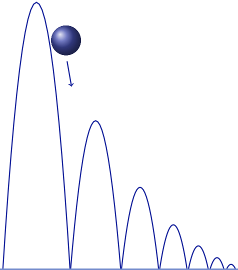
$$[]\wedge \quad [\alpha](P\wedge Q) \leftrightarrow [\alpha]P \wedge [\alpha]Q$$

$$[]\wedge\ \dfrac{\dfrac{}{2gx{=}2gH{-}v^2 \vdash [x''{=}{-}g \,\&\, x{\geq}0]2gx{=}2gH{-}v^2} \qquad \dfrac{}{\vdash [x''{=}{-}g \,\&\, x{\geq}0]x{\geq}0}}{2gx{=}2gH{-}v^2 \vdash [x'' = -g \,\&\, x{\geq}0](2gx{=}2gH{-}v^2 \wedge x{\geq}0)}$$

No solutions but still a proof.

Simple proof with simple arithmetic.

Independent proofs for independent questions.



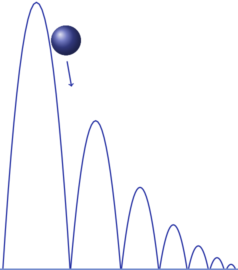
# Quantum's Back for a Differential Invariant Proof

$$\dfrac{\text{dI}\;\dfrac{\overline{x\geq0 \vdash [x':=v][v':=-g]2gx' = -2vv'}}{2gx=2gH-v^2 \vdash [x''=-g \,\&\, x\geq0]2gx=2gH-v^2} \qquad \dfrac{}{\vdash [x''=-g \,\&\, x\geq0]x\geq0}}{2gx=2gH-v^2 \vdash [x''=-g \,\&\, x\geq0](2gx=2gH-v^2 \land x\geq0)}\;[]\land$$

No solutions but still a proof.

Simple proof with simple arithmetic.

Independent proofs for independent questions.



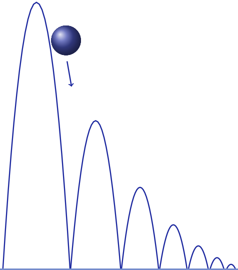
# Quantum's Back for a Differential Invariant Proof

$$
\dfrac{\dfrac{\dfrac{\overline{x\geq 0 \vdash 2g{\color{red}v} = -2v({\color{red}-g})}}{x\geq 0 \vdash [x':={\color{red}v}][v':={\color{red}-g}]2gx' = -2vv'}\,[:=]}{2gx=2gH-v^2 \vdash [x''=-g\,\&\,x\geq 0]2gx=2gH-v^2}\,\mathrm{dI} \qquad \overline{\vdash [x''=-g\,\&\,x\geq 0]x\geq 0}}{2gx=2gH-v^2 \vdash [x''=-g\,\&\,x\geq 0](2gx=2gH-v^2 \land x\geq 0)}\,[]\land
$$

No solutions but still a proof.

Simple proof with simple arithmetic.

Independent proofs for independent questions.



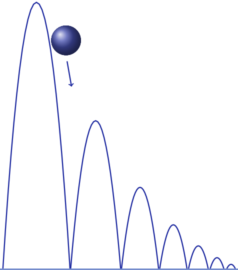
# Quantum's Back for a Differential Invariant Proof

$$
[]\wedge\;\dfrac{\text{dI}\;\dfrac{[:=]\;\dfrac{\mathbb{R}\;\dfrac{*}{x\geq 0 \vdash 2gv = -2v(-g)}}{x\geq 0 \vdash [x' := v][v' := -g]2gx' = -2vv'}}{2gx = 2gH - v^2 \vdash [x'' = -g \,\&\, x\geq 0]2gx = 2gH - v^2} \qquad \dfrac{}{\vdash [x'' = -g \,\&\, x\geq 0]x\geq 0}}{2gx = 2gH - v^2 \vdash [x'' = -g \,\&\, x\geq 0](2gx = 2gH - v^2 \wedge x\geq 0)}
$$

No solutions but still a proof.

Simple proof with simple arithmetic.

Independent proofs for independent questions.



# Quantum's Back for a Differential Invariant Proof

$$[]\land\,\dfrac{\mathrm{dI}\,\dfrac{[:=]\,\dfrac{\mathbb{R}\,\dfrac{*}{x\geq0 \vdash 2gv = -2v(-g)}}{x\geq0 \vdash [x':=v][v':=-g]2gx' = -2vv'}}{2gx=2gH-v^2 \vdash [x''=-g\,\&\,x\geq0]2gx=2gH-v^2} \qquad \mathrm{dW}\,\dfrac{\overline{x\geq0 \vdash x\geq0}}{\vdash [x''=-g\,\&\,x\geq0]x\geq0}}{2gx=2gH-v^2 \vdash [x''=-g\,\&\,x\geq0](2gx=2gH-v^2 \land x\geq0)}$$

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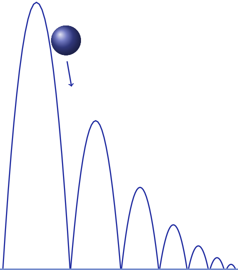

# Quantum's Back for a Differential Invariant Proof

$$
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$$

No solutions but still a proof.

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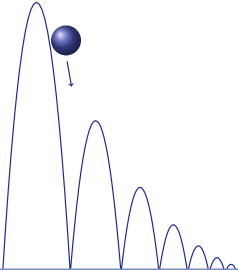
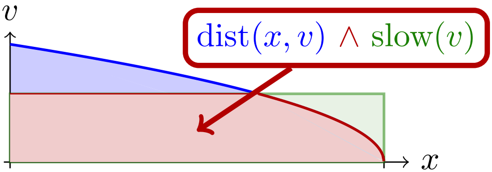


# Differential Invariant Conjunctions

**Differential Invariant**

$$\text{dI}\ \frac{\vdash [x' := f(x)]((A)' \land (B)')}{A \land B \vdash [x' = f(x)](A \land B)}$$

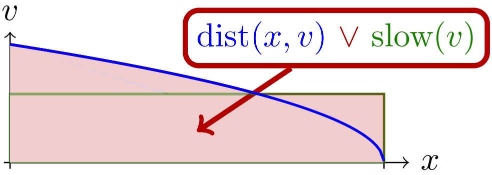



$$\text{DI}\ \big([x' = f(x)](A \land B) \leftrightarrow (A \land B)\big) \leftarrow [x' = f(x)]((A)' \land (B)')$$

# Differential Invariant Disjunctions

**Differential Invariant**

$$\text{dI}\ \frac{\vdash [x' := f(x)]((A)' \lor (B)')}{A \lor B \vdash [x' = f(x)](A \lor B)}$$

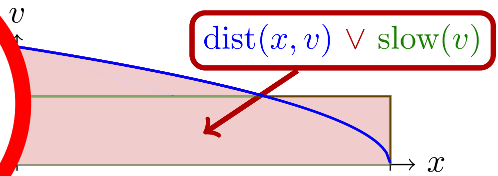



$$\text{DI}\ \big([x' = f(x)](A \lor B) \leftrightarrow (A \lor B)\big) \leftarrow [x' = f(x)]((A)' \lor (B)')$$

# Differential Invariant Disjunctions

**Differential Invariant**

$$\text{dI}\ \frac{\vdash [x' := f(x)]((A)' \lor (B)')}{A \lor B \vdash [x' = f(x)](A \lor B)}$$




$$\text{DI}\ \big([x' = f(x)](A \lor B) \leftrightarrow (A \lor B)\big) \leftarrow [x' = f(x)]((A)' \lor (B)')$$

# Differential Invariant Disjunctions

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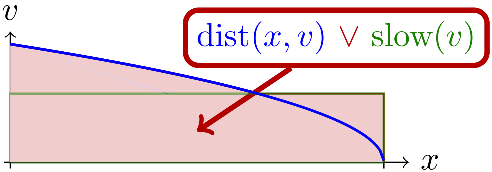


$$\text{DI}\ \big([x' = f(x)](A \lor B) \leftrightarrow (A \lor B)\big) \leftarrow [x' = f(x)]((A)' \land (B)')$$

# Differential Invariant Disjunctions

**Differential Invariant**

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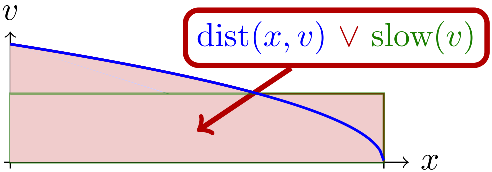



$$\text{DI}\ \big([x' = f(x)](A \lor B) \leftrightarrow (A \lor B)\big) \leftarrow [x' = f(x)]((A)' \land (B)')$$

**Proof (separately).**

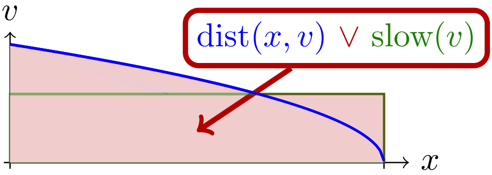
$$\vee\text{L}\ \dfrac{\text{MR}\ \dfrac{\dfrac{*}{A \vdash A \lor B} \quad \text{DI}\ \dfrac{\vdash [x'=f(x)](A)'}{A \vdash [x'=f(x)]A}}{A \vdash [x' = f(x)](A \lor B)} \quad \text{MR}\ \dfrac{\dfrac{*}{B \vdash A \lor B} \quad \text{DI}\ \dfrac{\vdash [x'=f(x)](B)'}{B \vdash [x'=f(x)]B}}{B \vdash [x' = f(x)](A \lor B)}}{A \lor B \vdash [x' = f(x)](A \lor B)}$$

□

# Differential Invariant Disjunctions

**Differential Invariant**

$$\text{dI}\ \frac{\vdash [x' := f(x)]((A)' \land (B)')}{A \lor B \vdash [x' = f(x)](A \lor B)}$$



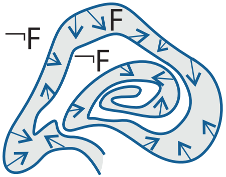

$$\text{DI}\ \big([x' = f(x)](A \lor B) \leftrightarrow (A \lor B)\big) \leftarrow [x' = f(x)]((A)' \land (B)')$$

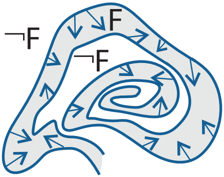
**Proof (separately).**

$$\text{∨L}\ \dfrac{\text{MR}\ \dfrac{\dfrac{*}{A \vdash A \lor B} \quad \text{DI}\ \dfrac{\vdash [x'=f(x)](A)'}{A \vdash [x'=f(x)]A}}{A \vdash [x' = f(x)](A \lor B)} \quad \text{MR}\ \dfrac{\dfrac{*}{B \vdash A \lor B} \quad \text{DI}\ \dfrac{\vdash [x'=f(x)](B)'}{B \vdash [x'=f(x)]B}}{B \vdash [x' = f(x)](A \lor B)}}{A \lor B \vdash [x' = f(x)](A \lor B)}$$

□

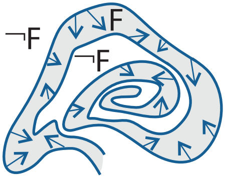
$$[]\land\ \ [\alpha](P \land Q) \leftrightarrow [\alpha]P \land [\alpha]Q$$



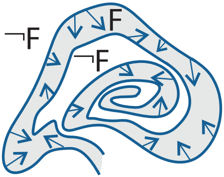




$$\frac{Q \rightarrow [x' := f(x)](F)'}{F \vdash [x' = f(x) \,\&\, Q]F} \qquad \frac{\color{red}{F} \land Q \rightarrow [x' := f(x)](F)'}{F \vdash [x' = f(x) \,\&\, Q]F}$$

$$\text{loop} \; \frac{\color{red}{F} \vdash [\alpha]F}{F \vdash [\alpha^*]F}$$



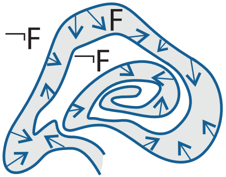

$$\frac{Q \rightarrow [x' := f(x)](F)'}{F \vdash [x' = f(x) \,\&\, Q]F}$$

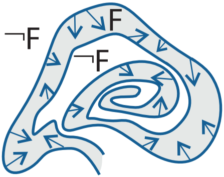



$$\frac{F \land Q \rightarrow [x' := f(x)](F)'}{F \vdash [x' = f(x) \,\&\, Q]F}$$

## Example (Restrictions)

$$\frac{}{v^2 - 2v + 1 = 0 \vdash [v' = w, w' = -v]\, v^2 - 2v + 1 = 0}$$



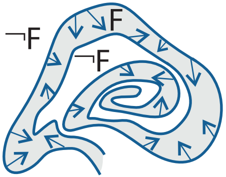

$$\frac{Q \to [x' := f(x)](F)'}{F \vdash [x' = f(x) \,\&\, Q]F}$$

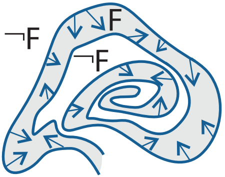
$$\frac{F \wedge Q \to [x' := f(x)](F)'}{F \vdash [x' = f(x) \,\&\, Q]F}$$

## Example (Restrictions)

$$\frac{\dfrac{}{v^2 - 2v + 1 = 0 \vdash [v' := w][w' := -v]\, 2vv' - 2v' = 0}}{v^2 - 2v + 1 = 0 \vdash [v' = w, w' = -v]\, v^2 - 2v + 1 = 0}$$

# Assuming Invariants



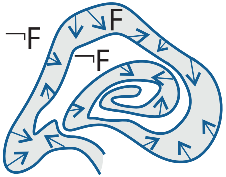



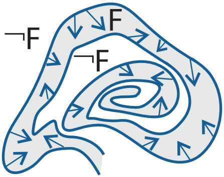

$$\frac{Q \to [x' := f(x)](F)'}{F \vdash [x' = f(x) \,\&\, Q]F}$$

$$\frac{F \land Q \to [x' := f(x)](F)'}{F \vdash [x' = f(x) \,\&\, Q]F}$$

## Example (Restrictions)

$$\cfrac{\cfrac{\cfrac{}{v^2 - 2v + 1 = 0 \vdash 2vw - 2w = 0}}{v^2 - 2v + 1 = 0 \vdash [v' := w][w' := -v]\, 2vv' - 2v' = 0}}{v^2 - 2v + 1 = 0 \vdash [v' = w, w' = -v]\, v^2 - 2v + 1 = 0}$$




$$\frac{Q \to [x' := f(x)](F)'}{F \vdash [x' = f(x) \,\&\, Q]F}$$

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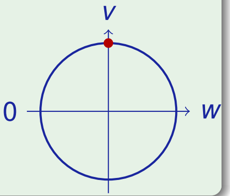
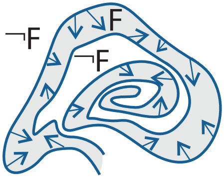
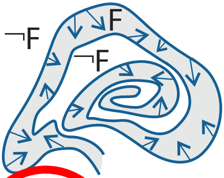





$$\frac{Q \to [x' := f(x)](F)'}{F \vdash [x' = f(x) \,\&\, Q]F}$$

$$\frac{{\color{red}F} \wedge Q \to [x' := f(x)](F)'}{F \vdash [x' = f(x) \,\&\, Q]F}$$

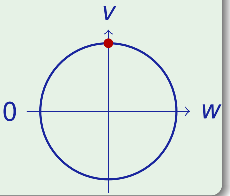
## Example (Restrictions are unsound!)

(unsound)

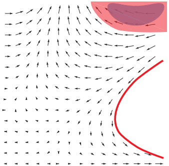
$$v^2 - 2v + 1 = 0 \vdash 2vw - 2w = 0$$

$$v^2 - 2v + 1 = 0 \vdash [v' := w][w' := -v]\, 2vv' - 2v' = 0$$

$$v^2 - 2v + 1 = 0 \vdash [v' = w, w' = -v]\, v^2 - 2v + 1 = 0$$



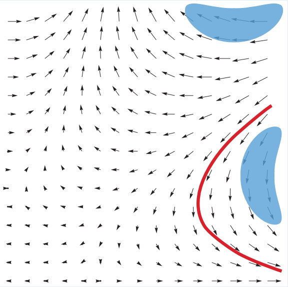
# Outline



## Differential Cut

$$\frac{}{F \vdash [x' = f(x)]F}$$

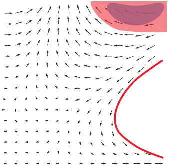
## Differential Cut



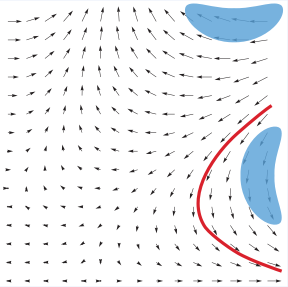
# Differential Cuts

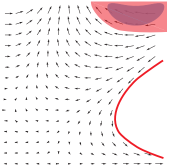
## Differential Cut

$$\frac{F \vdash [x' = f(x)]C}{F \vdash [x' = f(x)]F}$$



## Differential Cut

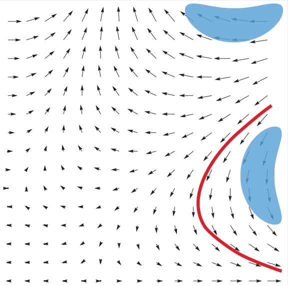


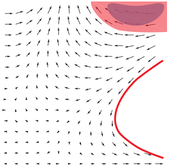


### Differential Cut

$$\frac{F \vdash [x' = f(x)]C \qquad F \vdash [x' = f(x)\,\&\,C]F}{F \vdash [x' = f(x)]F}$$

### Differential Cut

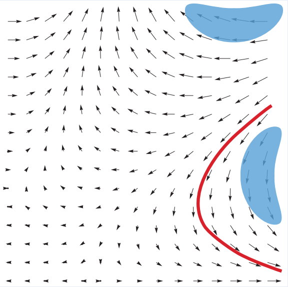




### Differential Cut

$$\frac{F \vdash [x' = f(x)\,\&\,Q]C \quad F \vdash [x' = f(x)\,\&\,Q \land C]F}{F \vdash [x' = f(x)\,\&\,Q]F}$$

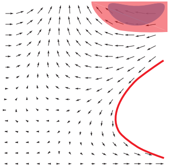
### Differential Cut



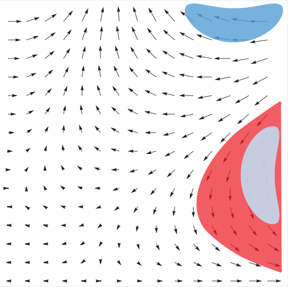
# Differential Cuts

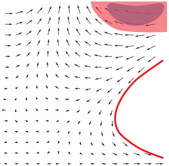
**Differential Cut**

$$\frac{F \vdash [x' = f(x) \,\&\, Q]C \quad F \vdash [x' = f(x) \,\&\, Q \land C]F}{F \vdash [x' = f(x) \,\&\, Q]F}$$



**Differential Cut**

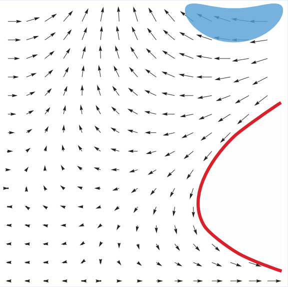


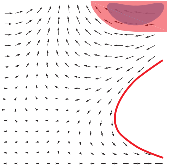


## Differential Cut

$$\frac{F \vdash [x' = f(x) \,\&\, Q]C \quad F \vdash [x' = f(x) \,\&\, Q \land C]F}{F \vdash [x' = f(x) \,\&\, Q]F}$$

## Differential Cut

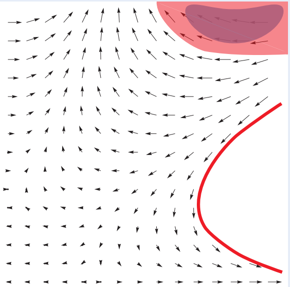




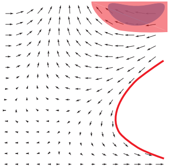
## Differential Cut

$$\frac{F \vdash [x' = f(x)\,\&\,Q]C \quad F \vdash [x' = f(x)\,\&\,Q \land C]F}{F \vdash [x' = f(x)\,\&\,Q]F}$$

## Differential Cut



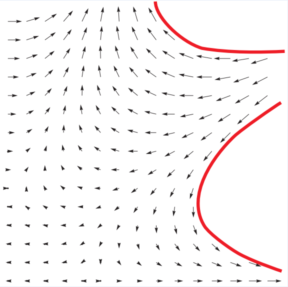
# Differential Cuts



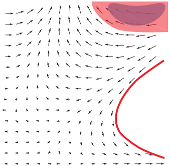
## Differential Cut

$$\frac{F \vdash [x' = f(x) \,\&\, Q]C \quad F \vdash [x' = f(x) \,\&\, Q \wedge C]F}{F \vdash [x' = f(x) \,\&\, Q]F}$$

## Differential Cut



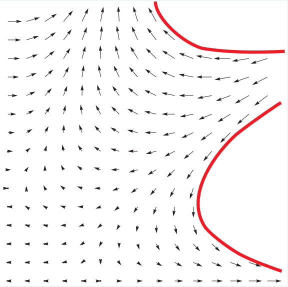
# Differential Cuts



**Differential Cut**

$$\frac{F \vdash [x' = f(x) \,\&\, Q]C \quad F \vdash [x' = f(x) \,\&\, Q \wedge C]F}{F \vdash [x' = f(x) \,\&\, Q]F}$$

**Differential Cut**



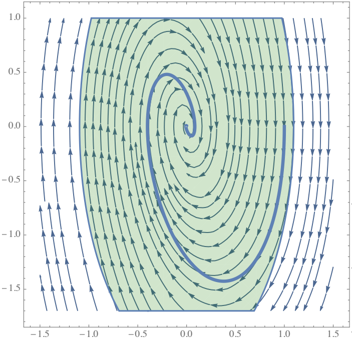
**Proof (Soundness).**

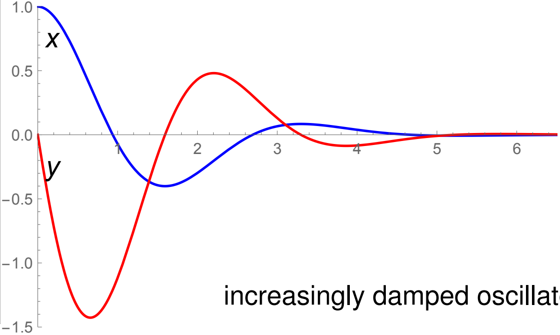
Let $\varphi \models x' = f(x) \wedge Q$ starting in $\omega \in [\![F]\!]$.
$\omega \in [\![[x' = f(x) \,\&\, Q]C]\!]$ by left premise.
Thus, $\varphi \models x' = f(x) \wedge Q \wedge C$.
Thus, $\varphi(r) \in [\![F]\!]$ by second premise. $\square$

# Differential Cut Example: Increasingly Damped Oscillator

$$\text{dC}\;\frac{}{\omega^2x^2+y^2\leq c^2 \vdash [x'=y, y'=-\omega^2x-2d\omega y, d'=7 \,\&\, \omega\geq 0]\,\omega^2x^2+y^2\leq c^2}$$

$$\text{dC}\ \frac{}{\omega^2x^2+y^2\le c^2 \vdash [x'=y, y'=-\omega^2x-2d\omega y, d'=7 \,\&\, \omega\ge 0]\,\omega^2x^2+y^2\le c^2}$$





# Differential Cut Example: Increasingly Damped Oscillator

$$\text{dI}\ \dfrac{}{\omega^2x^2+y^2\le c^2 \vdash [x'=y, y'=-\omega^2x-2d\omega y, d'=7\ \&\ \omega\ge 0\land d\ge 0]\, \omega^2x^2+y^2\le c^2}$$

$$\text{dC}\ \dfrac{}{\omega^2x^2+y^2\le c^2 \vdash [x'=y, y'=-\omega^2x-2d\omega y, d'=7\ \&\ \omega\ge 0]\, \omega^2x^2+y^2\le c^2}$$

increasingly damped oscillator

$$\text{dC}\,\dfrac{\text{dI}\,\dfrac{}{\omega^2x^2+y^2\le c^2 \vdash [x'=y, y'=-\omega^2x-2d\omega y, d'=7\,\&\,\omega\ge0\land d\ge0]\,\omega^2x^2+y^2\le c^2}}{\omega^2x^2+y^2\le c^2 \vdash [x'=y, y'=-\omega^2x-2d\omega y, d'=7\,\&\,\omega\ge0]\,\omega^2x^2+y^2\le c^2}$$

$$\text{dI}\,\dfrac{}{d\ge0 \vdash [x'=y, y'=-\omega^2x-2d\omega y, d'=7\,\&\,\omega\ge0]\,d\ge0}$$

increasingly damped oscillator

# Differential Cut Example: Increasingly Damped Oscillator

$$\dfrac{\mathrm{dI}\;\dfrac{}{\omega^2x^2+y^2\le c^2 \vdash [x'=y, y'=-\omega^2x-2d\omega y, d'=7 \,\&\, \omega\ge0 \land d\ge0]\, \omega^2x^2+y^2\le c^2}}{\omega^2x^2+y^2\le c^2 \vdash [x'=y, y'=-\omega^2x-2d\omega y, d'=7 \,\&\, \omega\ge0]\, \omega^2x^2+y^2\le c^2}\;\mathrm{dC}$$

$$\dfrac{[:=]\;\dfrac{}{\omega\ge0 \vdash [d':=7]\, d'\ge0}}{d\ge0 \vdash [x'=y, y'=-\omega^2x-2d\omega y, d'=7 \,\&\, \omega\ge0]\, d\ge0}\;\mathrm{dI}$$

increasingly damped oscillator

# Differential Cut Example: Increasingly Damped Oscillator

$$\dfrac{\dfrac{}{\omega^2x^2+y^2\le c^2 \vdash [x'=y, y'=-\omega^2x-2d\omega y, d'=7\,\&\,\omega\ge0\land d\ge0]\,\omega^2x^2+y^2\le c^2}\text{dI}}{\omega^2x^2+y^2\le c^2 \vdash [x'=y, y'=-\omega^2x-2d\omega y, d'=7\,\&\,\omega\ge0]\,\omega^2x^2+y^2\le c^2}\text{dC}$$

$$\dfrac{\dfrac{\dfrac{}{\omega\ge0 \vdash 7\ge0}\mathbb{R}}{\omega\ge0 \vdash [d':=7]\,d'\ge0}[:=]}{d\ge0 \vdash [x'=y, y'=-\omega^2x-2d\omega y, d'=7\,\&\,\omega\ge0]\,d\ge0}\text{dI}$$

increasingly damped oscillator

# Differential Cut Example: Increasingly Damped Oscillator

$$\text{dC}\;\dfrac{\text{dI}\;\dfrac{}{\omega^2x^2+y^2\le c^2 \vdash [x'=y, y'=-\omega^2x-2d\omega y, d'=7\,\&\,\omega\ge 0\land d\ge 0]\,\omega^2x^2+y^2\le c^2}}{\omega^2x^2+y^2\le c^2 \vdash [x'=y, y'=-\omega^2x-2d\omega y, d'=7\,\&\,\omega\ge 0]\,\omega^2x^2+y^2\le c^2}$$

ask

$$\text{dI}\;\dfrac{[:=]\;\dfrac{\mathbb{R}\;\dfrac{*}{\omega\ge 0 \vdash 7\ge 0}}{\omega\ge 0 \vdash [d':=7]\,d'\ge 0}}{d\ge 0 \vdash [x'=y, y'=-\omega^2x-2d\omega y, d'=7\,\&\,\omega\ge 0]\,d\ge 0}$$

increasingly damped oscillator

$$\dfrac{\dfrac{\dfrac{}{\omega\geq 0\land d\geq 0 \vdash [x':=y][y':=-\omega^2x-2d\omega y]\,2\omega^2 x x' + 2 y y' \leq 0}\;[:=]}{\omega^2x^2+y^2\leq c^2 \vdash [x'=y, y'=-\omega^2x-2d\omega y, d'=7\,\&\,\omega\geq 0\land d\geq 0]\,\omega^2x^2+y^2\leq c^2}\;\mathrm{dI}}{\omega^2x^2+y^2\leq c^2 \vdash [x'=y, y'=-\omega^2x-2d\omega y, d'=7\,\&\,\omega\geq 0]\,\omega^2x^2+y^2\leq c^2}\;\mathrm{dC}$$

$$\dfrac{\dfrac{\dfrac{*}{\omega\geq 0 \vdash 7\geq 0}\;\mathbb{R}}{\omega\geq 0 \vdash [d':=7]\,d'\geq 0}\;[:=]}{d\geq 0 \vdash [x'=y, y'=-\omega^2x-2d\omega y, d'=7\,\&\,\omega\geq 0]\,d\geq 0}\;\mathrm{dI}$$

increasingly damped oscillator

# Differential Cut Example: Increasingly Damped Oscillator

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\omega\geq 0\land d\geq 0 \vdash 2\omega^2 x y + 2y(-\omega^2 x - 2d\omega y) \leq 0}\mathbb{R}}{\omega\geq 0\land d\geq 0 \vdash [x':=y][y':=-\omega^2 x - 2d\omega y]\, 2\omega^2 x x' + 2 y y' \leq 0}[:=]}{\omega^2x^2+y^2\leq c^2 \vdash [x'=y, y'=-\omega^2x-2d\omega y, d'=7\ \&\ \omega\geq 0\land d\geq 0]\,\omega^2x^2+y^2\leq c^2}\mathrm{dI}}{\omega^2x^2+y^2\leq c^2 \vdash [x'=y, y'=-\omega^2x-2d\omega y, d'=7\ \&\ \omega\geq 0]\,\omega^2x^2+y^2\leq c^2}\mathrm{dC}$$

$$\dfrac{\dfrac{\dfrac{*}{\omega\geq 0 \vdash 7\geq 0}\mathbb{R}}{\omega\geq 0 \vdash [d':=7]\, d'\geq 0}[:=]}{d\geq 0 \vdash [x' = y, y' = -\omega^2 x - 2d\omega y, d'=7\ \&\ \omega\geq 0]\, d\geq 0}\mathrm{dI}$$

increasingly damped oscillator

# Differential Cut Example: Increasingly Damped Oscillator

$$\text{dC}\;\dfrac{\text{dI}\;\dfrac{[:=]\;\dfrac{\mathbb{R}\;\dfrac{*}{\omega\geq0\land d\geq0 \vdash 2\omega^2xy+2y(-\omega^2x-2d\omega y)\leq0}}{\omega\geq0\land d\geq0 \vdash [x':=y][y':=-\omega^2x-2d\omega y]\,2\omega^2xx'+2yy'\leq0}}{\omega^2x^2+y^2\leq c^2 \vdash [x'=y,y'=-\omega^2x-2d\omega y,d'=7\,\&\,\omega\geq0\land d\geq0]\,\omega^2x^2+y^2\leq c^2}}{\omega^2x^2+y^2\leq c^2 \vdash [x'=y,y'=-\omega^2x-2d\omega y,d'=7\,\&\,\omega\geq0]\,\omega^2x^2+y^2\leq c^2}$$

DC

$$\text{dI}\;\dfrac{[:=]\;\dfrac{\mathbb{R}\;\dfrac{*}{\omega\geq0 \vdash 7\geq0}}{\omega\geq0 \vdash [d':=7]\,d'\geq0}}{d\geq0 \vdash [x'=y,y'=-\omega^2x-2d\omega y,d'=7\,\&\,\omega\geq0]\,d\geq0}$$

increasingly damped oscillator

$$\text{dC}\;\dfrac{\text{dI}\;\dfrac{[:=]\;\dfrac{\mathbb{R}\;\dfrac{*}{\omega\geq0\land d\geq0 \vdash 2\omega^2xy+2y(-\omega^2x-2d\omega y)\leq 0}}{\omega\geq0\land d\geq0 \vdash [x':=y][y':=-\omega^2x-2d\omega y]\,2\omega^2xx'+2yy'\leq 0}}{\omega^2x^2+y^2\leq c^2 \vdash [x'=y,y'=-\omega^2x-2d\omega y, d'=7\,\&\,\omega\geq0\land d\geq0]\,\omega^2x^2+y^2\leq c^2}}{\omega^2x^2+y^2\leq c^2 \vdash [x'=y,y'=-\omega^2x-2d\omega y, d'=7\,\&\,\omega\geq0]\,\omega^2x^2+y^2\leq c^2}$$

init

$$\text{dI}\;\dfrac{[:=]\;\dfrac{\mathbb{R}\;\dfrac{*}{\omega\geq0 \vdash 7\geq0}}{\omega\geq0 \vdash [d':=7]\,d'\geq0}}{d\geq0 \vdash [x'=y,y'=-\omega^2x-2d\omega y, d'=7\,\&\,\omega\geq0]\,d\geq0}$$

# Differential Cut Example: Increasingly Damped Oscillator

$$
\begin{array}{rl}
 & \hspace{6em} * \\ \hline
\mathbb{R} & \omega\geq0\land d\geq0 \vdash 2\omega^2xy + 2y(-\omega^2x - 2d\omega y) \leq 0 \\ \hline
[:=] & \omega\geq0\land d\geq0 \vdash [x'{:=}y][y'{:=}-\omega^2x - 2d\omega y]\,2\omega^2xx' + 2yy' \leq 0 \\ \hline
\text{dI} & \omega^2x^2+y^2\leq c^2 \vdash [x'=y, y'=-\omega^2x-2d\omega y, d'=7\,\&\,\omega\geq0\land d\geq0]\,\omega^2x^2+y^2\leq c^2 \\ \hline
\text{dC} & \omega^2x^2+y^2\leq c^2 \vdash [x'=y, y'=-\omega^2x-2d\omega y, d'=7\,\&\,\omega\geq0]\,\omega^2x^2+y^2\leq c^2
\end{array}
$$

init

$$
\begin{array}{rl}
 & \hspace{6em} * \\ \hline
\mathbb{R} & \omega\geq0 \vdash 7\geq0 \\ \hline
[:=] & \omega\geq0 \vdash [d'{:=}7]\,d'\geq0 \\ \hline
\text{dI} & d\geq0 \vdash [x' = y, y' = -\omega^2x - 2d\omega y, d'=7\,\&\,\omega\geq0]\,d\geq0
\end{array}
$$

Could repeatedly diffcut in formulas to help the proof

# Ex: Differential Cuts

$$\text{dC}\ \frac{}{x^3 \geq -1 \land y^5 \geq 0 \vdash [x' = (x-2)^4 + y^5, y' = y^2]x^3 \geq -1}$$

$$\text{dC}\ \frac{}{x^3 \geq -1 \land y^5 \geq 0 \vdash [x' = (x-2)^4 + y^5, y' = y^2]x^3 \geq -1}$$

$$\text{dI}\ \frac{}{y^5 \geq 0 \vdash [x' = (x-2)^4 + y^5, y' = y^2]y^5 \geq 0}$$

$$\text{dC}\ \frac{}{x^3 \geq -1 \land y^5 \geq 0 \vdash [x' = (x-2)^4 + y^5, y' = y^2]x^3 \geq -1}$$

$$\text{dI}\ \dfrac{[:=]\ \dfrac{}{\vdash [x' := (x-2)^4 + y^5][y' := y^2]5y^4 y' \geq 0}}{y^5 \geq 0 \vdash [x' = (x-2)^4 + y^5, y' = y^2]y^5 \geq 0}$$

$$\text{dC}\ \frac{}{x^3 \geq -1 \land y^5 \geq 0 \vdash [x' = (x-2)^4 + y^5, y' = y^2]x^3 \geq -1}$$

$$\text{dI}\ \dfrac{[:=]\ \dfrac{\mathbb{R}\ \dfrac{}{\vdash 5y^4 y^2 \geq 0}}{\vdash [x':=(x-2)^4 + y^5][y':=y^2]5y^4 y' \geq 0}}{y^5 \geq 0 \vdash [x' = (x-2)^4 + y^5, y' = y^2]y^5 \geq 0}$$

$$\text{dC}\ \frac{}{x^3 \geq -1 \land y^5 \geq 0 \vdash [x' = (x-2)^4 + y^5, y' = y^2]x^3 \geq -1}$$

$$\text{dI}\ \dfrac{[:=]\ \dfrac{\mathbb{R}\ \dfrac{*}{\vdash 5y^4y^2 \geq 0}}{\vdash [x':=(x-2)^4+y^5][y':=y^2]5y^4y' \geq 0}}{y^5 \geq 0 \vdash [x' = (x-2)^4 + y^5, y' = y^2]\color{red}{y^5 \geq 0}}$$

$$\dfrac{\text{dI}\;\dfrac{}{x^3 \geq -1 \vdash [x' = (x-2)^4 + y^5, y' = y^2 \,\&\, y^5 \geq 0]x^3 \geq -1 \;\triangleright}}{\text{dC}\; x^3 \geq -1 \land {\color{red}y^5 \geq 0} \vdash [x' = (x-2)^4 + y^5, y' = y^2]x^3 \geq -1}$$

$$\dfrac{\text{[:=]}\;\dfrac{\mathbb{R}\;\dfrac{*}{\vdash 5y^4y^2 \geq 0}}{\vdash [x' := (x-2)^4 + y^5][y' := y^2]5y^4y' \geq 0}}{\text{dI}\; {\color{red}y^5 \geq 0} \vdash [x' = (x-2)^4 + y^5, y' = y^2]{\color{red}y^5 \geq 0}}$$

# Ex: Differential Cuts

$$
\dfrac{\dfrac{\dfrac{}{y^5 \geq 0 \vdash [x':=(x-2)^4+y^5][y':=y^2]3x^2x' \geq 0}\;[:=]}{x^3 \geq -1 \vdash [x' = (x-2)^4+y^5, y' = y^2 \,\&\, y^5 \geq 0]x^3 \geq -1}\;\mathrm{dI}\quad \triangleright}{x^3 \geq -1 \land y^5 \geq 0 \vdash [x' = (x-2)^4+y^5, y' = y^2]x^3 \geq -1}\;\mathrm{dC}
$$

$$
\dfrac{\dfrac{\dfrac{*}{\vdash 5y^4y^2 \geq 0}\;\mathbb{R}}{\vdash [x':=(x-2)^4+y^5][y':=y^2]5y^4y' \geq 0}\;[:=]}{y^5 \geq 0 \vdash [x' = (x-2)^4+y^5, y' = y^2]y^5 \geq 0}\;\mathrm{dI}
$$

$$
\dfrac{\dfrac{\dfrac{}{y^5 \geq 0 \vdash 3x^2((x-2)^4 + y^5) \geq 0}\;\mathbb{R}}{y^5 \geq 0 \vdash [x':=(x-2)^4 + y^5][y':=y^2]3x^2x' \geq 0}\;[:=]}{\dfrac{x^3 \geq -1 \vdash [x' = (x-2)^4 + y^5, y' = y^2 \,\&\, y^5 \geq 0]x^3 \geq -1 \;\triangleright}{x^3 \geq -1 \land y^5 \geq 0 \vdash [x' = (x-2)^4 + y^5, y' = y^2]x^3 \geq -1}\;\mathrm{dC}}\;\mathrm{dI}
$$

$$
\dfrac{\dfrac{\dfrac{*}{\vdash 5y^4y^2 \geq 0}\;\mathbb{R}}{\vdash [x':=(x-2)^4 + y^5][y':=y^2]5y^4y' \geq 0}\;[:=]}{y^5 \geq 0 \vdash [x' = (x-2)^4 + y^5, y' = y^2]y^5 \geq 0}\;\mathrm{dI}
$$

$$\dfrac{\dfrac{\dfrac{\dfrac{*}{y^5 \geq 0 \vdash 3x^2((x-2)^4 + y^5) \geq 0}\;\mathbb{R}}{y^5 \geq 0 \vdash [x' := (x-2)^4 + y^5][y' := y^2]3x^2 x' \geq 0}\;[:=]}{x^3 \geq -1 \vdash [x' = (x-2)^4 + y^5, y' = y^2 \,\&\, y^5 \geq 0]x^3 \geq -1 \;\triangleright}\;\mathrm{dI}}{x^3 \geq -1 \land y^5 \geq 0 \vdash [x' = (x-2)^4 + y^5, y' = y^2]x^3 \geq -1}\;\mathrm{dC}$$

$$\dfrac{\dfrac{\dfrac{*}{\vdash 5y^4 y^2 \geq 0}\;\mathbb{R}}{\vdash [x' := (x-2)^4 + y^5][y' := y^2]5y^4 y' \geq 0}\;[:=]}{y^5 \geq 0 \vdash [x' = (x-2)^4 + y^5, y' = y^2]{\color{red}y^5 \geq 0}}\;\mathrm{dI}$$

# Outline

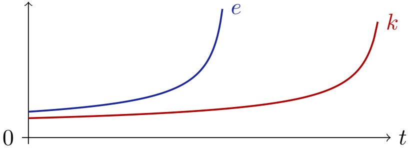
# Soundness Proof: Differential Invariants

**Lemma (Differential lemma)** (Differential value vs. Time-derivative)

$$\varphi \models x' = f(x) \land Q \text{ for } r > 0 \;\Rightarrow\; \forall 0 \le z \le r \;\; \varphi(z)[\![(e)']\!] = \frac{d\varphi(t)[\![e]\!]}{dt}(z)$$

**Differential Invariant**

$$\text{DI} \quad \begin{array}{l}\big([x' = f(x)]e \ge 0 \leftrightarrow e \ge 0\big) \\ \leftarrow [x' = f(x)](e)' \ge 0\end{array}$$




**Proof** ($\ge$ rate of change from $\ge$ initial value. Case $r = 0$ is easier.)

$h(t) \stackrel{\text{def}}{=} \varphi(t)[\![e]\!]$ is differentiable on $[0, r]$ if $r > 0$ by diff. lemma.

$\frac{dh(t)}{dt}(z) = \frac{d\varphi(t)[\![e]\!]}{dt}(z) = \varphi(z)[\![(e)']\!] \ge 0$ by lemma + assume for all $z$.

$h(r) - \underbrace{h(0)}_{\ge 0} = \underbrace{(r-0)}_{>0} \underbrace{\frac{dh(t)}{dt}(\xi)}_{\ge 0} \ge 0$ by mean-value theorem for some $\xi$. $\square$

# Outline

# Summary: Differential Invariants for Differential Equations

## Differential Weakening

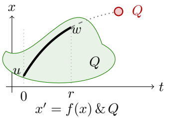
$$\frac{Q \vdash F}{\Gamma \vdash [x' = f(x) \,\&\, Q]F}$$

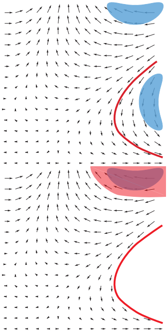
## Differential Invariant

$$\frac{Q \vdash [x' := f(x)]{\color{red}(F)'}}{F \vdash [x' = f(x) \,\&\, Q]F}$$

## Differential Cut

$$\frac{F \vdash [x' = f(x) \,\&\, Q]{\color{red}C} \quad F \vdash [x' = f(x) \,\&\, Q \land {\color{red}C}]F}{F \vdash [x' = f(x) \,\&\, Q]F}$$



## Differential Weakening

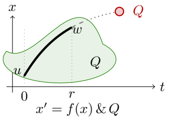
$$\frac{Q \vdash F}{\Gamma \vdash [x' = f(x) \,\&\, Q]F}$$

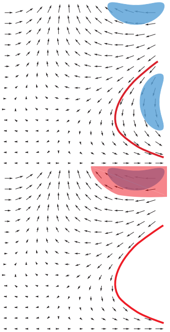
## Differential Invariant

$$\frac{Q \vdash [x' := f(x)](F)'}{F \vdash [x' = f(x) \,\&\, Q]F}$$

## Differential Cut

$$\frac{F \vdash [x' = f(x) \,\&\, Q]C \quad F \vdash [x' = f(x) \,\&\, Q \land C]F}{F \vdash [x' = f(x) \,\&\, Q]F}$$



DW $[x' = f(x) \,\&\, Q]F \leftrightarrow [x' = f(x) \,\&\, Q](Q \to F)$

DI $\big([x' = f(x) \,\&\, Q]F \leftrightarrow [?Q]F\big) \leftarrow (Q \to [x' = f(x) \,\&\, Q](F)')$

DC $\big([x' = f(x) \,\&\, Q]F \leftrightarrow [x' = f(x) \,\&\, Q \land C]F\big) \leftarrow [x' = f(x) \,\&\, Q]C$

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