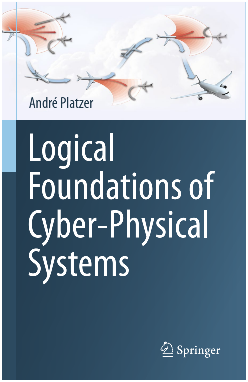
04: Safety & Contracts

Logical Foundations of Cyber-Physical Systems



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Outline

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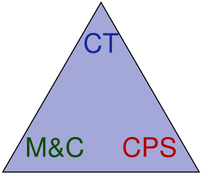
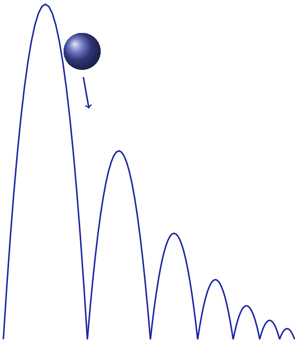
rigorous specification
contracts
preconditions
postconditions
differential dynamic logic



discrete+continuous
analytic specification

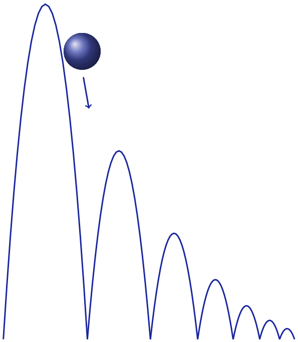
model semantics
reasoning principles

Outline



Example (Quantum the Bouncing Ball)

Quantum the Acrophobic Bouncing Ball



Example (Quantum the Bouncing Ball)

$$\{x' = v, v' = -g\}$$

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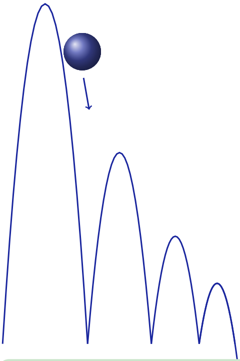

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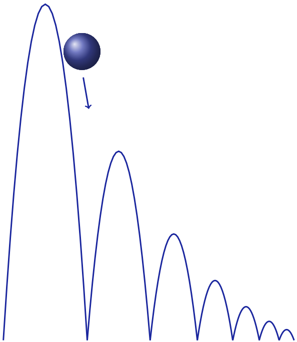

Example (Quantum the Bouncing Ball)

$$\{x' = v, v' = -g \,\&\, x \geq 0\}$$

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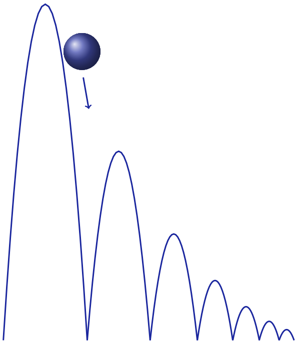

Example (Quantum the Bouncing Ball)

$$\{x' = v, v' = -g \,\&\, x \geq 0\};$$
$$\text{if}(x = 0)\ \ v := -cv$$

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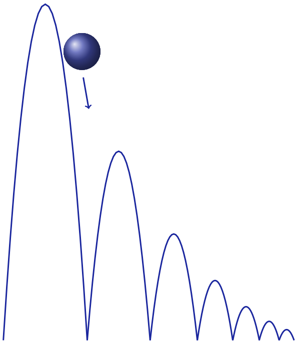

Example (Quantum the Bouncing Ball)

$$\left(\{x' = v, v' = -g \,\&\, x \geq 0\};\right.$$
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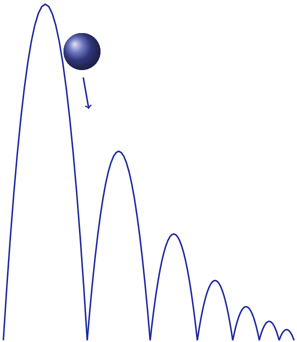
Quantum Discovered a Crack in the Fabric of Time



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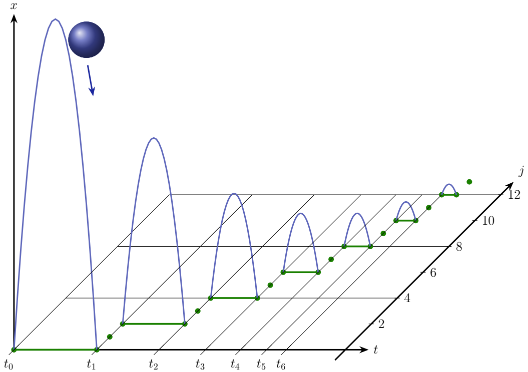

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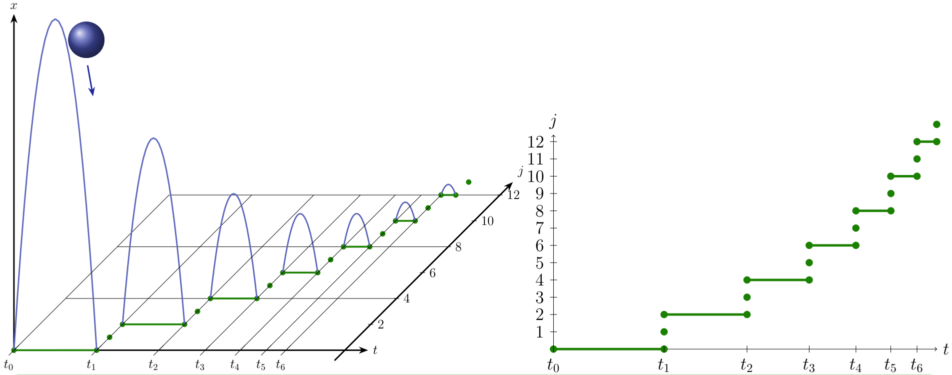

Example (Quantum the Bouncing Ball)

$$\big(\{x' = v, v' = -g \,\&\, x \geq 0\};$$
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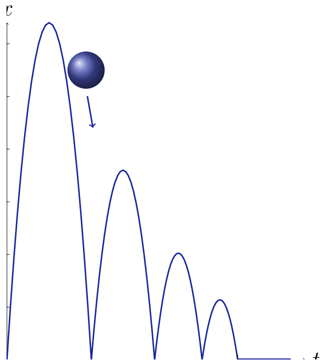
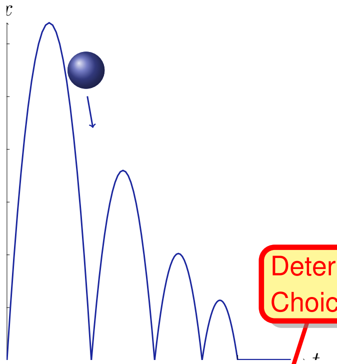
Quantum Learns to Deflate



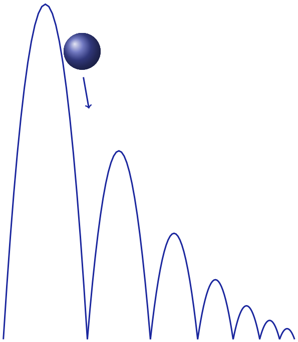
Example (Quantum the Bouncing Ball)

$$\big(\{x' = v, v' = -g \,\&\, x \geq 0\};$$
$$\text{if}(x = 0)\ \ v := -cv\big)^*$$



Example (Quantum the Bouncing Ball)

$$(\{x' = v, v' = -g \,\&\, x \geq 0\};$$
$$\text{if}(x = 0)\,(v := -cv \cup v := 0))^*$$



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$$\left(\{x' = v, v' = -g \,\&\, x \geq 0\};\right.$$
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Outline

Safety of Robots

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Three Laws of Robotics — Isaac Asimov 1942

1. A robot may not injure a human being or, through inaction, allow a human being to come to harm.
2. A robot must obey the orders given to it by human beings, except where such orders would conflict with the First Law.
3. A robot must protect its own existence as long as such protection does not conflict with the First or Second Law.

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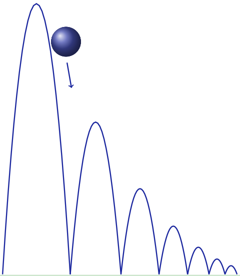
Three Laws of Robotics are not the answer.
They are the inspiration!

Contracts for Quantum the Acrophobic Bouncing Ball



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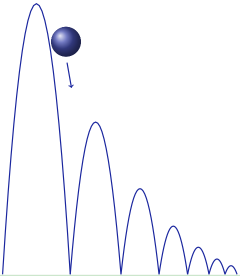

Example (Quantum the Bouncing Ball)

ensures$(0 \leq x)$

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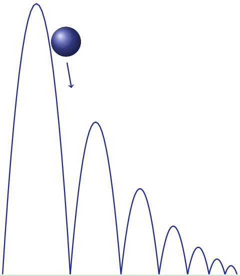
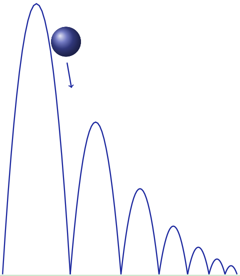


Example (Quantum the Bouncing Ball)

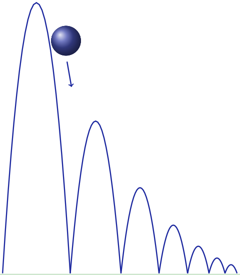
$$\mathbf{ensures}(0 \le x)$$
$$\mathbf{ensures}(x \le H)$$
$$\big(\{x' = v, v' = -g \,\&\, x \ge 0\};$$
$$\text{if}(x = 0)\, v := -cv\big)^*$$



Example (Quantum the Bouncing Ball)

requires$(x = H)$

ensures$(0 \leq x)$
ensures$(x \leq H)$
$\big(\{x' = v, v' = -g \,\&\, x \geq 0\};$
$\text{if}(x = 0)\, v := -cv\big)^*$



Example (Quantum the Bouncing Ball)

requires$(x = H)$

requires$(0 \leq H)$

ensures$(0 \leq x)$

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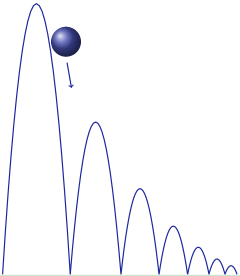
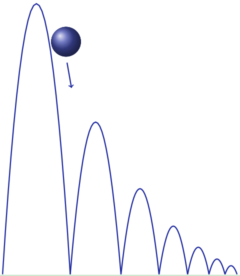


Example (Quantum the Bouncing Ball)

$$\mathbf{requires}(x = H)$$
$$\mathbf{requires}(0 \leq H)$$
$$\mathbf{ensures}(0 \leq x)$$
$$\mathbf{ensures}(x \leq H)$$
$$\big(\{x' = v, v' = -g \,\&\, x \geq 0\};$$
$$\mathrm{if}(x = 0)\, v := -cv\big)^* \mathbf{@invariant}(x \geq 0)$$



Example (Quantum the Bouncing Ball)

$$\textbf{requires}(x = H)$$
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Outline

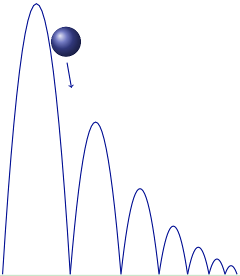
Contracts are Not Enough

CPS contracts are crucial for CPS safety.
We need to understand CPS programs and contracts and how we can convince ourselves that a CPS program respects its contract.

Contracts are at a disadvantage compared to full logic.

Logic is for Specification and Reasoning

1. Specification of a whole CPS program.
2. Analytic inspection of its parts.
3. Argumentative relations between contracts and program parts.
   "Yes, this CPS program meets its contract, and here's why ..."



Example (Quantum the Bouncing Ball)

$\textbf{requires}(x = H)$

$\textbf{requires}(0 \leq H)$

$\textbf{ensures}(0 \leq x)$

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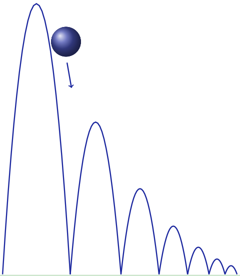


Example (Quantum the Bouncing Ball)

requires$(x = H)$

requires$(0 \le H)$

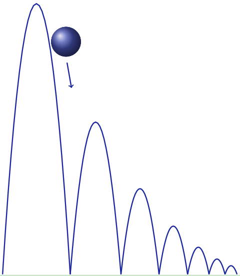
ensures$(0 \le x)$

ensures$(x \le H)$

$$\big(\{x' = v, v' = -g \,\&\, x \ge 0\};$$
$$\text{if}(x = 0)\, v := -cv\big)^*$$

Precondition:

$x = H \land 0 \le H$ in FOL



Example (Quantum the Bouncing Ball)

requires$(x = H)$
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$(\{x' = v, v' = -g \,\&\, x \geq 0\};$
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→ Precondition:
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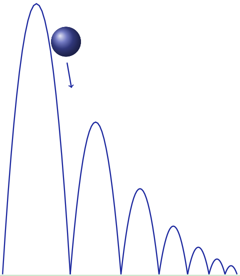
→ Postcondition:
$0 \leq x \land x \leq H$ in FOL



Example (Quantum the Bouncing Ball)

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$\big(\{x' = v, v' = -g \,\&\, x \geq 0\};$
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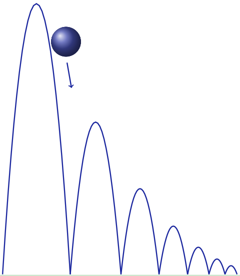
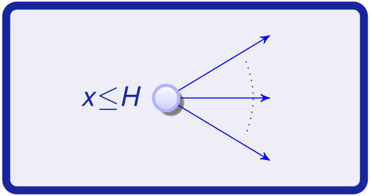
Precondition:
$x = H \land 0 \leq H$ in FOL
Postcondition:
$0 \leq x \land x \leq H$ in FOL
How to say post is true after all HP runs?

Contracts for Quantum the Acrophobic Bouncing Ball





Example (Quantum the Bouncing Ball)

requires$(x = H)$
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ensures$(0 \le x)$
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$\big(\{x' = v, v' = -g \,\&\, x \ge 0\};$
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Precondition:
$x = H \wedge 0 \le H$ in FOL

Postcondition:
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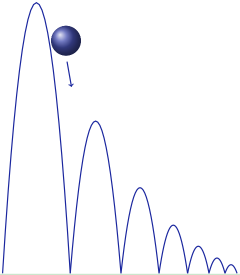

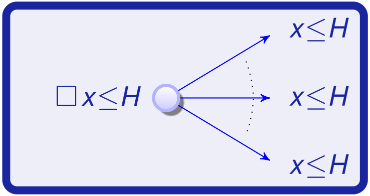

Example (Quantum the Bouncing Ball)

$\textbf{requires}(x = H)$

$\textbf{requires}(0 \le H)$

$\textbf{ensures}(0 \le x)$

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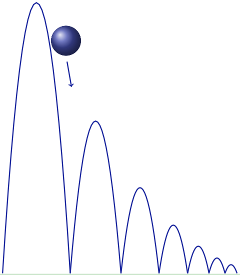

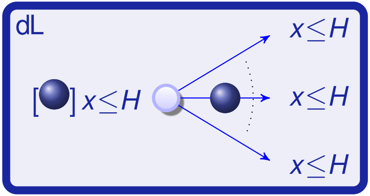

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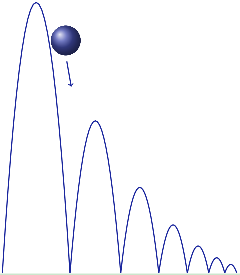

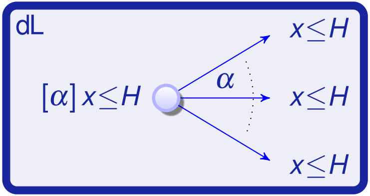

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Precondition:
$x = H \land 0 \leq H$ in FOL
Postcondition:
$0 \leq x \land x \leq H$ in FOL

$$[(\{x' = v, v' = -g \,\&\, x \geq 0\}; \mathrm{if}(x=0)\, v := -cv)^*]$$

Example (Quantum the Bouncing Ball)

requires$(x = H)$

requires$(0 \leq H)$

ensures$(0 \leq x)$

ensures$(x \leq H)$

$(\{x' = v, v' = -g \,\&\, x \geq 0\};$

$\mathrm{if}(x = 0)\, v := -cv)^*$

→ Precondition: $x = H \land 0 \leq H$ in FOL

→ Postcondition: $0 \leq x \land x \leq H$ in FOL

$$[(\{x' = v, v' = -g \,\&\, x \geq 0\}; \text{if}(x=0)\, v := -cv)^*](x \leq H)$$

Example (Quantum the Bouncing Ball)

requires$(x = H)$ → Precondition:

requires$(0 \leq H)$ → $x = H \land 0 \leq H$ in FOL

ensures$(0 \leq x)$ → Postcondition:

ensures$(x \leq H)$ → $0 \leq x \land x \leq H$ in FOL

$(\{x' = v, v' = -g \,\&\, x \geq 0\};$

$\text{if}(x = 0)\, v := -cv)^*$

Contracts for Quantum the Acrophobic Bouncing Ball

$$[(\{x' = v, v' = -g \,\&\, x \geq 0\}; \text{if}(x=0)\, v := -cv)^*](0 \leq x)$$

$$[(\{x' = v, v' = -g \,\&\, x \geq 0\}; \text{if}(x=0)\, v := -cv)^*](x \leq H)$$

Example (Quantum the Bouncing Ball)

requires$(x = H)$
requires$(0 \leq H)$
ensures$(0 \leq x)$
ensures$(x \leq H)$
$(\{x' = v, v' = -g \,\&\, x \geq 0\};$
$\text{if}(x = 0)\, v := -cv)^*$

Precondition:
$x = H \wedge 0 \leq H$ in FOL
Postcondition:
$0 \leq x \wedge x \leq H$ in FOL

$$[(\{x' = v, v' = -g \,\&\, x \geq 0\}; \text{if}(x=0)\, v := -cv)^*](0 \leq x)$$
$$[(\{x' = v, v' = -g \,\&\, x \geq 0\}; \text{if}(x=0)\, v := -cv)^*](x \leq H)$$
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requires$(x = H)$
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$(\{x' = v, v' = -g \,\&\, x \geq 0\};$
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Precondition:
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Postcondition:
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$$[(\{x' = v, v' = -g \,\&\, x \geq 0\}; \text{if}(x=0)\, v := -cv)^*](0 \leq x)$$
$$\land [(\{x' = v, v' = -g \,\&\, x \geq 0\}; \text{if}(x=0)\, v := -cv)^*](x \leq H)$$
$$\leftrightarrow [(\{x' = v, v' = -g \,\&\, x \geq 0\}; \text{if}(x=0)\, v := -cv)^*](0 \leq x \land x \leq H)$$

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Precondition:
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$$[(\{x' = v, v' = -g \,\&\, x \geq 0\}; \text{if}(x=0)\, v := -cv)^*](0 \leq x)$$

Example (Quantum the Bouncing Ball)

$\textbf{requires}(x = H)$ → Precondition:

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$\textbf{ensures}(0 \leq x)$ → Postcondition:

$\textbf{ensures}(x \leq H)$ → $0 \leq x \land x \leq H$ in FOL

$(\{x' = v, v' = -g \,\&\, x \geq 0\};$

$\text{if}(x = 0)\, v := -cv)^*$

Contracts for Quantum the Acrophobic Bouncing Ball

$$x{=}H \rightarrow [(\{x' = v, v' = -g \,\&\, x{\geq}0\}; \text{if}(x{=}0)\, v := -cv)^*](0 \leq x)$$

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Precondition:
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Postcondition:
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$$0 \leq x \land x = H \rightarrow [(\{x' = v, v' = -g \,\&\, x \geq 0\}; \text{if}(x = 0)\, v := -cv)^*](0 \leq x)$$

Example (Quantum the Bouncing Ball)

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Outline

Definition (Syntax of differential dynamic logic)

The *formulas* of *differential dynamic logic* are defined by the grammar:

$$P, Q ::= e \geq \tilde{e} \mid \neg P \mid P \wedge Q \mid P \vee Q \mid P \rightarrow Q \mid \forall x\, P \mid \exists x\, P \mid [\alpha]P \mid \langle\alpha\rangle P$$

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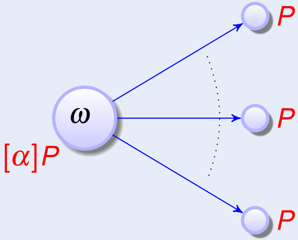
Not | And | Or | Imply | All reals | Some real | All runs | Some runs

Differential Dynamic Logic dL: Semantics

Definition (dL Formulas)



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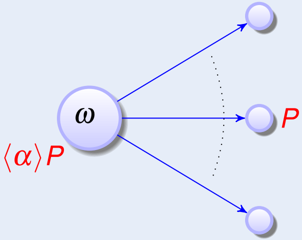
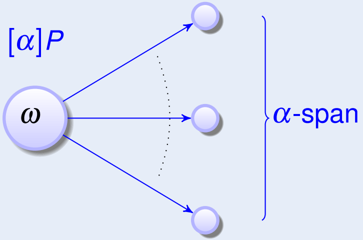
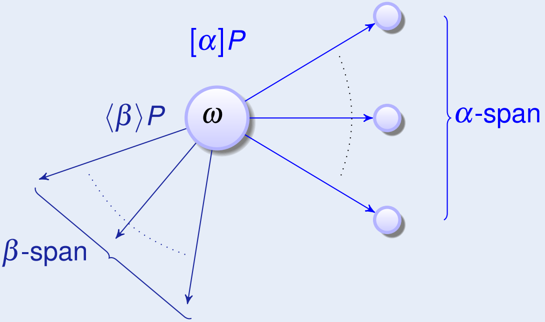


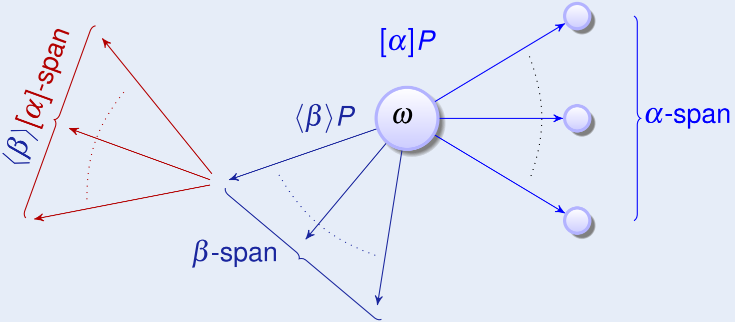
Definition (dL Formulas)



Definition (dL Formulas)



Definition (dL Formulas)



Differential Dynamic Logic: Syntax & Semantics

Definition (Syntax of differential dynamic logic)

The *formulas* of *differential dynamic logic* are defined by the grammar:

$$P, Q ::= e \geq \tilde{e} \mid \neg P \mid P \wedge Q \mid P \vee Q \mid P \rightarrow Q \mid \forall x\, P \mid \exists x\, P \mid [\alpha]P \mid \langle\alpha\rangle P$$

Definition (dL semantics) $([\![\cdot]\!] : \mathrm{Fml} \rightarrow \wp(\mathscr{S}))$

$$\begin{aligned}
[\![e \geq \tilde{e}]\!] &= \{\omega : \omega[\![e]\!] \geq \omega[\![\tilde{e}]\!]\} \\
[\![\neg P]\!] &= [\![P]\!]^{\complement} = \mathscr{S} \setminus [\![P]\!] \\
[\![P \wedge Q]\!] &= [\![P]\!] \cap [\![Q]\!] \\
[\![P \vee Q]\!] &= [\![P]\!] \cup [\![Q]\!] \\
[\![P \rightarrow Q]\!] &= [\![P]\!]^{\complement} \cup [\![Q]\!] \\
[\![\langle\alpha\rangle P]\!] &= [\![\alpha]\!] \circ [\![P]\!] = \{\omega : \nu \in [\![P]\!] \text{ for some } \nu : (\omega,\nu) \in [\![\alpha]\!]\} \\
[\![[\alpha]P]\!] &= [\![\neg\langle\alpha\rangle\neg P]\!] = \{\omega : \nu \in [\![P]\!] \text{ for all } \nu : (\omega,\nu) \in [\![\alpha]\!]\} \\
[\![\exists x\, P]\!] &= \{\omega : \omega_x^r \in [\![P]\!] \text{ for some } r \in \mathbb{R}\} \\
[\![\forall x\, P]\!] &= \{\omega : \omega_x^r \in [\![P]\!] \text{ for all } r \in \mathbb{R}\}
\end{aligned}$$

$$\omega_x^d(y) = \begin{cases} d & \text{if } y = x \\ \omega(y) & \text{if } y \neq x \end{cases}$$

Differential Dynamic Logic: Syntax & Semantics

$[\![P]\!]$	the set of states in which formula P is true
$\omega \models P$	formula P is true in state ω, alias $\omega \in [\![P]\!]$
$\models P$	formula P is valid, i.e., true in all states ω, i.e., $[\![P]\!] = \mathscr{S}$

Definition (dL semantics) $([\![\cdot]\!] : \mathrm{Fml} \to \wp(\mathscr{S}))$

$$
\begin{aligned}
[\![e \geq \tilde{e}]\!] &= \{\omega : \omega[\![e]\!] \geq \omega[\![\tilde{e}]\!]\} \\
[\![\neg P]\!] &= [\![P]\!]^{\complement} = \mathscr{S} \setminus [\![P]\!] \\
[\![P \land Q]\!] &= [\![P]\!] \cap [\![Q]\!] \\
[\![P \lor Q]\!] &= [\![P]\!] \cup [\![Q]\!] \\
[\![P \to Q]\!] &= [\![P]\!]^{\complement} \cup [\![Q]\!] \\
[\![\langle\alpha\rangle P]\!] &= [\![\alpha]\!] \circ [\![P]\!] = \{\omega : \nu \in [\![P]\!] \text{ for some } \nu : (\omega, \nu) \in [\![\alpha]\!]\} \\
[\![[\alpha]P]\!] &= [\![\neg\langle\alpha\rangle\neg P]\!] = \{\omega : \nu \in [\![P]\!] \text{ for all } \nu : (\omega, \nu) \in [\![\alpha]\!]\} \\
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[\![\forall x\, P]\!] &= \{\omega : \omega_x^r \in [\![P]\!] \text{ for all } r \in \mathbb{R}\}
\end{aligned}
$$

Differential Dynamic Logic: Syntax & Semantics

$[\![P]\!]$ the set of states in which formula P is true

$\omega \models P$ formula P is true in state ω, alias $\omega \in [\![P]\!]$

$\models P$ formula P is valid, i.e., true in all states ω, i.e., $[\![P]\!] = \mathscr{S}$

$\exists d\, [x := 1; x' = d] x \geq 0$ and $[x := x+1; x' = d] x \geq 0$ and $\langle x' = d \rangle x \geq 0$

Definition (dL semantics) $([\![\cdot]\!] : \mathrm{Fml} \to \wp(\mathscr{S}))$

$$
\begin{aligned}
[\![e \geq \tilde{e}]\!] &= \{\omega : \omega[\![e]\!] \geq \omega[\![\tilde{e}]\!]\} \\
[\![\neg P]\!] &= [\![P]\!]^{\complement} = \mathscr{S} \setminus [\![P]\!] \\
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\end{aligned}
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$[\![P]\!]$ the set of states in which formula P is true

$\omega \models P$ formula P is true in state ω, alias $\omega \in [\![P]\!]$

$\models P$ formula P is valid, i.e., true in all states ω, i.e., $[\![P]\!] = \mathscr{S}$

$\models \exists d\,[x := 1; x'=d]x\geq 0$ and $\not\models [x := x+1; x'=d]x\geq 0$ and $\not\models \langle x'=d\rangle x\geq 0$

Definition (dL semantics) $([\![\cdot]\!] : \mathrm{Fml} \to \wp(\mathscr{S}))$

$$
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\end{aligned}
$$

Notational Conventions: Precedence

Convention (Operator Precedence)

1. Unary operators (e.g., *, $\neg$, $\forall x, \exists x$, $[\alpha], \langle\alpha\rangle$) bind stronger than binary
2. $\wedge$ binds stronger than $\vee$, which binds stronger than $\rightarrow, \leftrightarrow$
3. ; binds stronger than $\cup$
4. Arithmetic operators $+, -, \cdot$ associate to the left
5. Logical and program operators associate to the right

Example (Operator Precedence)

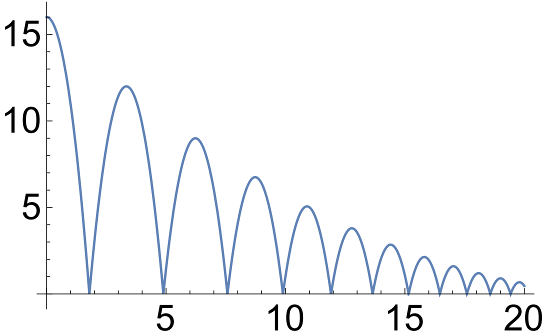
$[\alpha]P \wedge Q \equiv ([\alpha]P) \wedge Q$ $\quad \forall x\, P \wedge Q \equiv (\forall x\, P) \wedge Q$ $\quad \forall x\, P \rightarrow Q \equiv (\forall x\, P) \rightarrow Q$

$\alpha;\beta \cup \gamma \equiv (\alpha;\beta) \cup \gamma$ $\quad \alpha \cup \beta;\gamma \equiv \alpha \cup (\beta;\gamma)$ $\quad \alpha;\beta^* \equiv \alpha;(\beta^*)$

$P \rightarrow Q \rightarrow R \equiv P \rightarrow (Q \rightarrow R)$.

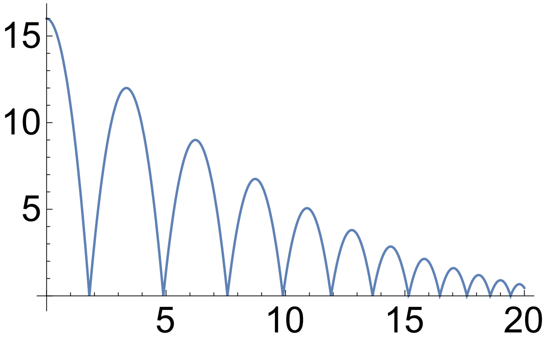
But $\rightarrow, \leftrightarrow$ expect explicit parentheses. Illegal: $P \rightarrow Q \leftrightarrow R$ $\quad P \leftrightarrow Q \rightarrow R$

Outline



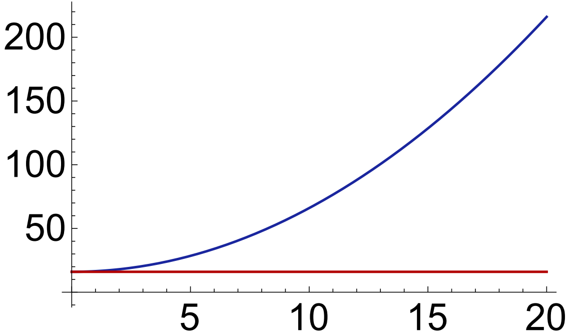
Example (Bouncing Ball)

$$\big(\{x' = v, v' = -g \,\&\, x \geq 0\};$$
$$\text{if}(x = 0)\, v := -cv\big)^*$$



Example (Bouncing Ball)

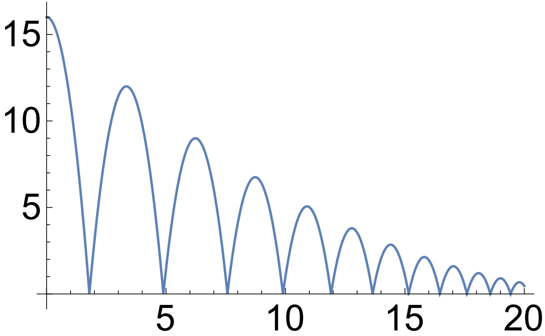
$$\textcolor{red}{H{=}x{\geq}0} \quad \rightarrow \textcolor{red}{[}\big(\{x' = v, v' = -g \,\&\, x \geq 0\};$$
$$\text{if}(x = 0)\, v := -cv\big)^*\textcolor{red}{]\; 0{\leq}x{\leq}H}$$



Not if $g < 0$ in anti-gravity

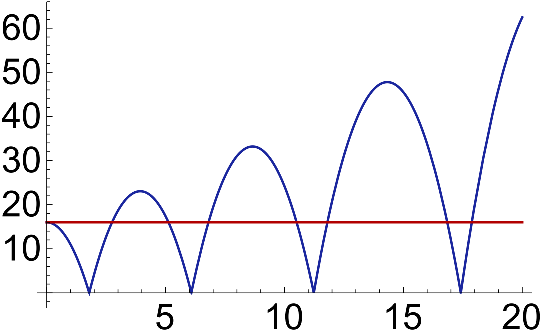
Example (Bouncing Ball)

$$H=x\geq 0 \rightarrow [(\{x'=v, v'=-g \,\&\, x\geq 0\};\ \text{if}(x=0)\, v:=-cv)^*]\, 0\leq x\leq H$$



Example (Bouncing Ball)

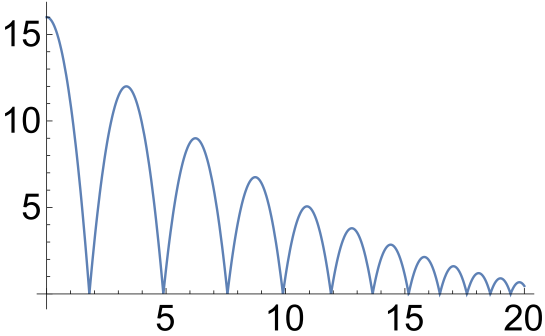
$$H{=}x{\geq}0 \land g{>}0 \rightarrow \big[\big(\{x' = v, v' = -g \,\&\, x \geq 0\};\ \text{if}(x = 0)\, v := -cv\big)^*\big]\, 0{\leq}x{\leq}H$$



Not if $c > 1$ for anti-damping

Example (Bouncing Ball)

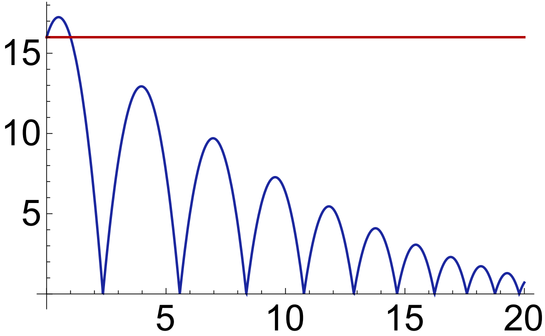
$$H = x \geq 0 \land g > 0 \rightarrow [(\{x' = v, v' = -g \,\&\, x \geq 0\};$$
$$\text{if}(x = 0)\, v := -cv)^*]\, 0 \leq x \leq H$$



Example (Bouncing Ball)

$$\color{red}{1 \geq c \geq 0} \land H = x \geq 0 \land g > 0 \rightarrow \big[\big(\{x' = v, v' = -g \,\&\, x \geq 0\};$$
$$\text{if}(x = 0)\, v := -cv\big)^*\big]\, 0 \leq x \leq H$$

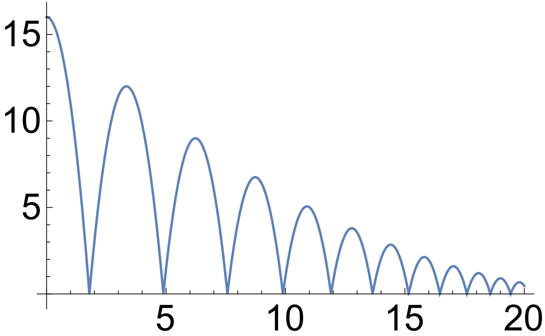
Quantum the Acrophobic Bouncing Ball



Not if $v > 0$ initial climbing

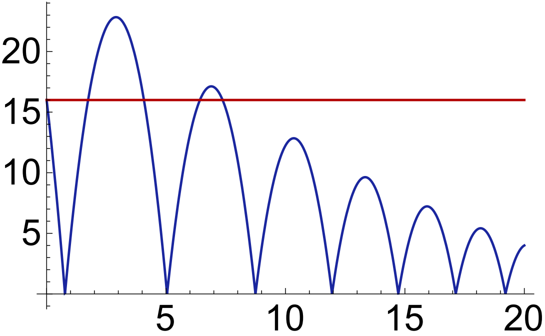
Example (Bouncing Ball)

$$1 \geq c \geq 0 \land H = x \geq 0 \land g > 0 \to [(\{x' = v, v' = -g \,\&\, x \geq 0\};$$
$$\text{if}(x = 0)\, v := -cv)^*]\, 0 \leq x \leq H$$



Example (Bouncing Ball)

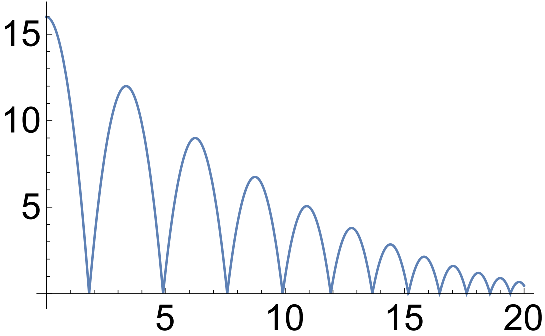
$$v\leq 0 \land 1\geq c\geq 0 \land H=x\geq 0 \land g>0 \rightarrow \left[\left(\{x' = v, v' = -g \,\&\, x \geq 0\};\right.\right.$$
$$\left.\left.\text{if}(x = 0)\, v := -cv\right)^*\right] 0\leq x\leq H$$



Not if $v \ll 0$ initial dribbling

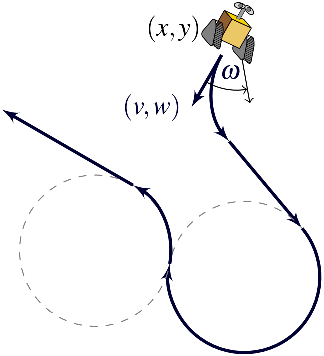
Example (Bouncing Ball)

$$v \leq 0 \land 1 \geq c \geq 0 \land H = x \geq 0 \land g > 0 \rightarrow \big[\big(\{x' = v, v' = -g \,\&\, x \geq 0\}; \text{if}(x = 0)\, v := -cv \big)^* \big]\, 0 \leq x \leq H$$

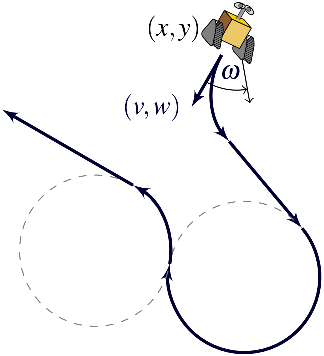


Example (Bouncing Ball)

$$v{=}0 \land 1{\geq}c{\geq}0 \land H{=}x{\geq}0 \land g{>}0 \rightarrow \Big[\big(\{x' = v, v' = -g \,\&\, x \geq 0\};\ \text{if}(x = 0)\, v := -cv\big)^*\Big]\, 0{\leq}x{\leq}H$$



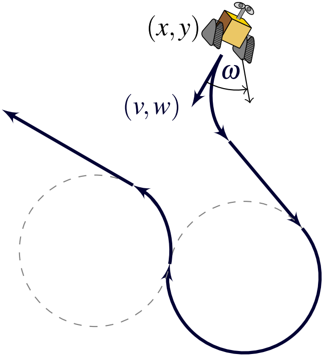
(x, y)
ω
(v, w)



Example (Runaround Robot)

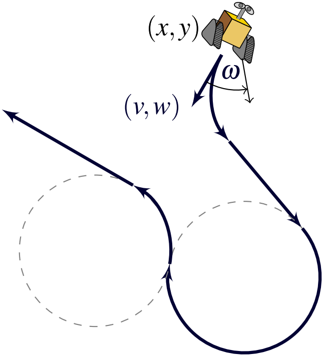
$$\Big((\omega := -1 \cup \omega := 1 \cup \omega := 0);$$
$$\{x' = v, y' = w, v' = \omega w, w' = -\omega v\}\Big)^*$$

Ex: Runaround Robot



Example (Runaround Robot)

$$(x,y) \neq o \rightarrow \big[\big((\omega := -1 \cup \omega := 1 \cup \omega := 0);$$
$$\{x' = v, y' = w, v' = \omega w, w' = -\omega v\}\big)^*\big]\, (x,y) \neq o$$



Example (Runaround Robot)

$$(x,y) \neq o \to \Big[\Big(\big(?Q_{-1};\omega := -1 \cup ?Q_1;\omega := 1 \cup ?Q_0;\omega := 0\big);$$
$$\{x' = v, y' = w, v' = \omega w, w' = -\omega v\}\Big)^*\Big](x,y) \neq o$$

Outline

Differential Dynamic Logic dL: Syntax

Definition (Hybrid program α)

$$\alpha, \beta \ ::= \ x := f(x) \mid ?Q \mid x' = f(x) \,\&\, Q \mid \alpha \cup \beta \mid \alpha ; \beta \mid \alpha^*$$

Definition (dL Formula P)

$$P, Q \ ::= \ e \geq \tilde{e} \mid \neg P \mid P \wedge Q \mid \forall x\, P \mid \exists x\, P \mid [\alpha]P \mid \langle \alpha \rangle P$$

Differential Dynamic Logic dL: Syntax

Discrete Assign | Test Condition | Differential Equation | Nondet. Choice | Seq. Compose | Nondet. Repeat

Definition (Hybrid program α)

$$\alpha, \beta \ ::= \ x := f(x) \mid ?Q \mid x' = f(x) \,\&\, Q \mid \alpha \cup \beta \mid \alpha ; \beta \mid \alpha^*$$

Definition (dL Formula P)

$$P, Q \ ::= \ e \geq \tilde{e} \mid \neg P \mid P \wedge Q \mid \forall x\, P \mid \exists x\, P \mid [\alpha] P \mid \langle \alpha \rangle P$$

All Reals | Some Reals | All Runs | Some Runs

Differential Dynamic Logic dL: Semantics

Definition (Hybrid program semantics) $([\![\cdot]\!] : \mathrm{HP} \to \wp(\mathcal{S} \times \mathcal{S}))$

$$
\begin{aligned}
[\![x := f(x)]\!] &= \{(\omega,\nu) : \nu = \omega \text{ except } \nu[\![x]\!] = \omega[\![f(x)]\!]\} \\
[\![?Q]\!] &= \{(\omega,\omega) : \omega \in [\![Q]\!]\} \\
[\![x' = f(x)]\!] &= \{(\varphi(0),\varphi(r)) : \varphi \models x' = f(x) \text{ for some duration } r\} \\
[\![\alpha \cup \beta]\!] &= [\![\alpha]\!] \cup [\![\beta]\!] \\
[\![\alpha;\beta]\!] &= [\![\alpha]\!] \circ [\![\beta]\!] \\
[\![\alpha^*]\!] &= [\![\alpha]\!]^* = \bigcup_{n\in\mathbb{N}} [\![\alpha^n]\!]
\end{aligned}
$$

compositional semantics

Definition (dL semantics) $([\![\cdot]\!] : \mathrm{Fml} \to \wp(\mathcal{S}))$

$$
\begin{aligned}
[\![e \geq \tilde{e}]\!] &= \{\omega : \omega[\![e]\!] \geq \omega[\![\tilde{e}]\!]\} \\
[\![\neg P]\!] &= [\![P]\!]^{\complement} \\
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[\![[\alpha]P]\!] &= [\![\neg\langle\alpha\rangle\neg P]\!] = \{\omega : \nu \in [\![P]\!] \text{ for all } \nu : (\omega,\nu) \in [\![\alpha]\!]\} \\
[\![\exists x\, P]\!] &= \{\omega : \omega_x^r \in [\![P]\!] \text{ for some } r \in \mathbb{R}\}
\end{aligned}
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