15-819/18-879: Logical Analysis of Hybrid Systems

09: Real Algebraic Geometry

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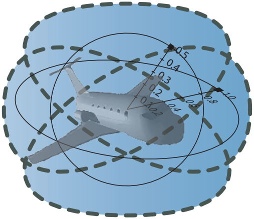


Outline

1. Connection with Real Algebraic Geometry
   - Algebraic Varieties
   - Semialgebraic Sets

What is the correspondence between logical formulas of reals and subsets in real space?

Outline

1. Connection with Real Algebraic Geometry
   - Algebraic Varieties
   - Semialgebraic Sets

You don't have to know all algebraic geometry, but develop intuition

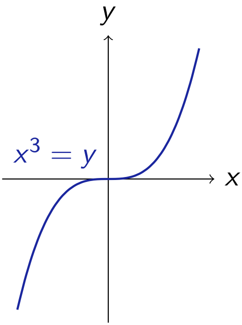
Definition (Real Affine Algebraic Variety)

$V \subseteq \mathbb{R}^n$ is an *affine variety* iff, for some set $F \subseteq \mathbb{R}[X_1, \dots, X_n]$ of polynomials over $\mathbb{R}$:

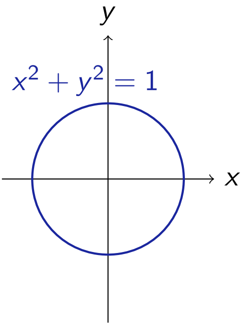
$$V = V(F) := \{x \in \mathbb{R}^n \ : \ f(x) = 0 \text{ for all } f \in F\}$$

i.e., affine varieties are subsets of $\mathbb{R}^n$ that are definable by a set of polynomial equations.

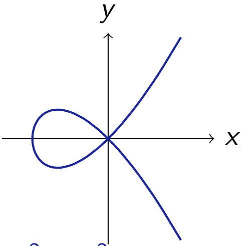
Algebraic Variety Examples



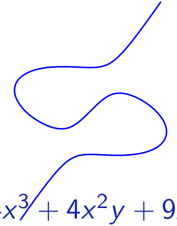
$x^3 = y$



$x^2 + y^2 = 1$

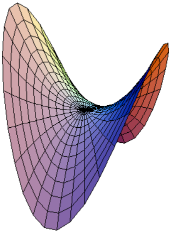


$y^2 = x^2(x+1)$



$4x^3 + 4x^2y + 9xy^2 - 9y^3 - 36x + 36y = 0$

$z = x^2 - y^2$



Definition (Semialgebraic Set)

$S \subseteq \mathbb{R}^n$ is an *semialgebraic set* iff it is defined by a finite intersection of polynomial equations *and inequalities* or any finite union of such sets.

$$S = \bigcup_{i=1}^{t} \bigcap_{j=1}^{s} \{x \in \mathbb{R}^n \ : \ p(x) \sim 0\} \quad \text{with any } \sim \in \{=, \geq, >\}$$

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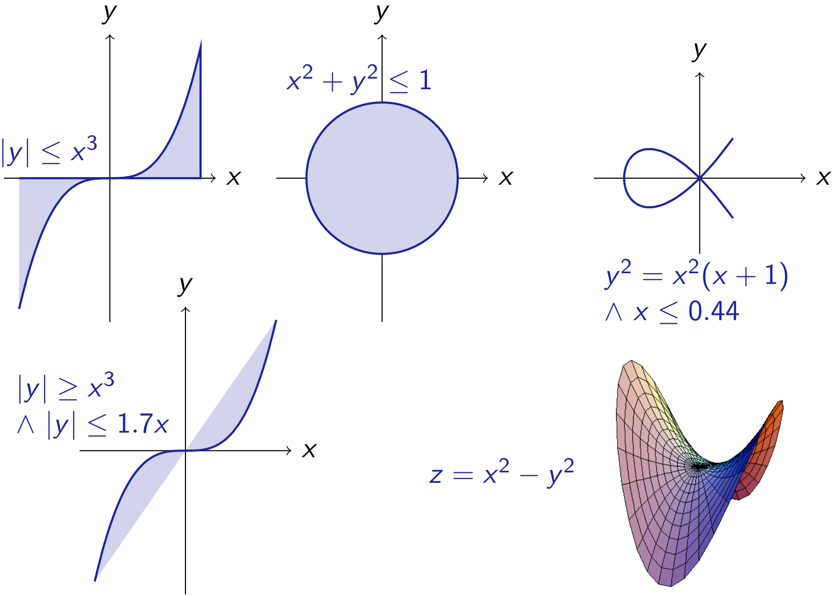
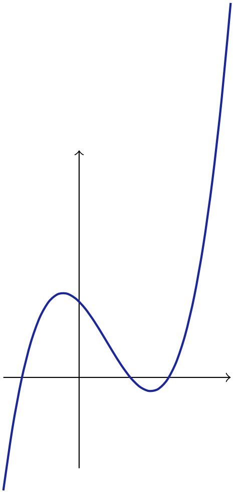
Theorem (Tarski'30,'51, Seidenberg'54)

Semialgebraic sets are closed under finite unions, finite intersections, complements and projection to linear subspaces.

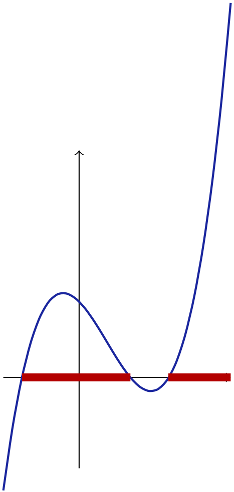


y
x
$|y| \leq x^3$
y
x
$x^2 + y^2 \leq 1$
y
x
$y^2 = x^2(x+1)$
$\land\ x \leq 0.44$
y
x
$|y| \geq x^3$
$\land\ |y| \leq 1.7x$
$z = x^2 - y^2$



$$\exists y\,(y \geq 0 \land 1 - x - 1.83x^2 + 1.66x^3 > y)$$

Quantifier Elimination and Projection



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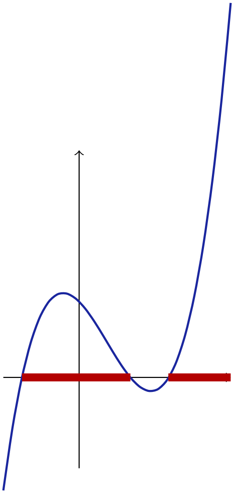
Quantifier Elimination and Projection

$\exists y\, (y \geq 0 \wedge 1 - x - 1.83x^2 + 1.66x^3 > y)$



$0.75 < x \wedge x < 0.68 \vee x > 1.17$