15-819/18-879: Logical Analysis of Hybrid Systems

28: Complete Axiomatization of Differential Dynamic Logic

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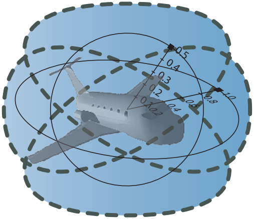

Outline

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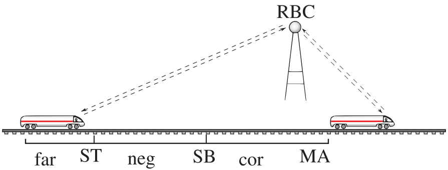
1. Verification Calculus for Differential Dynamic Logic d$\mathcal{L}$
   - Compositionality Motives
2. Soundness
3. Completeness
   - Incompleteness
   - Completeness
   - Expressibility and Rendition of Hybrid Programs
   - Relative Completeness of First-Order Assertions
   - Relative Completeness of Differential Logic Calculus

$\mathcal{A}$ d$\mathcal{L}$ Motives: The Logic of Hybrid Systems

differential dynamic logic

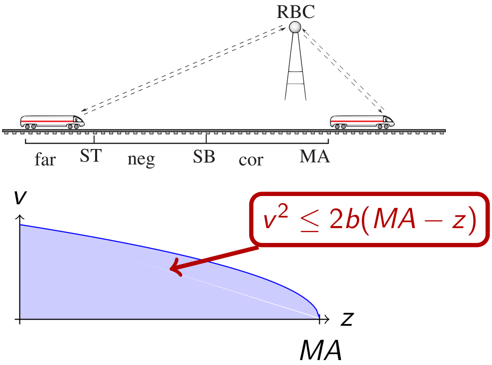
$$d\mathcal{L} = \quad DL + HP$$



d$\mathcal{L}$ Motives: Regions in First-order Logic

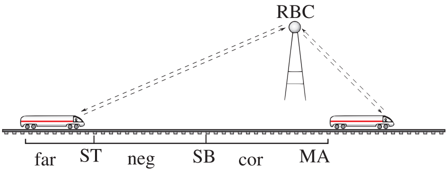
differential dynamic logic

$$\mathrm{d}\mathcal{L} = \mathrm{FOL}_{\mathbb{R}}$$



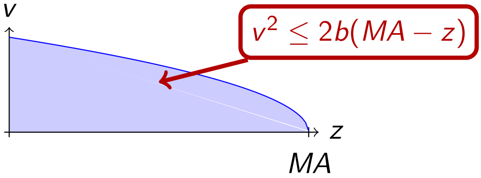
differential dynamic logic

$d\mathcal{L} = \mathrm{FOL}_{\mathbb{R}}$



$\forall MA\, \exists SB \ldots$

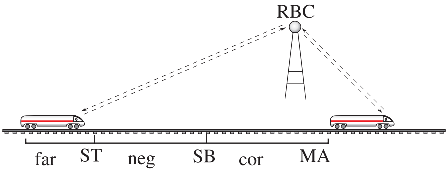
$\forall t{\geq}0 \ldots$



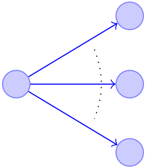
dℒ Motives: State Transitions in Dynamic Logic

differential dynamic logic

$\mathrm{d}\mathcal{L} = \mathrm{FOL}_{\mathbb{R}} +$

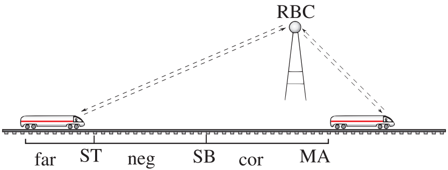


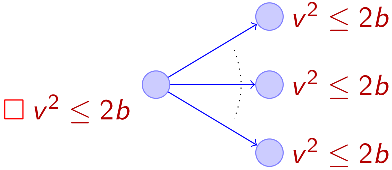
$v^2 \leq 2b$



differential dynamic logic

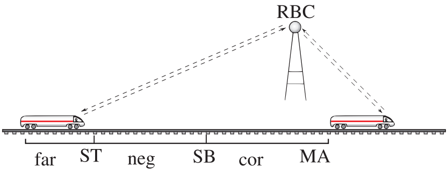
$d\mathcal{L} = \mathrm{FOL}_{\mathbb{R}} + \mathrm{ML}$

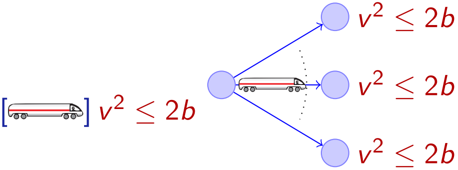




differential dynamic logic

$$\mathrm{d}\mathcal{L} = \mathrm{FOL}_{\mathbb{R}} + \mathrm{DL}$$

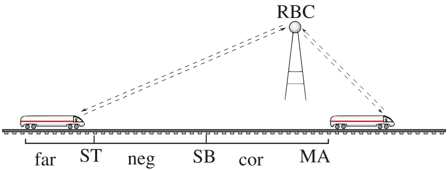




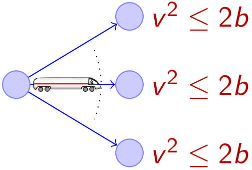
$d\mathcal{L}$ Motives: Hybrid Programs as Uniform Model

differential dynamic logic

$d\mathcal{L} = \mathrm{FOL}_{\mathbb{R}} + \mathrm{DL} + \mathrm{HP}$



$[z'' = a]\, v^2 \leq 2b$



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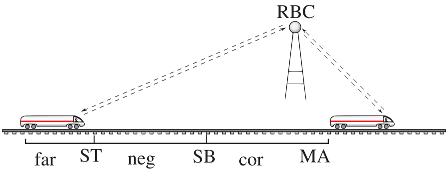
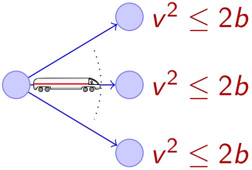


$[\text{if}(z > SB)\ a := -b;\ z'' = a]\ v^2 \le 2b$



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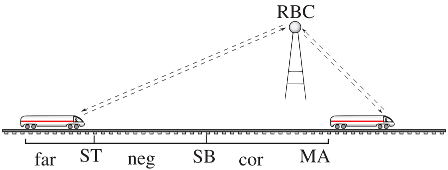
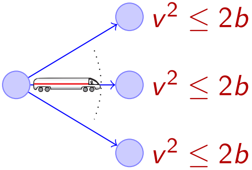


$$[\underbrace{\mathtt{if}(z > SB)\, a := -b;\ z'' = a}_{\text{hybrid program}}]\, v^2 \leq 2b$$



10 propositional rules

$$\frac{\vdash \phi}{\neg\phi \vdash} \qquad \frac{\phi, \psi \vdash}{\phi \land \psi \vdash} \qquad \frac{\phi \vdash \quad \psi \vdash}{\phi \lor \psi \vdash} \qquad \frac{\vdash \phi \quad \phi \vdash}{\vdash}$$

$$\frac{\phi \vdash}{\vdash \neg\phi} \qquad \frac{\vdash \phi \quad \vdash \psi}{\vdash \phi \land \psi} \qquad \frac{\vdash \phi, \psi}{\vdash \phi \lor \psi}$$

$$\frac{\phi \vdash \psi}{\vdash \phi \rightarrow \psi} \qquad \frac{\vdash \phi \quad \psi \vdash}{\phi \rightarrow \psi \vdash} \qquad \frac{}{\phi \vdash \phi}$$

$$\frac{\langle\alpha\rangle\langle\beta\rangle\phi}{\langle\alpha;\beta\rangle\phi} \qquad \frac{\phi \lor \langle\alpha\rangle\langle\alpha^*\rangle\phi}{\langle\alpha^*\rangle\phi} \qquad \frac{\phi_{x_1}^{\theta_1}\cdots{}_{x_n}^{\theta_n}}{\langle x_1 := \theta_1, \dots, x_n := \theta_n\rangle\phi}$$

$$\frac{[\alpha][\beta]\phi}{[\alpha;\beta]\phi} \qquad \frac{\phi \land [\alpha][\alpha^*]\phi}{[\alpha^*]\phi} \qquad \frac{\langle x_1 := \theta_1, \dots, x_n := \theta_n\rangle\phi}{[x_1 := \theta_1, \dots, x_n := \theta_n]\phi}$$

$$\frac{\langle\alpha\rangle\phi \lor \langle\beta\rangle\phi}{\langle\alpha\cup\beta\rangle\phi} \qquad \frac{\chi \land \psi}{\langle ?\chi\rangle\psi} \qquad \frac{\exists t{\geq}0\,\big((\forall 0{\leq}\tilde{t}{\leq}t\,\langle\mathcal{S}(\tilde{t})\rangle\chi) \land \langle\mathcal{S}(t)\rangle\phi\big)}{\langle x_1' = \theta_1, \dots, x_n' = \theta_n \land \chi\rangle\phi}$$

$$\frac{[\alpha]\phi \land [\beta]\phi}{[\alpha\cup\beta]\phi} \qquad \frac{\chi \rightarrow \psi}{[?\chi]\psi} \qquad \frac{\forall t{\geq}0\,\big((\forall 0{\leq}\tilde{t}{\leq}t\,\langle\mathcal{S}(\tilde{t})\rangle\chi) \rightarrow \langle\mathcal{S}(t)\rangle\phi\big)}{[x_1' = \theta_1, \dots, x_n' = \theta_n \land \chi]\phi}$$

$$\frac{\vdash \phi(s(X_1,\dots,X_n))}{\vdash \forall x\, \phi(x)} \qquad \frac{\vdash \phi(X)}{\vdash \exists x\, \phi(x)}$$

$$\frac{\phi(s(X_1,\dots,X_n)) \vdash}{\exists x\, \phi(x) \vdash} \qquad \frac{\phi(X) \vdash}{\forall x\, \phi(x) \vdash}$$

s new, $\{X_1,\dots,X_n\} = FV(\exists x\, \phi(x))$ $\qquad$ X new variable

$$\frac{\vdash \mathrm{QE}(\forall X\, (\Phi(X) \vdash \Psi(X)))}{\Phi(s(X_1,\dots,X_n)) \vdash \Psi(s(X_1,\dots,X_n))} \qquad \frac{\vdash \mathrm{QE}(\exists X \bigwedge_i (\Phi_i \vdash \Psi_i))}{\Phi_1 \vdash \Psi_1 \quad \dots \quad \Phi_n \vdash \Psi_n}$$

X new variable $\qquad$ X only in branches $\Phi_i \vdash \Psi_i$

QE needs to be defined in premiss

$$\frac{\vdash \forall^\alpha(\phi \to \psi)}{[\alpha]\phi \vdash [\alpha]\psi}$$

$$\frac{\vdash \forall^\alpha(\phi \to \psi)}{\langle\alpha\rangle\phi \vdash \langle\alpha\rangle\psi}$$

$$\frac{\vdash \forall^\alpha(\phi \to [\alpha]\phi)}{\phi \vdash [\alpha^*]\phi}$$

$$\frac{\vdash \forall^\alpha \forall v{>}0\,(\varphi(v) \to \langle\alpha\rangle\varphi(v-1))}{\exists v\,\varphi(v) \vdash \langle\alpha^*\rangle\exists v{\leq}0\,\varphi(v)}$$

Outline

Soundness

Theorem (Soundness)

d$\mathcal{L}$ *calculus is sound, i.e.,*

$$\vdash \phi \quad \Rightarrow \quad \vDash \phi$$

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Challenges (Soundness Proof)

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- $x' = f(x)$
- Side deductions

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Challenges (Soundness Proof)

- $x' = f(x)$
- Side deductions
- Free variables & Skolemization

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Completeness

Can we prove all valid formulas of d$\mathcal{L}$?

$$\vDash \phi \quad \Rightarrow \quad \vdash \phi?$$

Incompleteness

Theorem (Incompleteness)

Both the discrete fragment and the continuous fragment of $\mathrm{d}\mathcal{L}$ are not effectively axiomatisable, i.e., they have no sound and complete effective calculus, because natural numbers are definable in both fragments.

Incompleteness

Theorem (Incompleteness)

Both the discrete fragment and the continuous fragment of $\mathrm{d}\mathcal{L}$ are not effectively axiomatisable, i.e., they have no sound and complete effective calculus, because natural numbers are definable in both fragments.

Theorem (Gödels's Incompleteness'31)

First-order logic with (non-linear) arithmetic of natural numbers has no sound and complete effective calculus.

Incompleteness

Proof (Incompleteness).

Discrete fragment:

$$\langle (x := x+1)^* \rangle\ x = n$$

$$\xrightarrow{+1}\xrightarrow{+1}\xrightarrow{+1}\xrightarrow{+1}\xrightarrow{+1}$$

$\square$

Incompleteness

Proof (Incompleteness).

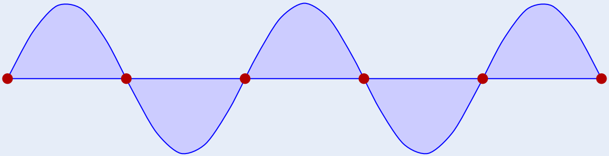
Discrete fragment:

$$\langle (x := x + 1)^* \rangle \; x = n$$

$$\xrightarrow{+1}\xrightarrow{+1}\xrightarrow{+1}\xrightarrow{+1}\xrightarrow{+1}$$

Continuous fragment:

$$\langle s'' = -s, \tau' = 1 \rangle (s = 0 \land \tau = n) \qquad \rightsquigarrow s = \sin$$



$\square$

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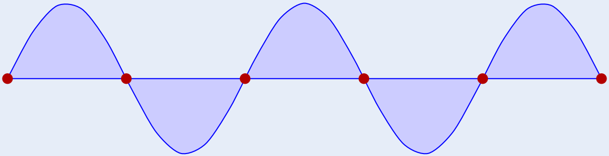

What's missing in characterization? $\square$

Incompleteness

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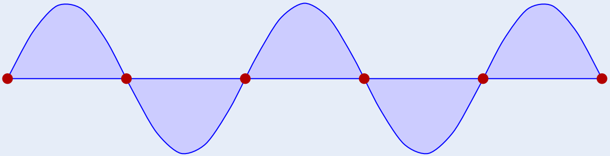

What's missing in characterization? $s \neq 0 \lor s'(0) \neq 0$ $\square$

Incomplete! But are we missing proof rules?

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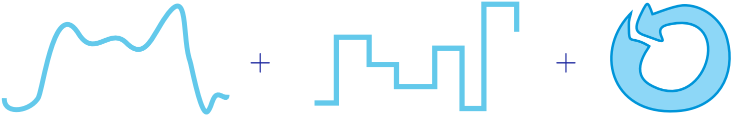
Relativity

Cook,Harel: discrete-DL/data$_{\mathbb{N}}$ hybrid-d$\mathcal{L}$/data$_{\mathbb{R}}$??

Sources of Incompleteness



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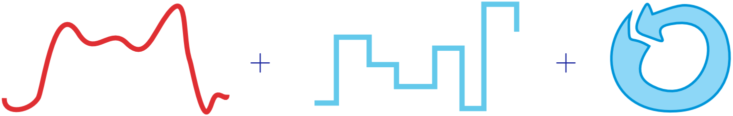
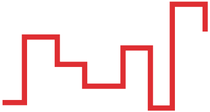
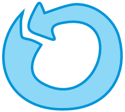


continuous

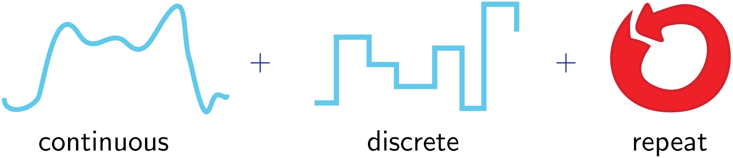
Sources of Incompleteness

 +  + 

continuous

discrete

Sources of Incompleteness



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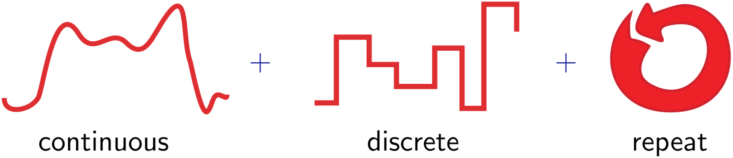
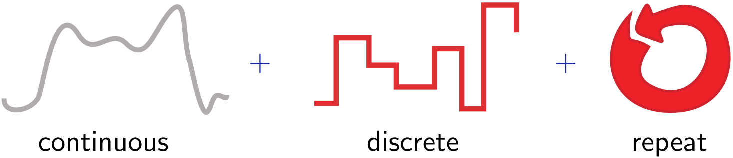


Relative Completeness



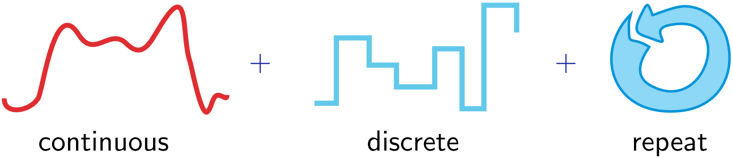
Relative Completeness

continuous + discrete + repeat

Theorem (Relative Completeness)

d$\mathcal{L}$ *calculus is a sound & complete axiomatisation of hybrid systems relative to differential equations.*

Proof Outline 15p



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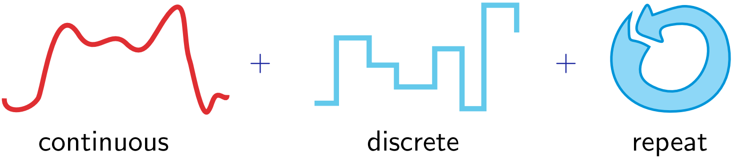

$\Downarrow$

Relativity

Cook,Harel: discrete-DL/data P.: hybrid-d$\mathcal{L}$/differential equations

First-Order Logic of Differential Equations FOD

Definition (First-Order Logic of Differential Equations)

$$\mathsf{FOD} = \mathsf{FOL}_{\mathbb{R}} + [x_1' = \theta_1, \ldots, x_n' = \theta_n]F$$

$$\mathsf{FOD}\ \phi ::= \theta_1 \geq \theta_2 \mid \neg\phi \mid \phi_1 \land \phi_2 \mid \forall x\, \phi \mid \exists x\, \phi \mid [x_1' = \theta_1, \ldots, x_n' = \theta_n]\phi$$

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with $\mathrm{FOL}_{\mathbb{R}}$-formula F

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with $\text{FOL}_{\mathbb{R}}$-formula F

both will do

Relative Completeness

Theorem (Relative Completeness)

d$\mathcal{L}$ *calculus is complete relative to first-order logic of differential equations.*

$$\vDash \phi \quad \textit{iff} \quad \mathit{Taut}_{\mathit{FOD}} \vdash \phi$$

where $\quad \mathit{FOD} = \mathit{FOL}_{\mathbb{R}} + [x_1' = \theta_1, \ldots, x_n' = \theta_n]F$

▸ Proof Outline 15p

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▸ Proof Outline 15p

Corollary (Proof-theoretical Alignment)

verification of hybrid systems = verification of dynamical systems!

Relative Completeness Proof

$$\vDash \phi \quad \text{iff} \quad Taut_{\mathrm{FOD}} \vdash \phi$$

where $\quad \mathrm{FOD} = \mathrm{FOL}_{\mathbb{R}} + [x_1' = \theta_1, \dots, x_n' = \theta_n]F$

Return

Proof (Relative Completeness, 15 pages).

1. Strong enough invariants and variants expressible in d$\mathcal{L}$
2. d$\mathcal{L}$ expressible in FOD
3. valid d$\mathcal{L}$ formulas d$\mathcal{L}$-derivable from corresponding FOD axioms
4. finite FOD formula characterising unbounded hybrid repetition
5. FOD characterises $\mathbb{R}$-Gödel encoding
6. First-order expressible & program rendition:
   for each ϕ there is $F \in \mathrm{FOD}$ $\vDash \phi \leftrightarrow F$
7. Propositionally & first-order complete
8. Relative complete for first-order safety $F \to [\alpha]G$
9. Relative complete for first-order liveness $F \to \langle\alpha\rangle G$

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Relative Completeness Proof

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◂ Return

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$$\vDash \phi \quad \text{iff} \quad Taut_{\mathsf{FOD}} \vdash \phi$$

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Proof (Relative Completeness, 15 pages).

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2. d$\mathcal{L}$ expressible in FOD
3. valid d$\mathcal{L}$ formulas d$\mathcal{L}$-derivable from corresponding FOD axioms
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$$\vDash \phi \quad \text{iff} \quad Taut_{\mathsf{FOD}} \vdash \phi$$

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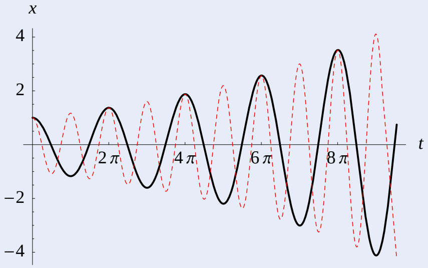
Proof ($\mathbb{R}$-Gödel encoding).

FOD characterises constructive bijection $\mathbb{R} \to \mathbb{R}^2$



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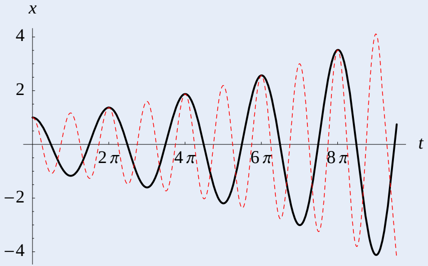

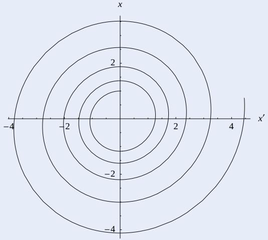

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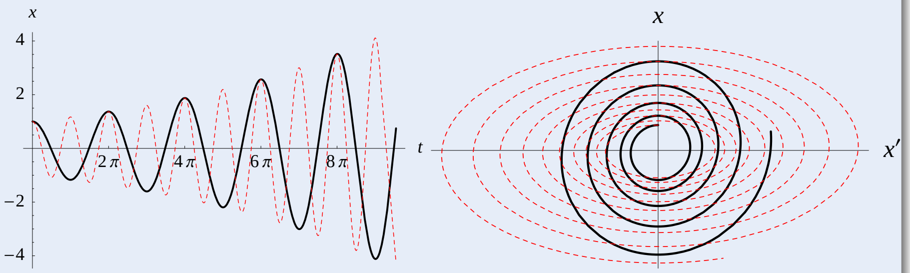

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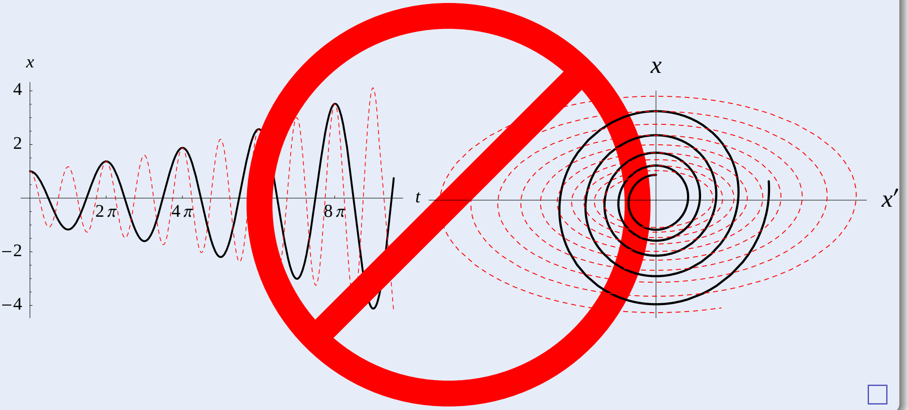
FOD characterises constructive bijection $\mathbb{R} \to \mathbb{R}^2$ not differentiable!



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$$\begin{aligned}
\text{at}(Z,n,j,z) &\leftrightarrow \forall i\!:\!\mathbb{Z}\ \ \text{digit}(z,i) = \text{digit}(Z, n(i-1)+j) \wedge n > 0 \wedge n,j \in \mathbb{N} \\
\text{digit}(a,i) &= \text{intpart}(2\,\text{frac}(2^{i-1}a)) \\
\text{intpart}(a) &= a - \text{frac}(a) \\
\text{frac}(a) = z &\leftrightarrow \exists i\!:\!\mathbb{Z}\ \ z = a - i \wedge -1 < z \wedge z < 1 \wedge az \geq 0 \quad \text{“keep sign”}
\end{aligned}$$

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$$\begin{aligned} 2^i = z \leftrightarrow\ & i \geq 0 \land \langle x := 1; t := 0; x' = x \ln 2, t' = 1 \rangle (t = i \land x = z) \\ \lor\ & i < 0 \land \langle x := 1; t := 0; x' = -x \ln 2, t' = -1 \rangle (t = i \land x = z) \end{aligned}$$

$$\ln 2 = z \leftrightarrow \langle x := 1; t := 0; x' = x, t' = 1 \rangle (x = 2 \land t = z)$$

$\square$

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Program Rendition in FOD

Lemma (Program rendition)

For every HP α with variables among $\vec{x} = x_1, \ldots, x_k$ there is a FOD-formula $\mathcal{S}_\alpha(\vec{x}, \vec{v})$ with variables among the $2k$ distinct variables $\vec{x} = x_1, \ldots, x_k$ and $\vec{v} = v_1, \ldots, v_k$ such that

$$\vDash \mathcal{S}_\alpha(\vec{x}, \vec{v}) \leftrightarrow \langle\alpha\rangle \vec{x} = \vec{v}$$

or, equivalently, for every I, η, v,

$$I, \eta, v \models \mathcal{S}_\alpha(\vec{x}, \vec{v}) \text{ iff } (v, v[\vec{x} \mapsto [\![\vec{v}]\!]_{I,v,\eta}]) \in \rho_{I,\eta}(\alpha) \ .$$

Program Rendition Proof

Proof.

$$\mathcal{S}_{x_1:=\theta_1,..,x_k:=\theta_k}(\vec{x},\vec{v}) \equiv \bigwedge_{i=1}^{k}(v_i = \theta_i)$$

$$\mathcal{S}_{x_1'=\theta_1,..,x_k'=\theta_k}(\vec{x},\vec{v}) \equiv \langle x_1' = \theta_1,\ldots,x_k' = \theta_k\rangle\, \vec{v} = \vec{x}$$

$$\mathcal{S}_{x_1'=\theta_1,..,x_k'=\theta_k\wedge\chi}(\vec{x},\vec{v}) \equiv \langle t := 0; x_1'{=}\theta_1,\ldots,x_k'{=}\theta_k, t'{=}1\rangle\big(\vec{v} = \vec{x}$$
$$\wedge\, [x_1' = -\theta_1,\ldots,x_k' = -\theta_k, t' = -1](t \geq 0 \to \chi)\big)$$

$$\mathcal{S}_{?\chi}(\vec{x},\vec{v}) \equiv \vec{v} = \vec{x} \wedge \chi$$

$$\mathcal{S}_{\beta\cup\gamma}(\vec{x},\vec{v}) \equiv \mathcal{S}_\beta(\vec{x},\vec{v}) \vee \mathcal{S}_\gamma(\vec{x},\vec{v})$$

$$\mathcal{S}_{\beta;\gamma}(\vec{x},\vec{v}) \equiv \exists\vec{z}\,(\mathcal{S}_\beta(\vec{x},\vec{z}) \wedge \mathcal{S}_\gamma(\vec{z},\vec{v}))$$

$$\mathcal{S}_{\beta^*}(\vec{x},\vec{v}) \equiv \exists Z\,\exists n:\mathbb{N}\ \big(Z_1^{(n)} = \vec{x} \wedge Z_n^{(n)} = \vec{v}$$
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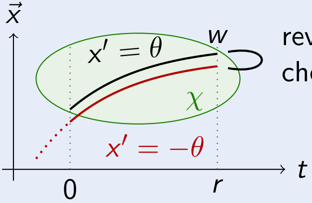

revert flow and time
check χ backwards

$\square$

Expressibility

Lemma (Expressibility)

d$\mathcal{L}$ expressible in FOD: for all d$\mathcal{L}$ formulas $\phi \in \mathrm{Fml}$ there is a FOD-formula $\phi^{\#} \in \mathrm{Fml}_{FOD}$ that is equivalent, i.e., $\vDash \phi \leftrightarrow \phi^{\#}$.

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The proof follows an induction on the structure of formula ϕ.

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Sequent Derivability

Lemma (Derivability of sequents)

$\vdash_{\mathcal{D}} \phi \rightarrow \psi$ *iff the sequent* $\phi \vdash \psi$ *is derivable from* $\mathcal{D}$*, denoted by* $\phi \vdash_{\mathcal{D}} \psi$*.*

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- When sequents are abbreviations for formulas, both sides are identical.

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$$\text{cut}\,\dfrac{\dfrac{*}{\phi \vdash \phi \to \psi, \psi} \qquad \to\text{l}\,\dfrac{\text{Ax}\,\dfrac{*}{\phi \vdash \phi, \psi} \qquad \text{Ax}\,\dfrac{*}{\psi, \phi \vdash \psi}}{\phi, \phi \to \psi \vdash \psi}}{\phi \vdash \psi}$$

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- The converse direction is by an application of $\rightarrow$r.

$\square$

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If $\vdash_{\mathcal{D}} \phi$ *is provable without free logical variables, then so are* $\vdash_{\mathcal{D}} \forall x\, \phi$ *and* $\vdash_{\mathcal{D}} \langle x_1 := \theta_1, \ldots x_n := \theta_n \rangle \phi$.

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Proof Sketch.

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- For reducing the first part of this lemma to the second, let s be a Skolem constant for state variable x.

$\square$

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If $\vdash_{\mathcal{D}} \phi$ is provable without free logical variables, then so are $\vdash_{\mathcal{D}} \forall x\, \phi$ and $\vdash_{\mathcal{D}} \langle x_1 := \theta_1, \dots x_n := \theta_n \rangle \phi$.

Proof Sketch.

- Second part: Induction on the structure of proofs with inductive jump prefix transformation (1page proof).
- For reducing the first part of this lemma to the second, let s be a Skolem constant for state variable x.
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- Using $\forall$r, continue derivation to a proof of $\forall X\, \langle x := X \rangle \phi$, which we abbreviate as $\forall x\, \phi$.
- Rule $\forall$r is applicable for Skolem constant s as no free logical variables occur in the proof.

□

Relative Completeness on First-Order Safety Assertions

Proposition (Relative completeness of first-order safety)

For every $\alpha \in \mathsf{HP}(\Sigma)$ *and each* $F, G \in \mathsf{Fml}_{\mathsf{FOL}}$

$$\vDash F \rightarrow [\alpha]G \textit{ implies } \vdash_{\mathcal{D}} F \rightarrow [\alpha]G \quad (\textit{thus } F \vdash_{\mathcal{D}} [\alpha]G)$$

Proof (α of the form $x_1 := \theta_1, \dots, x_n := \theta_n$, $?\chi$, $\beta \cup \gamma$, or $\beta;\gamma$).

- This follows from soundness of symmetric rules (equivalent transformations):

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- Thus, we can derive $F \rightarrow [\alpha]G$ by applying the respective rule.
- $\vDash F \rightarrow [x_1' = f(x_1)_1, \ldots, x_n' = f(x_n)_n]G$ is a FOD-formula and hence derivable as a $\mathcal{D}$ axiom.

$\square$

Relative Completeness on First-Order Safety Assertions

Proof (α of the form $\beta;\gamma$).

- $\vDash F \to [\beta;\gamma]G$, which implies $\vDash F \to [\beta][\gamma]G$.

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- $\vDash F \rightarrow [\beta;\gamma]G$, which implies $\vDash F \rightarrow [\beta][\gamma]G$.
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- from which composition $[;]$ yields $F \vdash_{\mathcal{D}} [\beta;\gamma]G$.

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Relative Completeness of First-Order Liveness Assertions

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- $\vDash F \to \langle\beta^*\rangle G$ derivable by variant convergence:
-

$$\exists \vec{v}\, \exists Z\, \left(Z_1^{(n)} = \vec{x} \land Z_n^{(n)} = \vec{v} \land \forall i : \mathbb{N}\, (1 \le i < n \to \mathcal{S}_\beta(Z_i^{(n)}, Z_{i+1}^{(n)})) \land G_{\vec{x}}^{\vec{v}}\right)$$

- Thus $\exists v\, \varphi(v) \vdash_{\mathcal{D}} \langle\beta^*\rangle \exists v{\le}0\, \varphi(v)$ by convergence con.
- From assumption, conclude valid FOD-formulas, hence $\mathcal{D}$-axioms:
  - $\vDash F \to \exists v\, \varphi(v)$, because $\vDash F \to \langle\beta^*\rangle G$
  - $\vDash (\exists v{\le}0\, \varphi(v)) \to G$, because $v{\le}0$ and the fact, that, by Gödel, $\varphi(v)$ only holds true for natural numbers, imply $\varphi(0)$. Further, $\varphi(0)$ entails G, because zero repetitions of β have no effect.
- Derive $\vdash_{\mathcal{D}} \forall^\beta(\exists v{\le}0\, \varphi(v) \to G)$ by Gen
- Extend to $\langle\beta^*\rangle \exists v{\le}0\, \varphi(v) \vdash_{\mathcal{D}} \langle\beta^*\rangle G$ by $\langle\rangle$gen.
- From $\vdash_{\mathcal{D}} F \to \exists v\, \varphi(v)$ conclude $F \vdash_{\mathcal{D}} \exists v\, \varphi(v)$.

Relative Completeness of First-Order Liveness Assertions

Proof (α of the form β^*).

- $\vDash F \to \langle\beta^*\rangle G$ derivable by variant convergence:
- $$\exists\vec{v}\,\exists Z\,\left(Z_1^{(n)} = \vec{x} \land Z_n^{(n)} = \vec{v} \land \forall i : \mathbb{N}\,(1 \le i < n \to \mathcal{S}_\beta(Z_i^{(n)}, Z_{i+1}^{(n)})) \land G_{\vec{x}}^{\vec{v}}\right)$$
- Thus $\exists v\,\varphi(v) \vdash_{\mathcal{D}} \langle\beta^*\rangle\exists v{\le}0\,\varphi(v)$ by convergence con.
- From assumption, conclude valid FOD-formulas, hence $\mathcal{D}$-axioms:
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- Derive $\vdash_{\mathcal{D}} \forall^\beta(\exists v{\le}0\,\varphi(v) \to G)$ by Gen
- Extend to $\langle\beta^*\rangle\exists v{\le}0\,\varphi(v) \vdash_{\mathcal{D}} \langle\beta^*\rangle G$ by $\langle\rangle$gen.
- From $\vdash_{\mathcal{D}} F \to \exists v\,\varphi(v)$ conclude $F \vdash_{\mathcal{D}} \exists v\,\varphi(v)$.
- Combine propositionally by a cut to $F \vdash_{\mathcal{D}} \langle\beta^*\rangle G$.

Relative Completeness of Differential Logic Calculus

Theorem (Relative Completeness)

d$\mathcal{L}$ *calculus is complete relative to first-order logic of differential equations.*

$$\vDash \phi \quad \textit{iff} \quad Taut_{FOD} \vdash \phi$$

$$\textit{where} \quad FOD = FOL_{\mathbb{R}} + [x_1' = \theta_1, \ldots, x_n' = \theta_n]F$$

Relative Completeness of Differential Logic Calculus

Proof Idea.

- By propositional recombination, inductively identify fragments of ϕ that correspond to $\phi_1 \to [\alpha]\phi_2$ or $\phi_1 \to \langle\alpha\rangle\phi_2$ logically.

$\square$

Relative Completeness of Differential Logic Calculus

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- By propositional recombination, inductively identify fragments of ϕ that correspond to $\phi_1 \to [\alpha]\phi_2$ or $\phi_1 \to \langle\alpha\rangle\phi_2$ logically.
- Express subformulas ϕ_i equivalently in FOD and resolve these first-order safety or liveness assertions by previous propositions.

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- Express subformulas ϕ_i equivalently in FOD and resolve these first-order safety or liveness assertions by previous propositions.
- Finally, prove that the original d$\mathcal{L}$ formula can be re-derived from the subproofs.

$\square$

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- Remainder of proof follows induction on a measure $|\phi|$ defined as the number of modalities in ϕ.
- For a simple and uniform proof, assume quantifiers to be abbreviations for modal formulas:

$$\exists x\,\phi \equiv \langle x' = 1\rangle\phi \vee \langle x' = -1\rangle\phi$$
$$\forall x\,\phi \equiv [x' = 1]\phi \wedge [x' = -1]\phi$$

Proof.

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Relative Completeness of Differential Logic Calculus

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- ϕ is of the form $\phi_1 \wedge \phi_2$, then individually deduce the simpler proofs for $\vdash_{\mathcal{D}} \phi_1$ and $\vdash_{\mathcal{D}} \phi_2$ by IH, which can be combined by $\wedge$r.

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- ϕ disjunction, hence (otherwise use associativity and commutativity):

$$\begin{array}{ccc} \phi_1 & \lor & [\alpha]\phi_2 \\ \phi_1 & \lor & \langle\alpha\rangle\phi_2 \end{array}$$

Proof.

- Unified notation: $\phi_1 \lor \langle\![\alpha]\!\rangle\phi_2$.

Relative Completeness of Differential Logic Calculus

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Relative Completeness of Differential Logic Calculus

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- There are equivalent FOD-formulas $\phi_1^\#, \phi_2^\#$ with $\vDash \phi_i \leftrightarrow \phi_i^\#$.
- By congruence, $\vDash \phi$ yields $\vDash \phi_1^\# \lor \langle\![\alpha]\!\rangle\phi_2^\#$, thus $\vDash \neg\phi_1^\# \to \langle\![\alpha]\!\rangle\phi_2^\#$.

Relative Completeness of Differential Logic Calculus

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- By previous propositions derive

$$\neg\phi_1^\# \vdash_{\mathcal{D}} \langle\!\![\alpha]\!\!\rangle \phi_2^\# \tag{1}$$

Proof.

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- Then, $|\phi_2| < |\phi|$, since ϕ_2 has less modalities.
- $|\phi_1| < |\phi|$ as $\langle\![\alpha]\!\rangle\phi_2$ contributes one modality to $|\phi|$ that is not in ϕ_1.
- There are equivalent FOD-formulas $\phi_1^\#, \phi_2^\#$ with $\vDash \phi_i \leftrightarrow \phi_i^\#$.
- By congruence, $\vDash \phi$ yields $\vDash \phi_1^\# \lor \langle\![\alpha]\!\rangle\phi_2^\#$, thus $\vDash \neg\phi_1^\# \to \langle\![\alpha]\!\rangle\phi_2^\#$.
- By previous propositions derive

$$\neg\phi_1^\# \vdash_{\mathcal{D}} \langle\![\alpha]\!\rangle\phi_2^\# \tag{1}$$

- $\vDash \phi_1 \leftrightarrow \phi_1^\#$ implies $\vDash \neg\phi_1 \to \neg\phi_1^\#$, which is derivable by IH, because $|\phi_1| < |\phi|$. By lemma, $\neg\phi_1 \vdash_{\mathcal{D}} \neg\phi_1^\#$, which we combine with (1) by a cut with $\neg\phi_1^\#$ to

$$\neg\phi_1 \vdash_{\mathcal{D}} \langle\![\alpha]\!\rangle\phi_2^\# \quad . \tag{2}$$

Proof.

- $\vDash \phi_2 \leftrightarrow \phi_2^{\#}$ implies $\vDash \phi_2^{\#} \to \phi_2$, which is derivable by IH, as $|\phi_2| < |\phi|$.

$\square$

Relative Completeness of Differential Logic Calculus

Proof.

- $\vDash \phi_2 \leftrightarrow \phi_2^{\#}$ implies $\vDash \phi_2^{\#} \to \phi_2$, which is derivable by IH, as $|\phi_2| < |\phi|$.
- Extend derivation of $\vdash_{\mathcal{D}} \phi_2^{\#} \to \phi_2$ to one of $\vdash_{\mathcal{D}} \forall^{\alpha}(\phi_2^{\#} \to \phi_2)$ by Gen

$\square$

Relative Completeness of Differential Logic Calculus

Proof.

- $\vDash \phi_2 \leftrightarrow \phi_2^\#$ implies $\vDash \phi_2^\# \to \phi_2$, which is derivable by IH, as $|\phi_2| < |\phi|$.
- Extend derivation of $\vdash_{\mathcal{D}} \phi_2^\# \to \phi_2$ to one of $\vdash_{\mathcal{D}} \forall^\alpha(\phi_2^\# \to \phi_2)$ by Gen
- Thus $\langle\!\![\alpha]\!\!\rangle\phi_2^\# \vdash_{\mathcal{D}} \langle\!\![\alpha]\!\!\rangle\phi_2$ by []gen or ⟨⟩gen.

$\square$

Proof.

- $\vDash \phi_2 \leftrightarrow \phi_2^{\#}$ implies $\vDash \phi_2^{\#} \to \phi_2$, which is derivable by IH, as $|\phi_2| < |\phi|$.
- Extend derivation of $\vdash_{\mathcal{D}} \phi_2^{\#} \to \phi_2$ to one of $\vdash_{\mathcal{D}} \forall^{\alpha}(\phi_2^{\#} \to \phi_2)$ by Gen
- Thus $\langle\!\![\alpha]\!\!\rangle\phi_2^{\#} \vdash_{\mathcal{D}} \langle\!\![\alpha]\!\!\rangle\phi_2$ by []gen or $\langle\rangle$gen.
- Combine propositionally with (2) by a cut with $\langle\!\![\alpha]\!\!\rangle\phi_2^{\#}$ to derive $\neg\phi_1 \vdash_{\mathcal{D}} \langle\!\![\alpha]\!\!\rangle\phi_2$

$\square$

Relative Completeness of Differential Logic Calculus

Proof.

- $\vDash \phi_2 \leftrightarrow \phi_2^\#$ implies $\vDash \phi_2^\# \rightarrow \phi_2$, which is derivable by IH, as $|\phi_2| < |\phi|$.
- Extend derivation of $\vdash_{\mathcal{D}} \phi_2^\# \rightarrow \phi_2$ to one of $\vdash_{\mathcal{D}} \forall^\alpha(\phi_2^\# \rightarrow \phi_2)$ by Gen
- Thus $⟨\![\alpha]\!⟩\phi_2^\# \vdash_{\mathcal{D}} ⟨\![\alpha]\!⟩\phi_2$ by []gen or $\langle\rangle$gen.
- Combine propositionally with (2) by a cut with $⟨\![\alpha]\!⟩\phi_2^\#$ to derive $\neg\phi_1 \vdash_{\mathcal{D}} ⟨\![\alpha]\!⟩\phi_2$
- Conclude $\vdash_{\mathcal{D}} \phi_1 \vee ⟨\![\alpha]\!⟩\phi_2$ with a cut.

$\square$

Relative Completeness

Theorem (Relative Completeness)

d$\mathcal{L}$ *calculus is a sound & complete axiomatisation of hybrid systems relative to differential equations.*

▸ Proof Outline 15p

continuous discrete repeat

$\Downarrow$

Relativity

Cook,Harel:	discrete-DL/data	P.:	hybrid-d$\mathcal{L}$/differential equations