15-819/18-879: Logical Analysis of Hybrid Systems

03: Hybrid Systems

André Platzer

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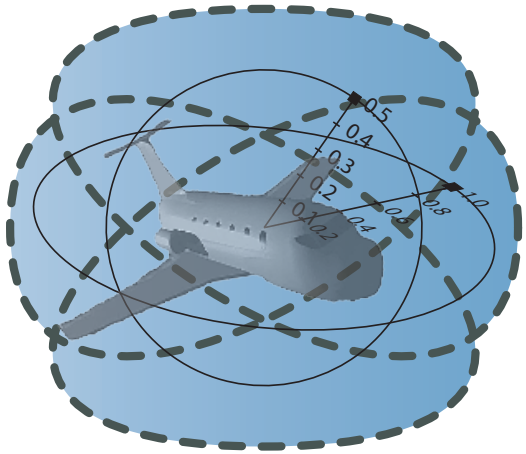

Outline

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Labeled Transition Systems

Definition (Labeled Transition System)

- Transition relation on $Q \times A \times Q$, denoted as $q \xrightarrow{a} q^+$, along with
- (possibly infinite) set A of transition actions,
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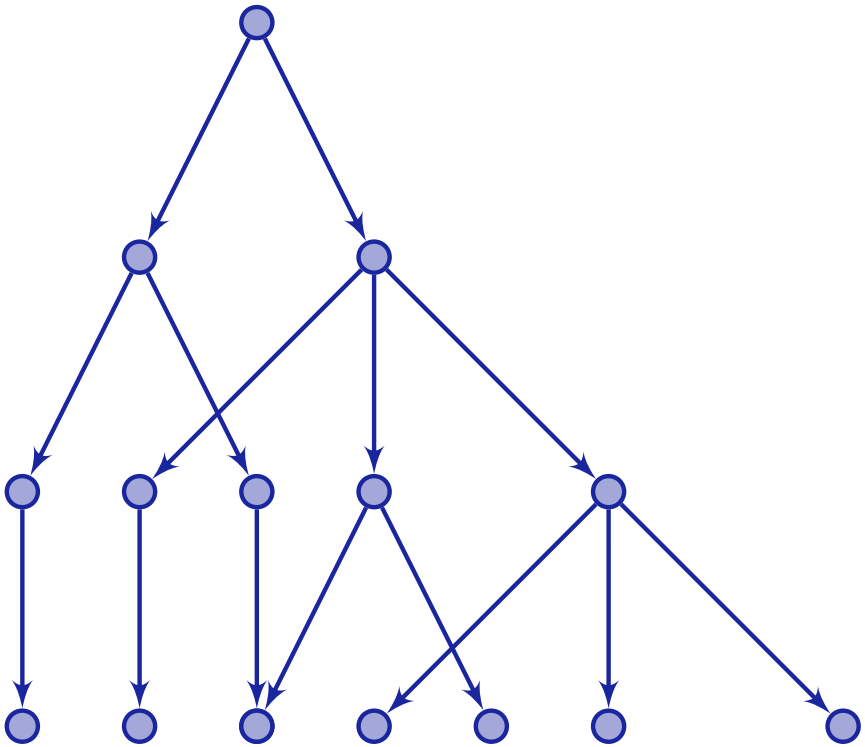
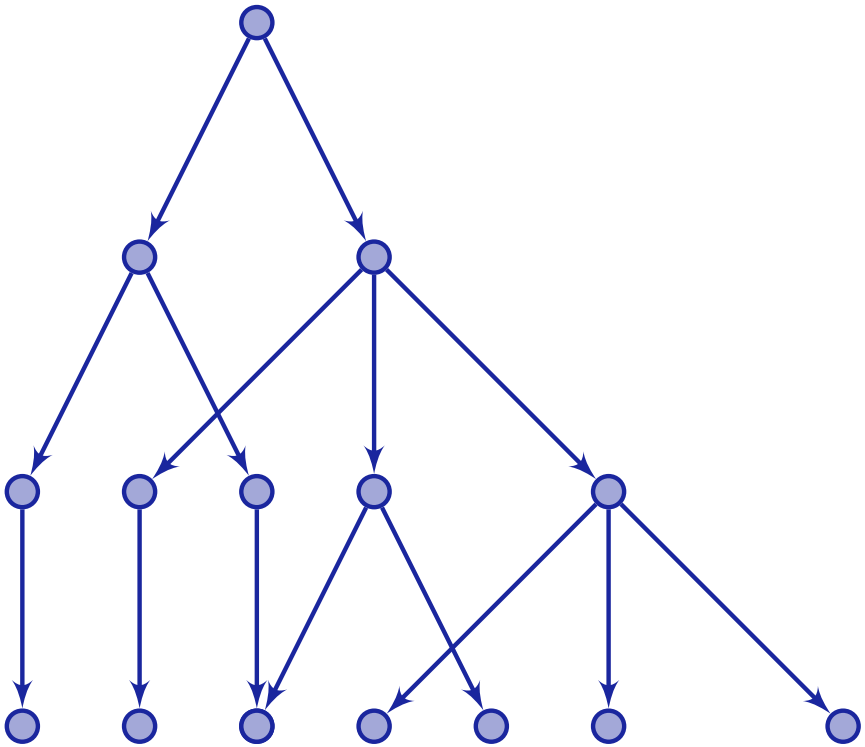


Labeled Transition Systems

Definition (Trace)

(In)Finite series $q_0, q_1, q_2, \ldots \in Q$ such that for some $a_i \in A$

$$q_0 \xrightarrow{a_0} q_1 \xrightarrow{a_1} q_2 \xrightarrow{a_2} q_3 \xrightarrow{a_3} q_4 \xrightarrow{a_4} \ldots$$



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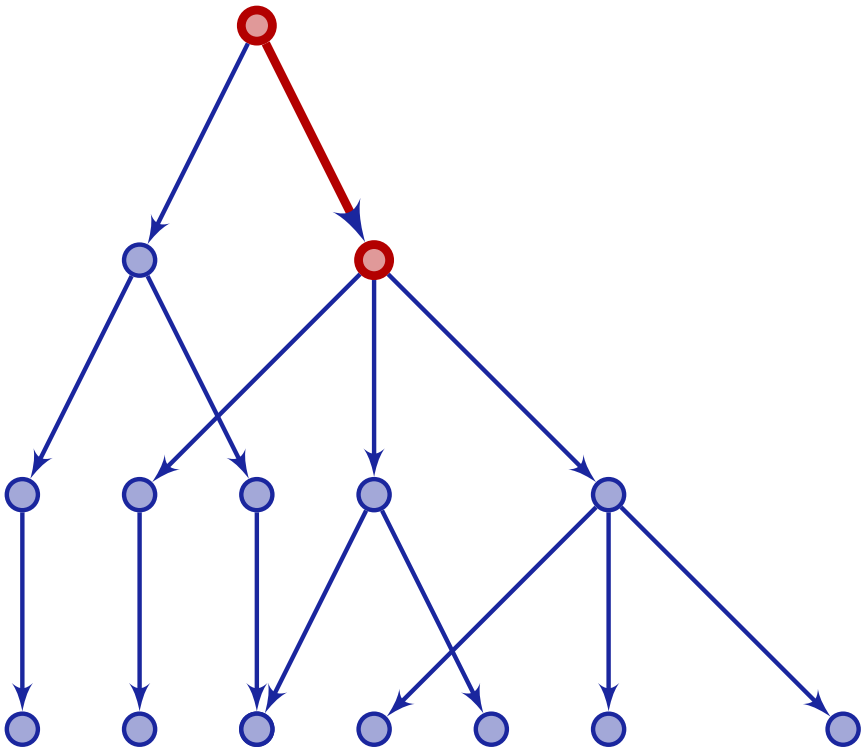

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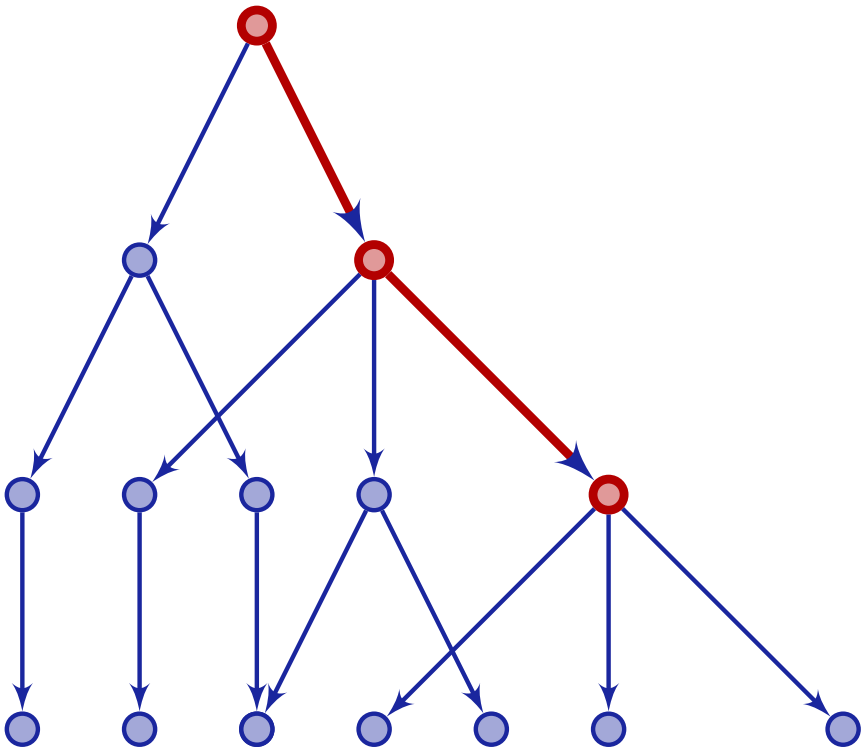

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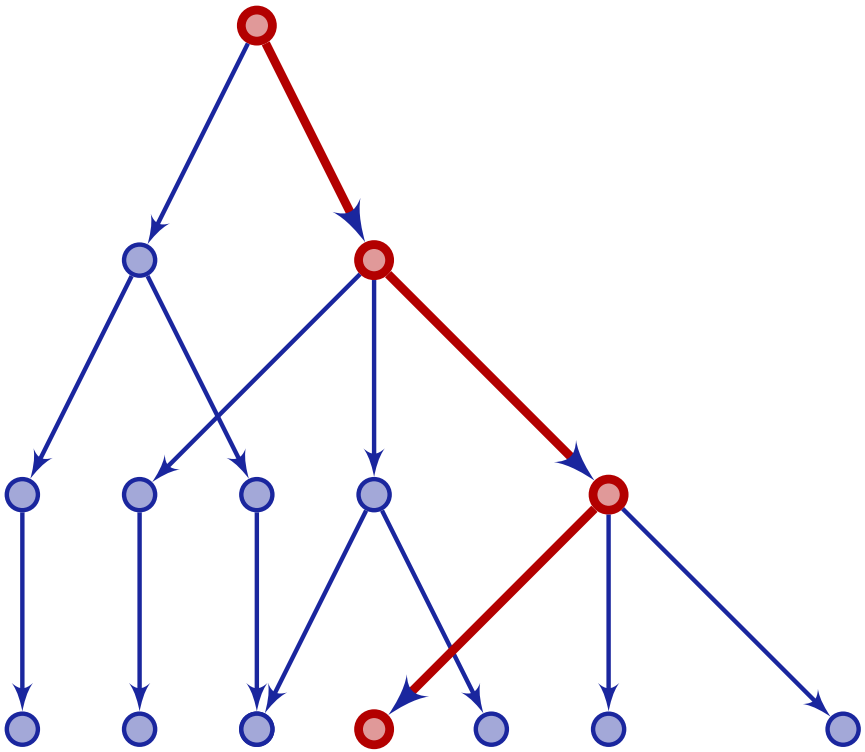

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Definition (Model Checking Problem for Reachability)

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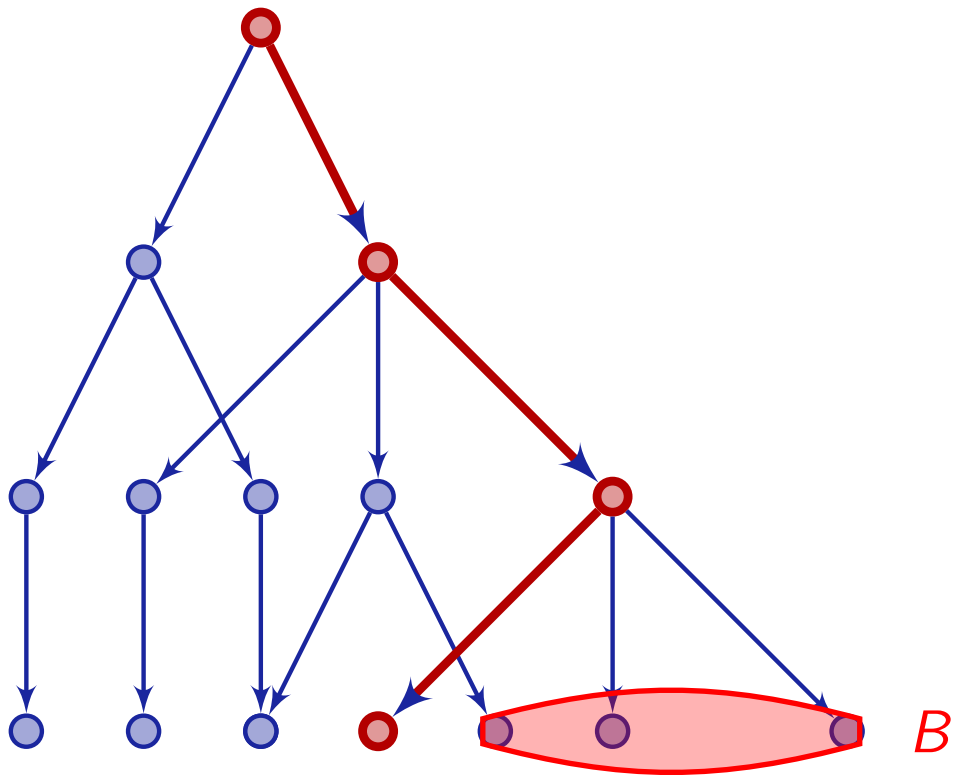

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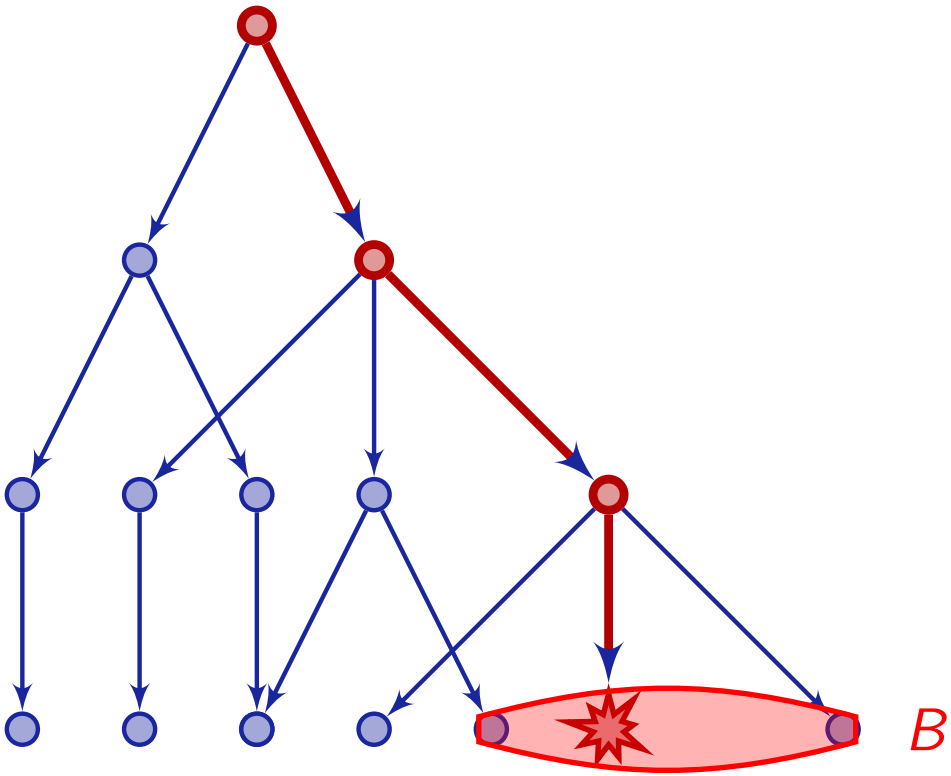
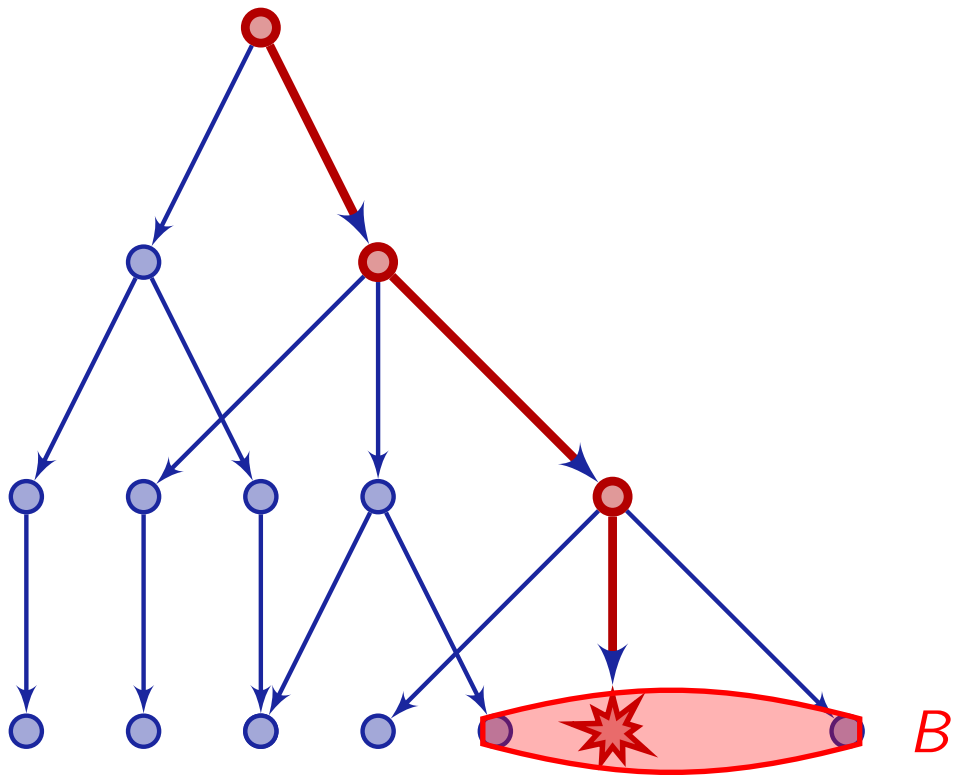


Labeled Transition Systems

Definition (Image Computation)

$$Post_A(Y) := \{q^+ \in Q : q \xrightarrow{a} q^+ \text{ for some } q \in Y, a \in A\}$$



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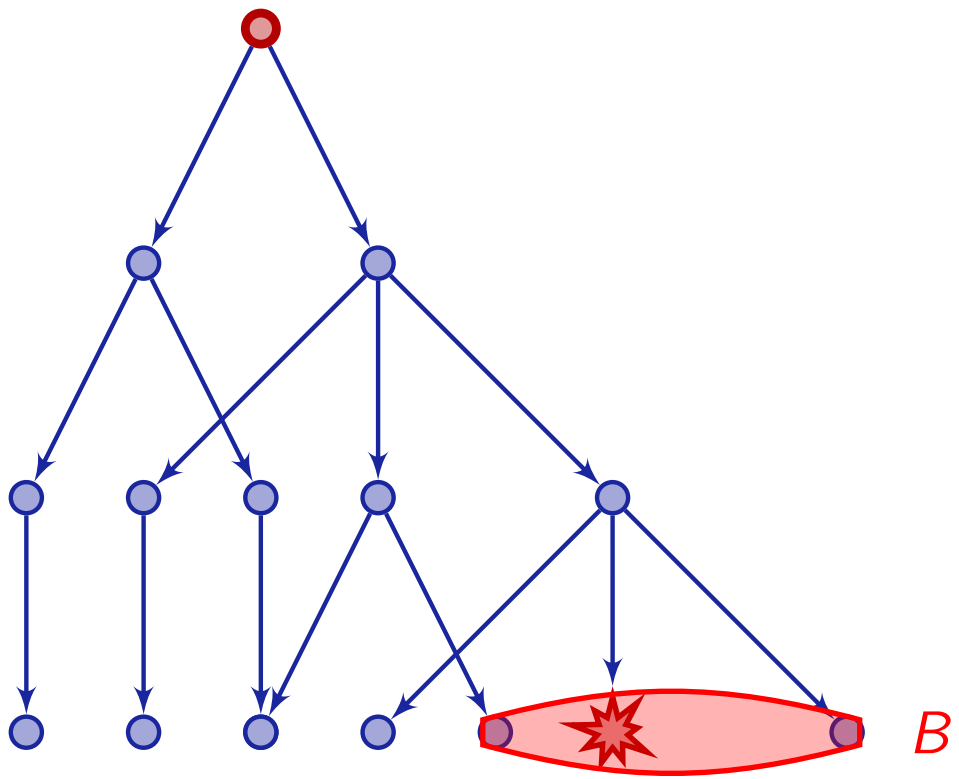

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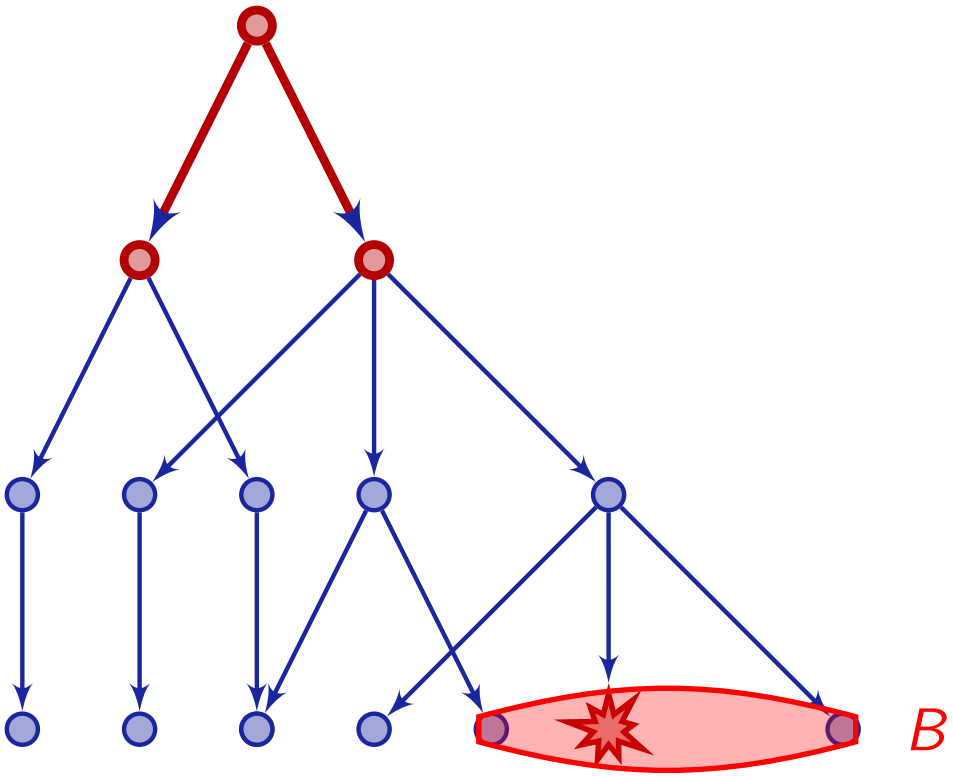

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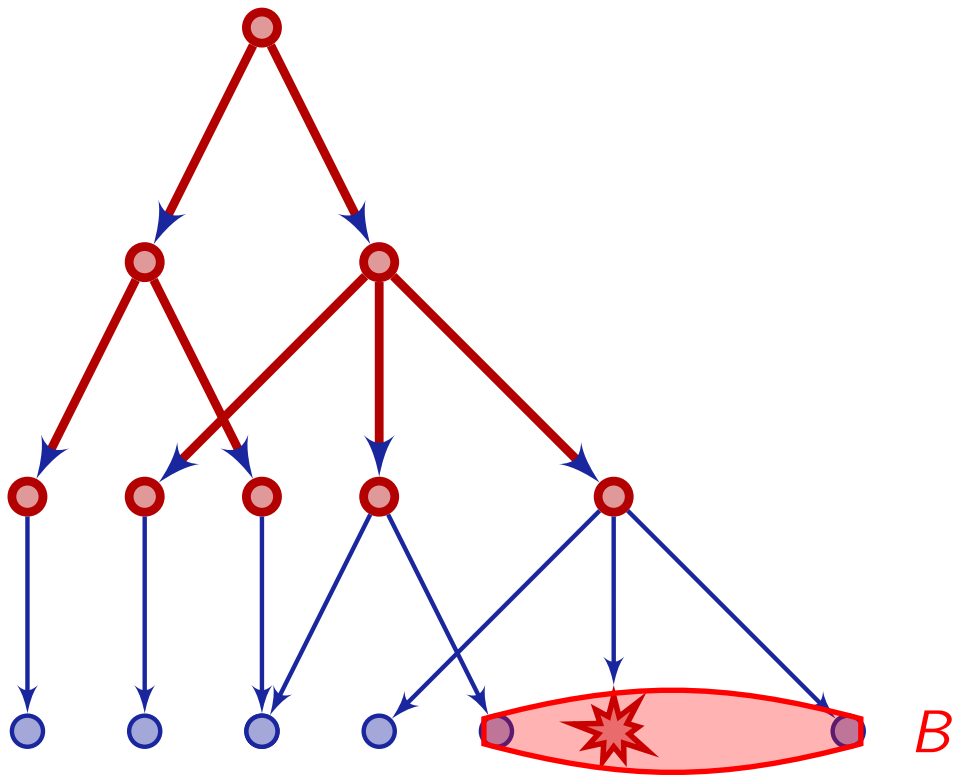

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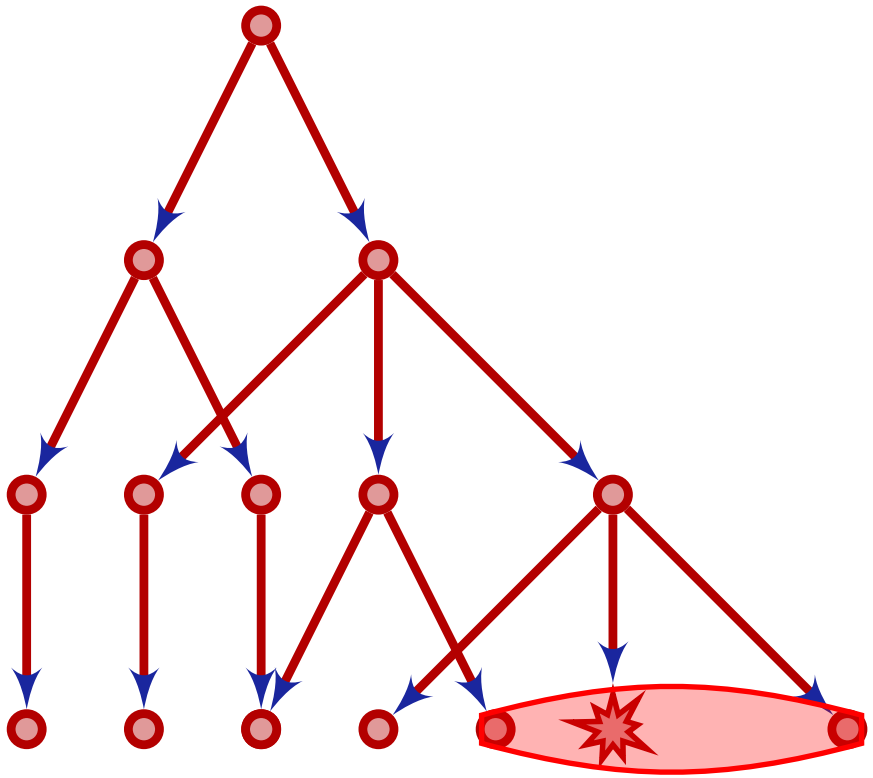

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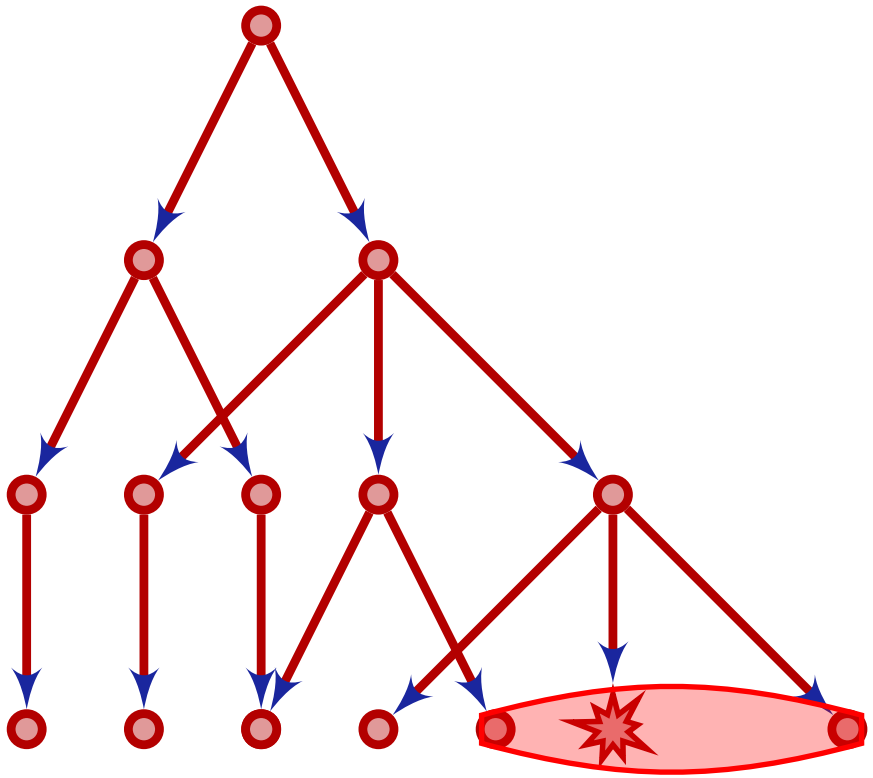
$$Post_A^*(Y) := \bigcup_{n \in \mathbb{N}} Post_A^n(Y) = \mu Z.(Y \cup Z \cup Post_A(Z))$$



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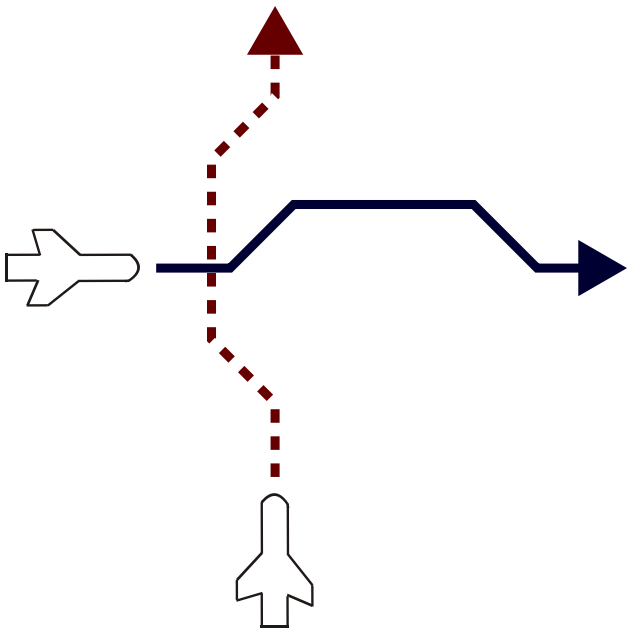
Finite Automata

Definition (Nondeterministic Finite Automata $\rightsquigarrow$ LTS)

- Transition relation on $Q \times A \times Q$, denoted as $q \xrightarrow{a} q^+$, along with
- finite set A of transition actions,
- finite set Q of states, initial states $Q_0 \subseteq Q$, accepting states $F \subseteq Q$.

Finite Automaton for Collision Avoidance



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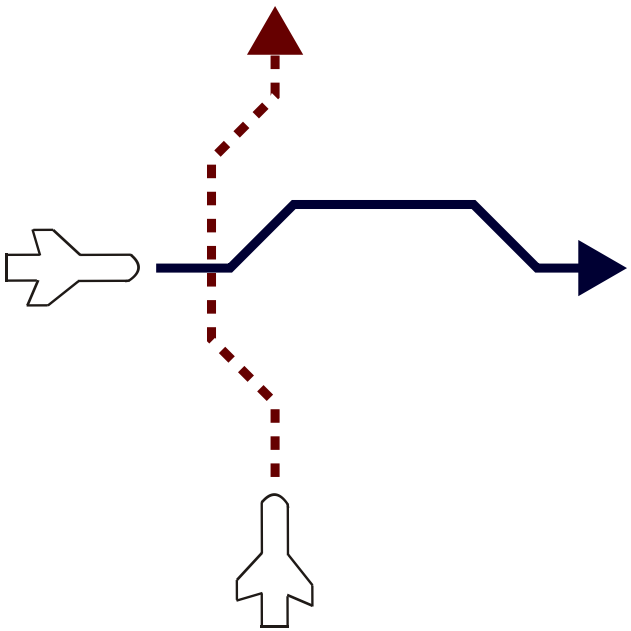

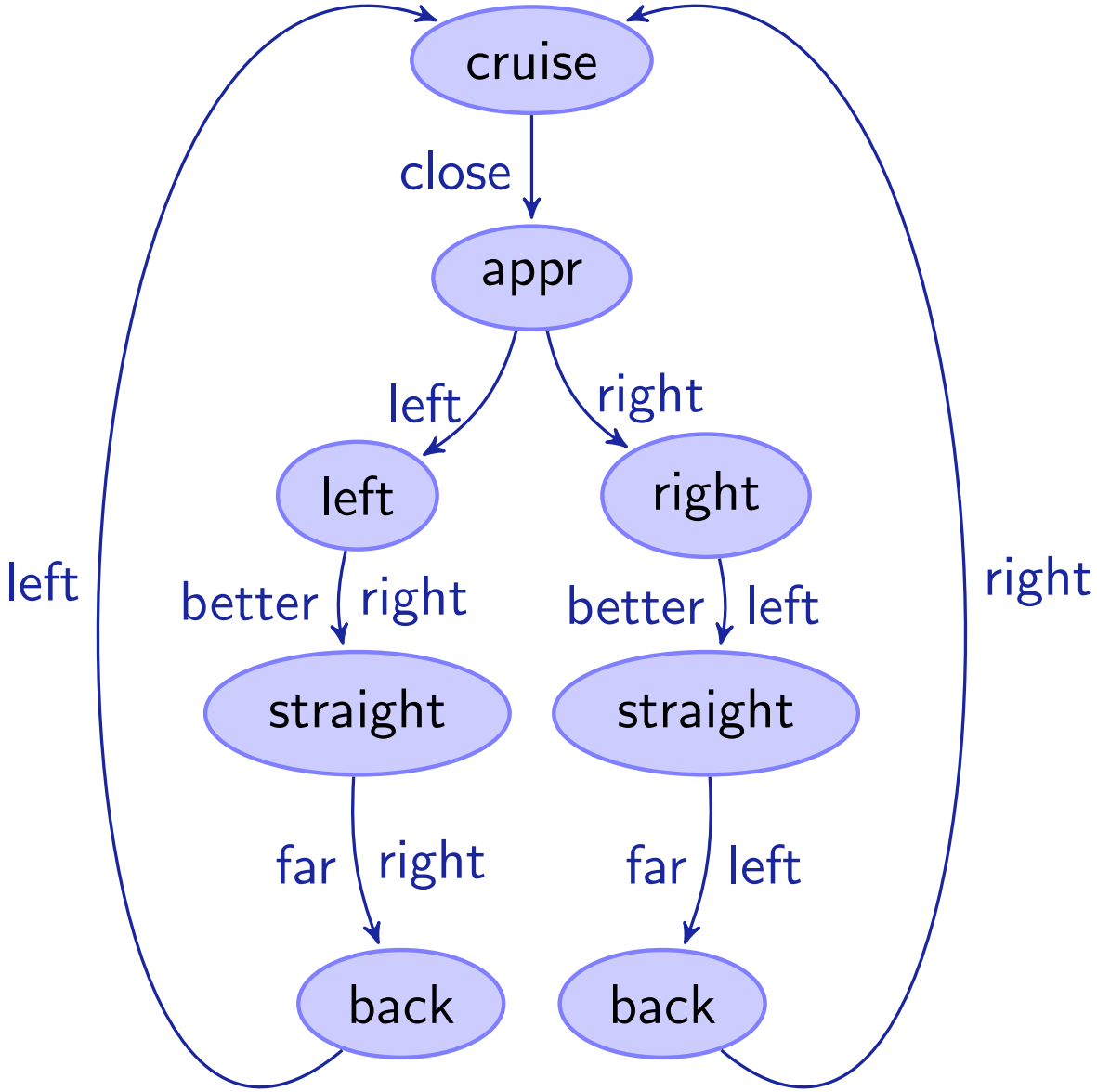

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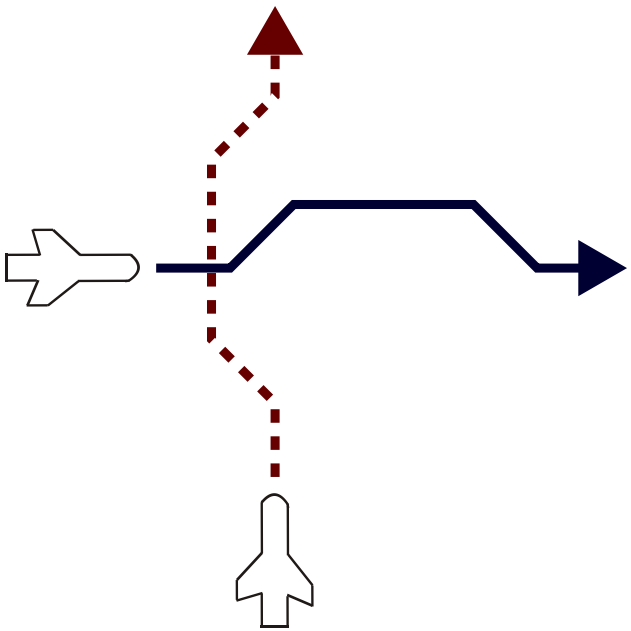
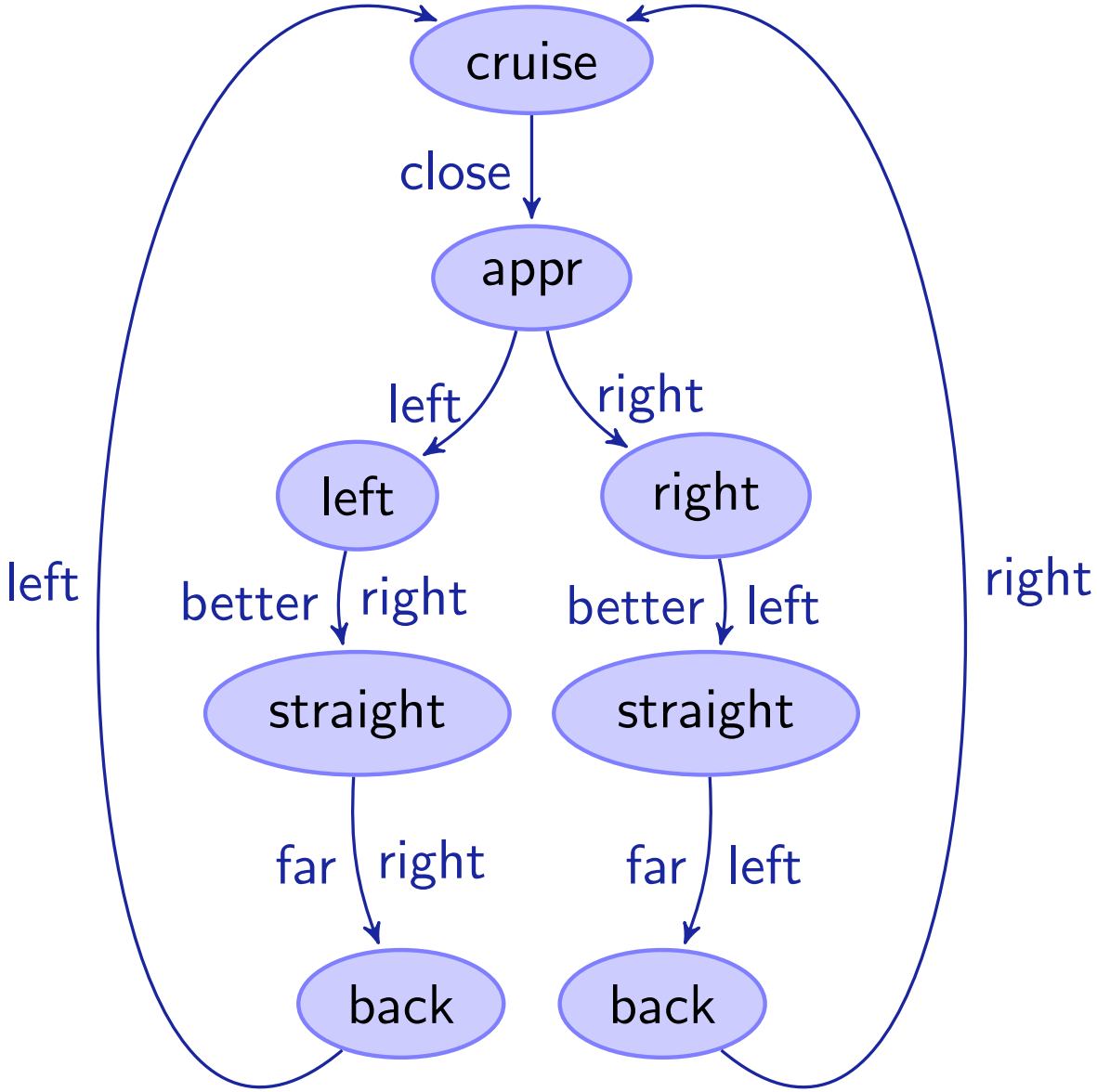


good model?



Collision avoidance is a property of controlled movement!

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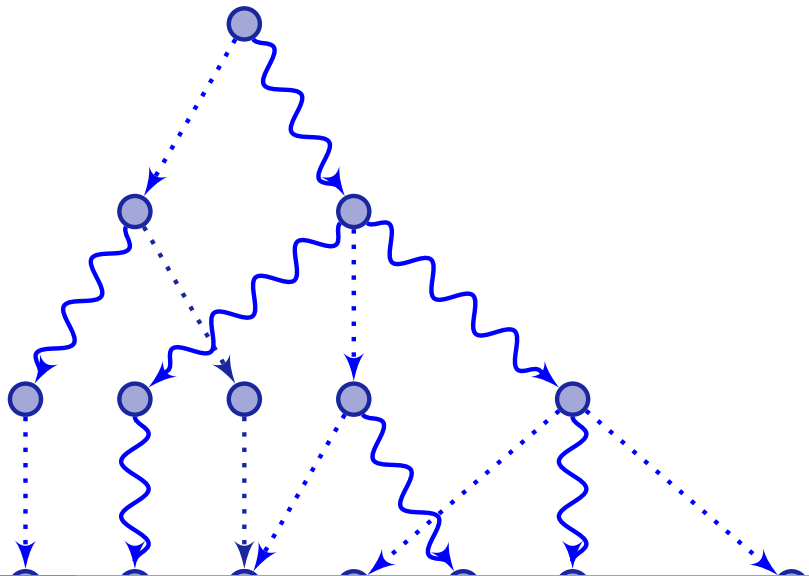

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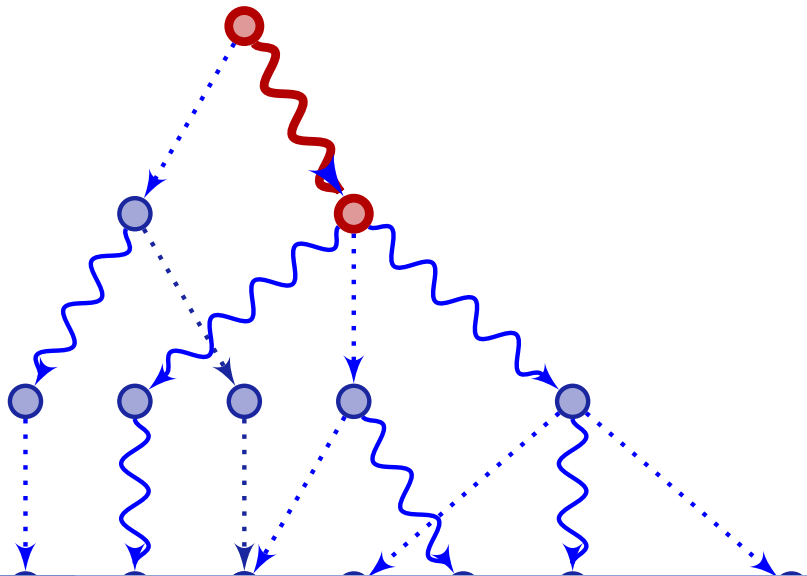

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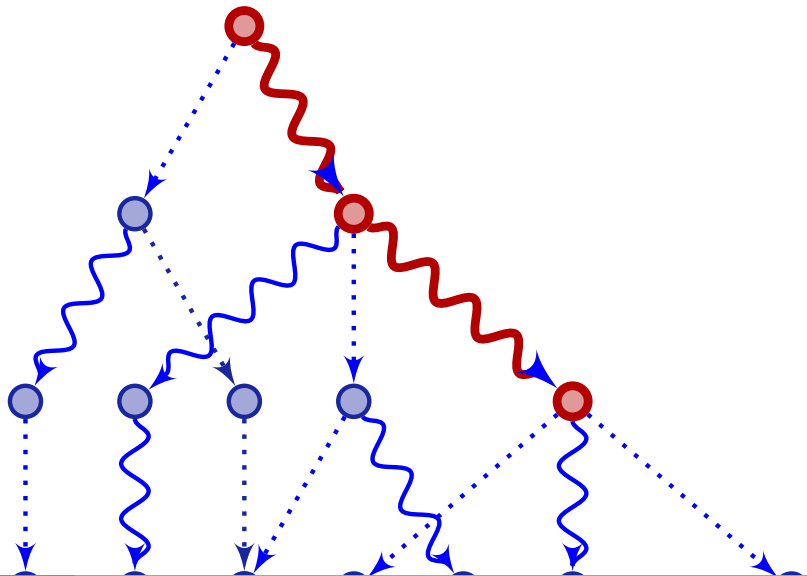

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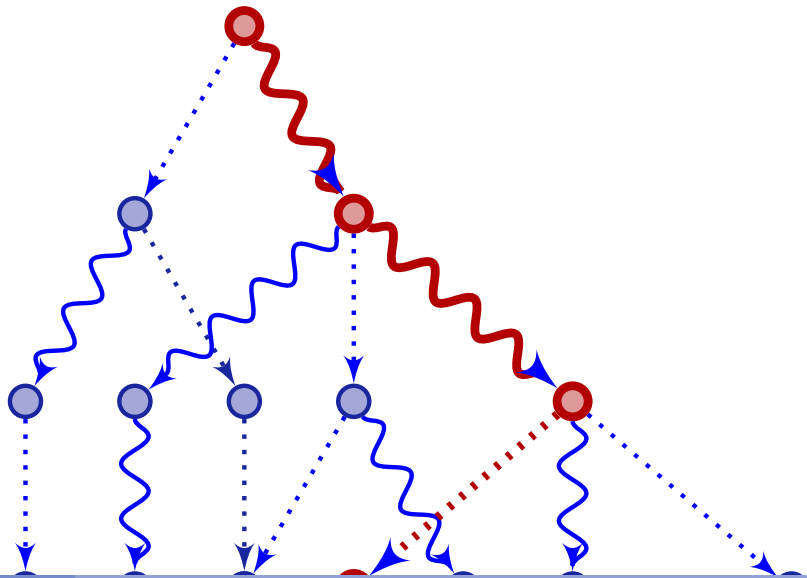

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Okay but how do we "justify" the respective transitions from our model??

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"Graph switching between continuous dynamical systems"

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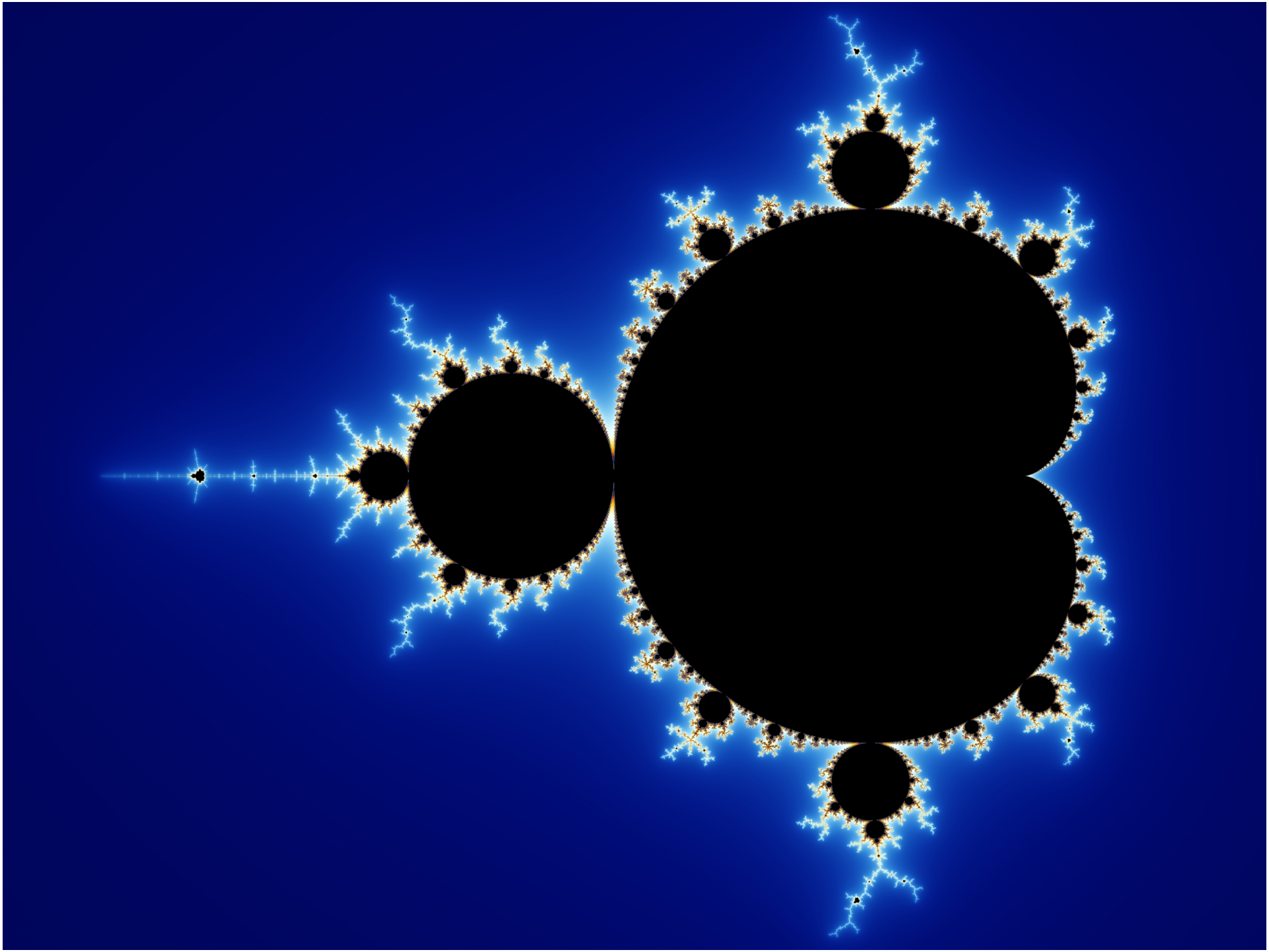
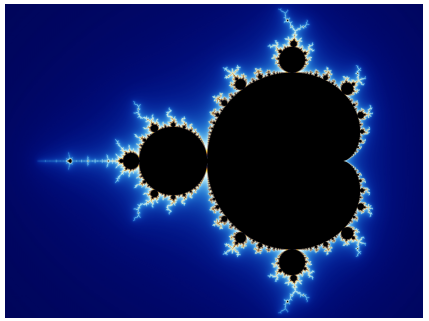


Mandelbrot Fractal



Example (Mandelbrot Set)

For complex numbers $c \in \mathbb{C}$ define $f_0(c) = c$ and $f_{n+1}(c) = f_n(c)^2 + c$. Then the Mandelbrot set is

$$\{c \in \mathbb{C} \ : \ f_n(c) \not\to \infty \text{ as } n \to \infty\}$$

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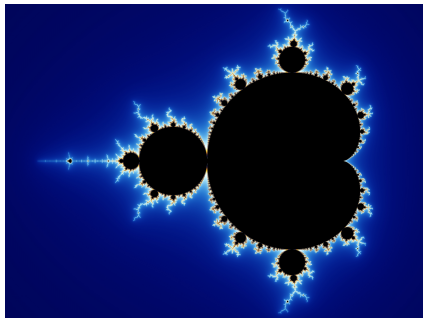

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Theorem (Lenore Blum, Cucker, Shub, Smale'90...98)

"The Mandelbrot set is undecidable over $\mathbb{R}$ / in Real Turing Machines"

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What if inv_q is a Mandelbrot set?

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And all relations decidable / definable in first-order real arithmetic etc.

Computationally relevant output needs computational input!


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