15-819/18-879: Logical Analysis of Hybrid Systems

08: First-Order Real Arithmetic

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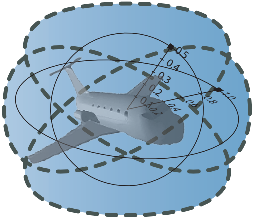

Outline

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1. First-Order Real Arithmetic
   - Axioms of Reals
   - Real-Closed Fields

In hybrid systems, there is significant logical structure in the properties, the system, the reasoning, ...

We need to understand the reals first

Axioms of Reals

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15. Sup "Non-empty subsets with upper bounds have supremum"

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But the first-order "view" of the reals is still fairly amazing

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- What's missing?
- That we have enough elements in R.

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5. R is a real field such that no proper algebraic extension is a formally real field.

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Example (Real-closed fields)

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- Computable numbers, i.e., those that can be approximated by a computable function up to any desired precision. π lives here!
- ZFC-Definable numbers, i.e., those real numbers $a \in \mathbb{R}$ for which there is a first-order formula φ in set theory with one free variable such that a is the unique real number for which φ holds true.

$$I, \beta \models \varphi \text{ iff } I(x) = a$$

The advantages of implicit definition over construction are roughly those of theft over honest toil [Russell]

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10. $\neg\exists x_1 \ldots \exists x_n\, (-1 = x_1^2 + \cdots + x_n^2)$ for any n
11. $\forall x\, \forall y\, (x < y \land p(x)p(y) < 0 \to \exists z\, (x < z < y \land p(z) = 0))$ for polynomial p (intermediate value property)

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