15-819/18-879: Logical Analysis of Hybrid Systems

02: Dynamical Systems

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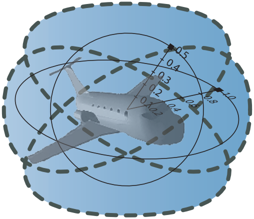

Outline

1. Dynamical Systems
   - Discrete Dynamical Systems
   - Continuous Dynamical Systems
   - Hybrid Dynamical Systems

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Dynamical System

Definition (Dynamical System)

One fixed rule describing temporal evolution of a point in state space $\mathcal{X}$.
That is, time T *acting on* $\mathcal{X}$ by $\varphi : T \times \mathcal{X} \to \mathcal{X}$, i.e.,

- T is a monoid for time (associative, neutral), e.g., $\mathbb{R}, \mathbb{Z}, \mathbb{N}, \mathbb{R}_{\geq 0}$
- $\varphi_0(x) = x$ "no time, no evolution"
- $\varphi_s(\varphi_t(x)) = \varphi_{t+s}(x)$ "piecewise evolution is okay"

Dynamical System

"One law to rule them all, and in the darkness bind them"

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Discrete Dynamical System

Definition (Discrete Dynamical System)

Discrete dynamical system $\varphi : T \times \mathcal{X} \to \mathcal{X}$ with $T = \mathbb{Z}$ or $T = \mathbb{N}$. Thus,

$$x_{n+1} = f(x_n)$$

for some generator/transition function $f : \mathcal{X} \to \mathcal{X}$.

$$\varphi_{n+1}(x) := f(\varphi_n(x)) = f^{n+1}(x)$$

Evolution/trace of discrete dynamical system from initial state $x_0 \in \mathcal{X}$

$$x_0 \mapsto f(x_0) \mapsto f^2(x_0) \mapsto f^3(x_0) \mapsto f^4(x_0) \mapsto \ldots$$

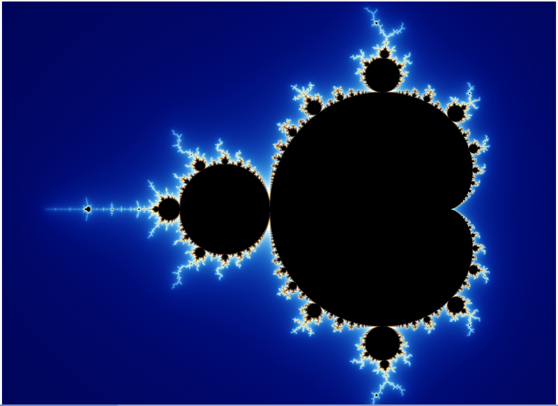
Definition (Difference equation $\rightsquigarrow$ change rate on unit grid)

$$x_{n+1} - x_n = f(x_n) - x_n =: h(x_n) * 1$$

Example (Mandelbrot set)

$x_{n+1} = \varphi_{n+1}(x_n) := x_n^2 + c$ on $\mathcal{X} = \mathbb{C}$ for a given $c \in \mathbb{C}$.

$$M := \mathbb{C} \setminus \{c \in \mathbb{C} \ : \ |\varphi_t(0)| \xrightarrow{t\to\infty} \infty\} \overset{(!)}{=} \{c \in \mathbb{C} \ : \ \forall t \ |\varphi_t(0)| \leq 2\}$$



Finite Automata

Definition (Finite Automaton)

- Σ finite input alphabet
- Q finite set of locations
- $\tau \subseteq Q \times \Sigma \times Q$ transition relation, written $q \xrightarrow{a} q^+$ for $(q, a, q^+) \in \tau$, or
  $\tau : Q \times \Sigma \to Q$ transition function if deterministic

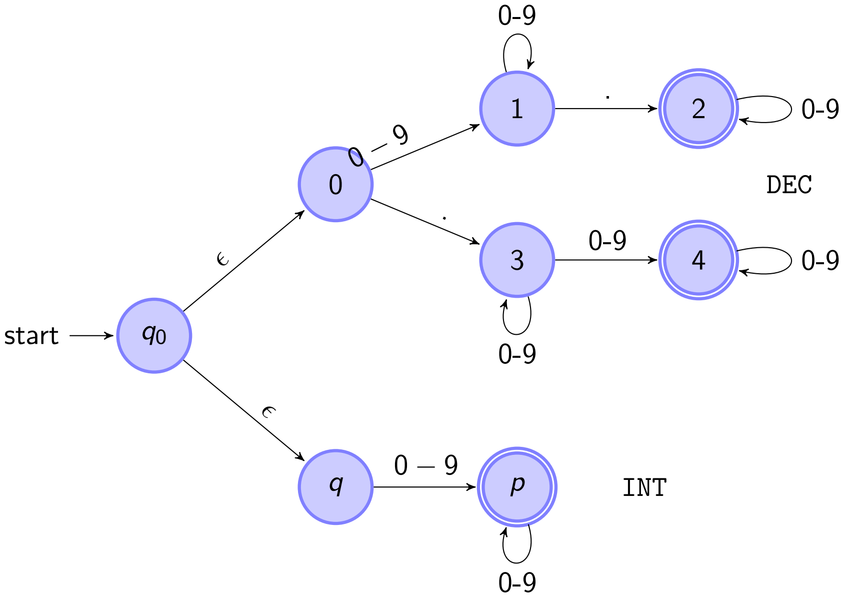
Often initial state $q_0 \in Q$ and accepting states $F \subseteq Q$ are given too.

Definition (Accepted Language)

Accepts input word $w = a_1 a_2 \dots a_k \in \Sigma^*$ iff $\exists n \, \exists q_1, q_2, \dots, q_n \in Q$ with:

$$q_0 \xrightarrow{a_1} q_1 \xrightarrow{a_2} q_2 \xrightarrow{a_3} q_3 \xrightarrow{a_4} \dots \xrightarrow{a_{n-1}} q_{n-1} \xrightarrow{a_n} q_n \in F$$

Finite Automaton Accepting Numbers



Finite Automata vs. Discrete Dynamical Systems

How do they align?

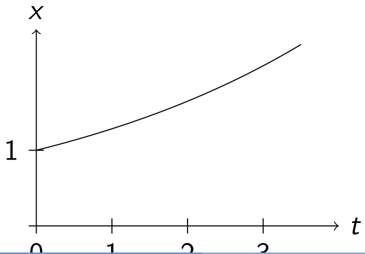
Continuous Dynamical System

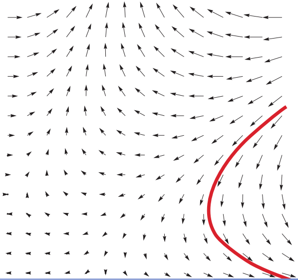
Definition (Continuous Dynamical System)

Continuous dynamical system $\varphi : T \times \mathcal{X} \to \mathcal{X}$ with $T = \mathbb{R}$ or $T = \mathbb{R}_{\geq 0}$ or interval and, e.g., $\mathcal{X} = \mathbb{R}^n$ and φ continuous.
Usually φ defined by a differential equation with (continuous) function $f : \mathcal{X} \to \mathcal{X}$ such that $\varphi.(x_0)$ solves the initial-value problem

$$x' = f(x) \quad x(0) = x_0$$

Evolution/flow of continuous dynamical system from initial state $x_0 \in \mathcal{X}$





Hybrid Dynamical System

System that evolves both discretely and continuously

Definition (Hybrid Dynamical System)

Continuous dynamical system $\varphi : T \times \mathcal{X} \to \mathcal{X}$ with $T =$ $\mathcal{X} =$

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