15-819/18-879 Hybrid Systems Analysis & Theorem Proving

Assignment 1 due by Thu 2/5/2009 hand in WEH 7120/7109

André Platzer

Carnegie Mellon University, Computer Science Department, Pittsburgh, PA aplatzer@cs.cmu.edu

Disclaimer: No solution will be accepted that comes without an **explanation**!

Exercise 1 Hybrid Systems (6p)

1. Give three new examples of hybrid systems (not from class or Bestiarium collection). Explain where their discrete and continuous dynamics and their hybrid interactions come from.
2. Discuss safety-critical properties of these systems and explain to what extent hybridness is crucial in establishing or analyzing these properties faithfully.

Exercise 2 Differential Equations (9p)

1. Find one solution of the following initial value problem (for $y_0, z_0 \in \mathbb{R}$), explain and prove why it is a solution:

$$\left[\begin{array}{cccc} x' = y & y' = z + u & z' = a & u' = b \\ x(0) = 0 & y(0) = y_0 & z(0) = z_0 & u(0) = 0 \end{array}\right] \tag{1}$$

2. How many solutions does equation (1) have?
3. Show that every function f on a domain $D \subseteq \mathbb{R} \times \mathbb{R}^n$ for which $\frac{\mathsf{d}f(x,y)}{\mathsf{d}y}$ exists and is bounded on D is Lipschitz-continuous. What is the corresponding Lipschitz-constant?
4. Show that continuously differentiable functions are locally Lipschitz-continuous.
5. Prove or disprove the following conjecture:
   Let f be a continuously differentiable function defined on $\mathbb{R} \times \mathbb{R}^n$. Then there is a solution of $x' = f(x)$ that is defined on $\mathbb{R}$.

Exercise 3 First-order Logic (7p)

1. Are the following formulas valid/invalid/satisfiable/unsatisfiable?
   a) $(a \leftrightarrow b) \leftrightarrow (a \rightarrow \neg b) \wedge (\neg b \rightarrow a)$
   b) $(c \leftrightarrow d) \rightarrow ((a \leftrightarrow c) \leftrightarrow (a \leftrightarrow d))$
   c) $\forall a \forall b \forall c (r(a,b) \wedge r(b,c) \rightarrow r(a,c)) \wedge \forall a \neg r(a,a) \rightarrow \forall a \forall b (r(a,b) \rightarrow \neg r(b,a))$

2. Show that there is a unique bijection between truth-assignments and valuation functions of propositional logic. You are allowed to use the language of abstract algebra or category theory if you happen to be familiar with that.

Exercise 4 Logic and the Reals (8p)

1. Give a quantifier-free formula in $\mathrm{FOL}_{\mathbb{R}}$ of the same vocabulary that is equivalent to
   a) $\exists x\,(ax^2 + bx + c = 0)$
   b) $\forall x\,(y < x^2) \rightarrow \exists z\,(a = yz^2)$
   c) $\forall x\,(\exists y\,(ax < y^2) \rightarrow bx < z)$
2. Prove that the sets definable using $\wedge, \vee, \neg, >, =, \geq, \leq, <, \neq$ with polynomial terms are exactly the semialgebraic sets.
3. Show that the logic $\mathrm{FOL}_{\mathbb{R}}[+, \cdot, =]$ and $\mathrm{FOL}_{\mathbb{R}}[+, \cdot, =, <]$ are equally expressive, i.e., any formula in one logic can be stated equivalently using a formula in the other.
4. Can you give a quantifier free formula in $\mathrm{FOL}_{\mathbb{R}}[+, \cdot, =]$ that is equivalent to $\exists x\,(ax > b)$? If so, prove equivalence. Otherwise, explain why.

** Exercise 5 Hybrid Systems Modeling (20p)

TCAS (Traffic Collision and Avoidance System) is an onboard unit installed on aircraft. It is responsible for detecting upcoming possible collisions and for giving resolution advisories (RA) to prevent them. Possible RA consist of either climbing or descending actions.

1. Develop a hybrid automaton modeling (simplified) TCAS.
2. Specify desirable properties of the TCAS system in natural language and try to formalize them in logic as good as possible.
3. Analyze the reachable state space of simplified TCAS in PHAVer[1] and try to answer one of your system properties based on that information.
   Hint: At all times are you allowed to simplify TCAS whenever you explain why your simplification is actually necessary and helpful provided that you argue why/under what circumstances your simplifications are adequate.

[1] `http://www-verimag.imag.fr/~frehse/phaver_web/`