15-819/18-879: Logical Analysis of Hybrid Systems

07: Quantifier Elimination

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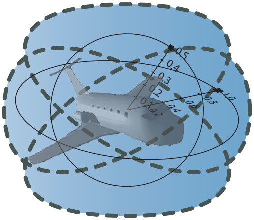

Outline

1. Interpreted Logic: First-Order Real Arithmetic
   - Syntax
   - Semantics
   - Quantifier Elimination

In hybrid systems, there is significant logical structure in the properties, the system, the reasoning, ...
We need to understand the reals first

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   - Syntax
   - Semantics
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Interpreted First-order Logic

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$$(x_1 - y_1)^2 + (x_2 - y_2)^2 > p^2$$

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- Interpreted first-order logic is like first-order logic, except that some symbols have a fixed semantics (all interpretations agree on the semantics of those symbols).
- Our primary focus: first-order real arithmetic $\mathrm{FOL}_{\mathbb{R}}$

First-order Logic of Real Arithmetic: Syntax

Definition (Interpreted $FOL_{\mathbb{R}}$ Term t)

$t ::=$

x	for variable $x \in V$
r	for rational number r
$t_1 + t_2$	(infix notation)
$t_1 - t_2$	(infix notation)
$t_1 \cdot t_2$	(infix notation)
$f(t_1, \ldots, t_n)$	for function $f/n \in \Sigma$ of arity $n \geq 0$

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Definition (Interpreted $FOL_{\mathbb{R}}$ Formula F, G)

$F ::=$

$t_1 \geq t_2$	(infix notation)
$t_1 > t_2$	(infix notation)
$t_1 = t_2$	(infix notation)
$p(t_1, \ldots, t_n)$	for predicate $p/n \in \Sigma$ of arity $n \geq 0$
$\neg F$	"not"
$(F \land G)$	"and"
$(F \lor G)$	"or"
$(F \to G)$	"implies"
$(F \leftrightarrow G)$	"equivalent/bi-implies"
$\forall x\, F$	"universal quantifier/forall" for $x \in V$
$\exists x\, F$	"existential quantifier/exists" for $x \in V$

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Quiz

Is this a formula of first-order real arithmetic?

1. $F \lor (G \land (H \leftrightarrow \neg F) \to J)$
2. $\forall x\, (p(x) \to \exists y\, (p(y) \land \exists x\, \neg r(x, y)))$
3. $\forall x\, \forall y\, (x > y \leftrightarrow x - y > 0)$
4. $x < y \land \exists z\, x > z^2$
5. $x > 0 \land \forall y\, \exists z\, (x > z^2 + y \cdot z - 5)$
6. $\forall x\, \exists y\, x > x^y$
7. $\exists x\, \forall y\, x > y + \pi$
8. $(\exists x\, \neg \exists y\, x > y + 3.1415926) \to \forall x\, (x^2 > x^3)$

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First-order Logic of Real Arithmetic: Semantics

Definition ($FOL_{\mathbb{R}}$ Interpretation I)

1. $D = \mathbb{R}$
2. I assigns relations and functions on $\mathbb{R}$ to all symbols in Σ
   - function $I(f) : \mathbb{R}^n \to \mathbb{R}$ for each function symbol f of arity n
   - relation $I(p) \subseteq \mathbb{R}^n$ for each predicate symbol p of arity n
   - element $I(c) \in \mathbb{R}$ for each constant symbol (function of arity 0)
   - truth-value $I(p) \in \{true, false\}$ for each predicate symbol of arity 0

   such that

   - $I(+)$ is addition on $\mathbb{R}$
   - $I(-)$ is subtraction on $\mathbb{R}$
   - $I(\cdot)$ is multiplication on $\mathbb{R}$
   - $I(=)$ is equality on $\mathbb{R}$
   - $I(>)$ is the greater relation on $\mathbb{R}$
   - $I(\geq)$ is the greater-equals relation on $\mathbb{R}$
   - $I(r) = r$ for all numbers $r \in \mathbb{Q}$

Quiz

(Validity of) which of the following logics is
decidable/semidecidable/undecidable/not semidecidable?

1. PL_0
2. FOL
3. $FOL_{\mathbb{N}}[+, \cdot, =]$
4. $FOL_{\mathbb{R}}[+, \cdot, =, <]$
5. $FOL_{\mathbb{Q}}[+, \cdot, =]$
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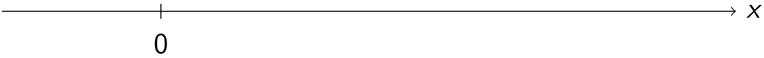
Quantifier Elimination by Example



Can we get rid of the quantifier without changing the semantics of the formula?

$$\exists x (x > 2 \land x < \tfrac{17}{3})$$

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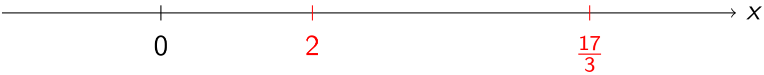

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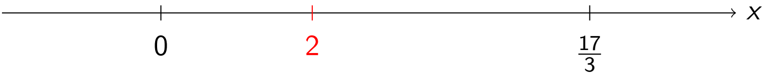

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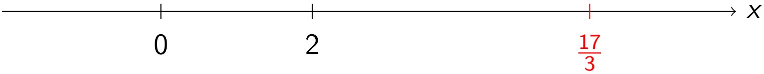
$$\begin{array}{lll} & \exists x(x > 2 \land x < \frac{17}{3}) & \\ \equiv & (2 > 2 \land 2 < \frac{17}{3}) & \quad \text{border case "}x = 2\text{"} \end{array}$$



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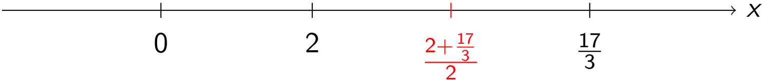

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\lor & (\frac{2+\frac{17}{3}}{2} > 2 \land \frac{2+\frac{17}{3}}{2} < \frac{17}{3}) & \text{intermediate case "} x = \frac{2+\frac{17}{3}}{2} \text{"}
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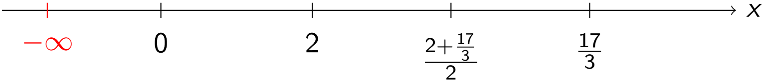

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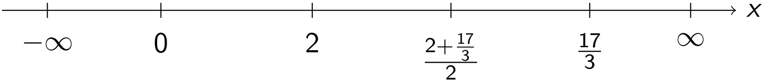

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\equiv & \mathit{true} & \text{evaluate}
\end{array}
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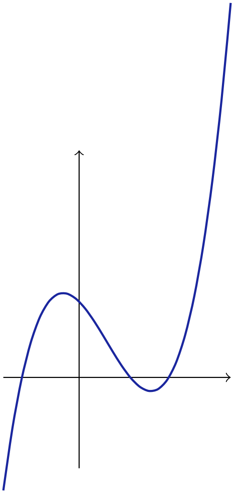
Quantifier Elimination and Projection



$$\exists y\, (y \geq 0 \land 1 - x - 1.83x^2 + 1.66x^3 > y)$$

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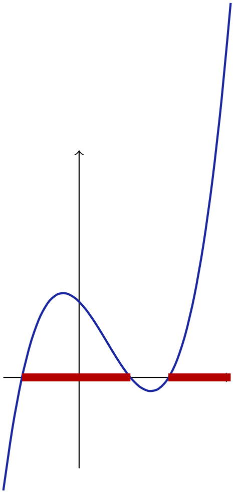
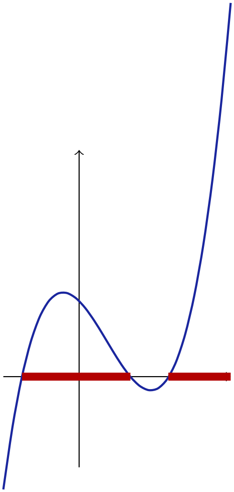


Quantifier Elimination and Projection

$\exists y\,(y \geq 0 \land 1 - x - 1.83x^2 + 1.66x^3 > y)$



$0.75 < x \land x < 0.68 \lor x > 1.17$

Quantifier Elimination in Real-Closed Fields

Definition (Quantifier elimination)

A first-order theory admits *quantifier elimination* if to each formula ϕ, a quantifier-free formula $QE(\phi)$ can be effectively associated that is equivalent (i.e., $\phi \leftrightarrow QE(\phi)$ is valid) and has no other free variables. The operation QE is further assumed to evaluate ground formulas (i.e., without variables), yielding a decision procedure for this theory.

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Theorem (Tarski'30,'51, Seidenberg'54)

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Theorem (Complexity, Davenport&Heintz'88,Weispfenning'88)

(Time and space) complexity of QE *for* $\mathbb{R}$ *is doubly exponential in the number of quantifier (alternations).*

Quantifier Elimination Examples

- $\text{QE}(\exists x\,(x^2 = 2)) \equiv$

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Quantifier Elimination Examples

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Quantifier Elimination Examples

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- $\text{QE}(\exists x\, (64 + 10x - 27x^2 - 11x^3 + 3x^4 + x^5 = 0 \land y = x)) \equiv$ $64 + 10y - 27y^2 - 11y^3 + 3y^4 + y^5 = 0$

Quiz

(Validity of) which of the following logics is decidable/semidecidable/undecidable/not semidecidable?

- ✓ PL_0 decidable
- ✓ FOL undecidable but semidecidable
- × $FOL_{\mathbb{N}}[+, \cdot, =]$ not semidecidable "Peano arithmetic" [Gödel'31]
- ✓ $FOL_{\mathbb{R}}[+, \cdot, =, <]$ decidable [Tarski'51]
- 5. $FOL_{\mathbb{Q}}[+, \cdot, =]$
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