15-819/18-879: Hybrid Systems Analysis & Theorem Proving

11: Differential-algebraic Dynamic Logic

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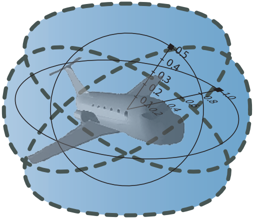
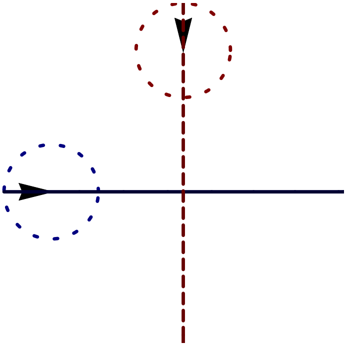
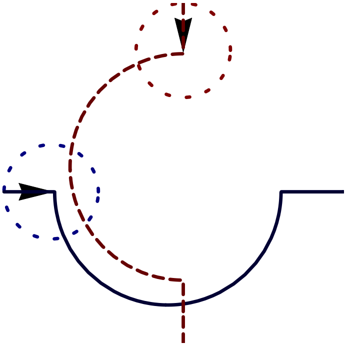


Outline

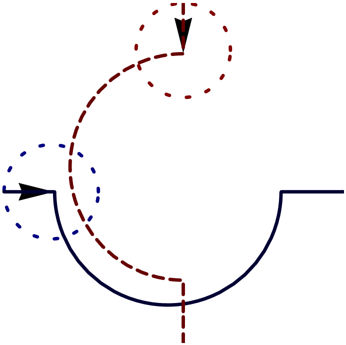
Outline

Hybrid Systems & Air Traffic Control





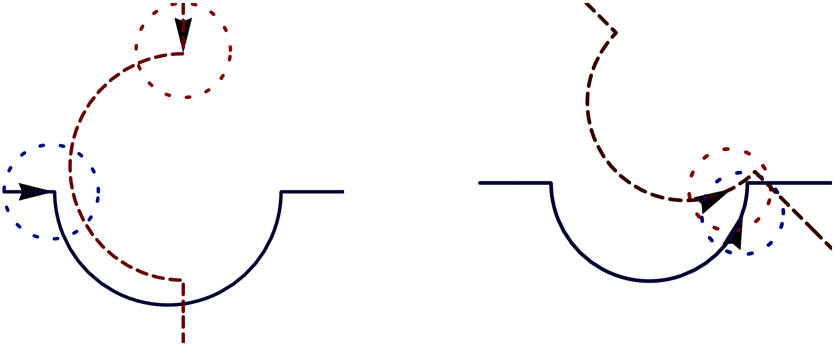
Hybrid Systems & Air Traffic Control



Hybrid Systems

continuous evolution along differential equations + discrete change

Verification of Hybrid Systems & Air Traffic Control



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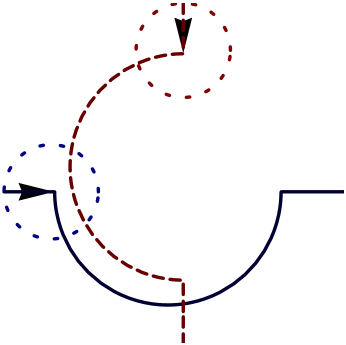

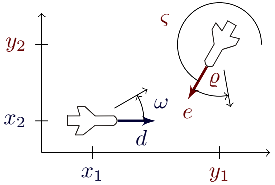

$$\begin{bmatrix} x_1' = -v_1 + v_2 \cos\vartheta + \omega x_2 \\ x_2' = v_2 \sin\vartheta - \omega x_1 \\ \vartheta' = \varpi - \omega \end{bmatrix}$$

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continuous evolution along differential equations + discrete change

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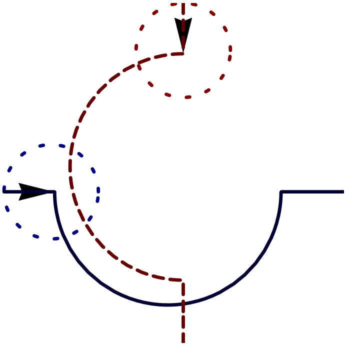
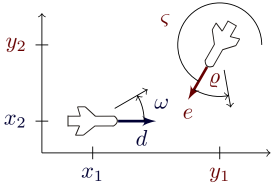
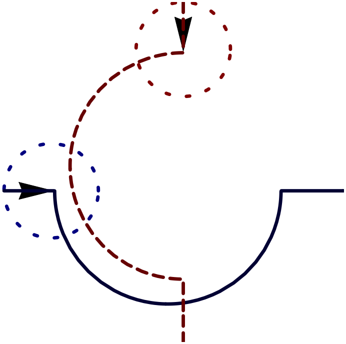


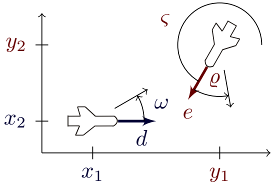


$$\begin{bmatrix} x_1' = -v_1 + v_2 \cos\vartheta + \omega x_2 \\ x_2' = v_2 \sin\vartheta - \omega x_1 \\ \vartheta' = \varpi - \omega \end{bmatrix}$$

Example ("Solving" differential equations)

$$x_1(t) = \frac{1}{\omega\varpi}\big(x_1\omega\varpi\cos t\omega - v_2\omega\cos t\omega\sin\vartheta + v_2\omega\cos t\omega\cos t\varpi\sin\vartheta - v_1\varpi\sin t\omega$$
$$+ x_2\omega\varpi\sin t\omega - v_2\omega\cos\vartheta\cos t\varpi\sin t\omega - v_2\omega\sqrt{1-\sin\vartheta^2}\sin t\omega$$
$$+ v_2\omega\cos\vartheta\cos t\omega\sin t\varpi + v_2\omega\sin\vartheta\sin t\omega\sin t\varpi\big)\ldots$$





$$\begin{bmatrix} x_1' = & -v_1 + v_2 \cos\vartheta + \omega x_2 \\ x_2' = & v_2 \sin\vartheta - \omega x_1 \\ \vartheta' = & \varpi - \omega \end{bmatrix}$$

Example ("Solving" differential equations)

$$\forall t \geq 0 \quad \frac{1}{\omega\varpi}\big(x_1\omega\varpi \cos t\omega - v_2\omega \cos t\omega \sin\vartheta + v_2\omega \cos t\omega \cos t\varpi \sin\vartheta - v_1\varpi \sin t\omega$$
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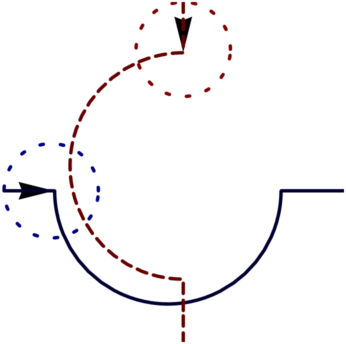
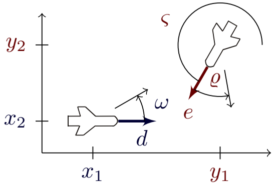
Verification of Hybrid Systems & Air Traffic Control

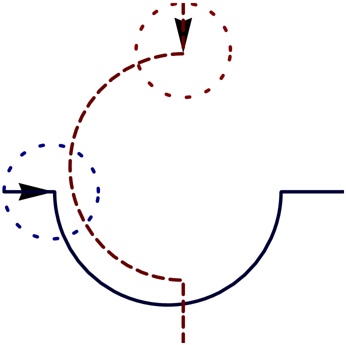


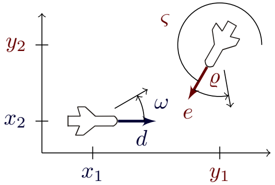


$$\begin{bmatrix} x_1' = -v_1 + v_2 \cos\vartheta + \omega x_2 \\ x_2' = v_2 \sin\vartheta - \omega x_1 \\ \vartheta' = \varpi - \omega \end{bmatrix}$$

Symbolic Verification	Numerical Verification
× constant/nilpotent dynamics	✓ challenging dynamics
× otherwise "no" solutions	× approximation errors
✓ sound	× unsound, . . . see [PC07]

Verification of Hybrid Systems & Air Traffic Control





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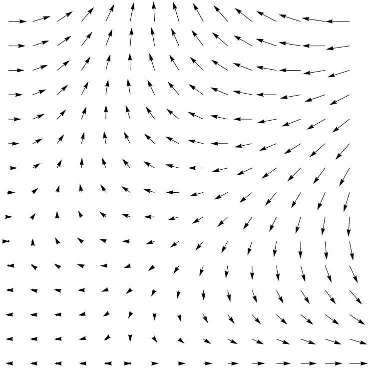
How To Get What We Really Need?

- ✓ challenging dynamics, e.g., curved flight
- ✓ automatic verification
- ✓ sound

Idea: Exploit Vector Field of Differential Equations

"Definition" (Differential Invariant)

"Property that remains true in the direction of the dynamics"



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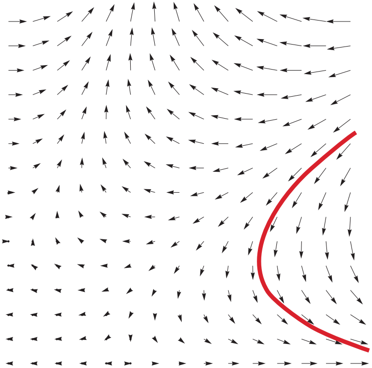
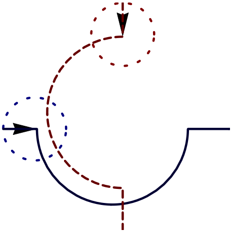
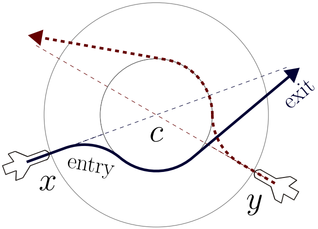
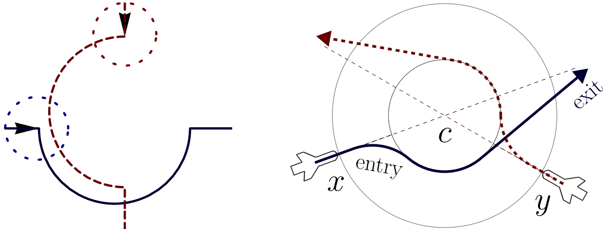






problem	technique	Op	Par	T	CI	Aut
$TRM \models z < m$	TL-MC	✓	×	✓	×	✓
$\models (\mathrm{Ax}(TRM) \to z < m)$	TL-calculus	×	×	✓	..	×
$\models [TRM]\, z < m$	DL-calculus	✓	✓	×	✓	×

Verification Approaches for Hybrid Systems



problem	technique	Op	Par	T	CI	Aut
$TRM \models z < m$	TL-MC	✓	×	✓	×	✓
$\models (\mathrm{Ax}(TRM) \rightarrow z < m)$	TL-calculus	×	×	✓	..	×
$\models [TRM]\, z < m$	DL-calculus	✓	✓	×	✓	?

differential-algebraic dynamic logic

$$\mathrm{DAL} = \mathrm{DL} + \mathrm{DAP}$$

Outline

1. Motivation
2. Differential-algebraic Programs
   - Design Motives
   - Syntax
     - Discrete Jump Constraints
     - Differential-algebraic Constraints
     - Differential-algebraic Programs
   - Semantics
     - Differential-algebraic Constraints
     - Discrete Jump Constraints
     - Differential-algebraic Programs
3. Air Traffic Control
4. Differential-algebraic Dynamic Logic DAL
   - Syntax
   - Semantics

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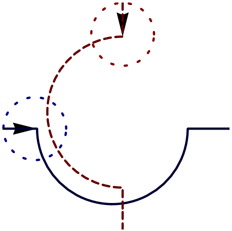
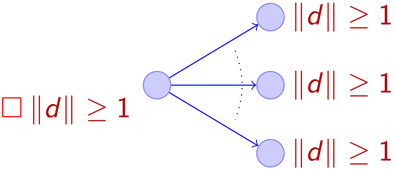
DAL Motives: State Transitions in Dynamic Logic

differential-algebraic dynamic logic

$\mathsf{DAL} = \mathsf{FOL}_{\mathbb{R}} + \mathsf{ML}$

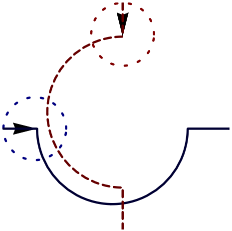


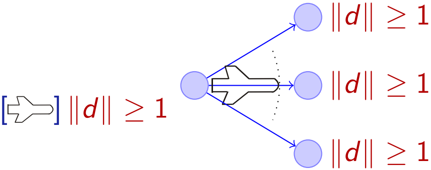


DAL Motives: State Transitions in Dynamic Logic

differential-algebraic dynamic logic

$\mathrm{DAL} = \mathrm{FOL}_{\mathbb{R}} + \mathrm{DL}$

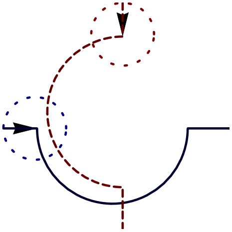




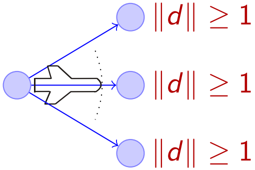
DAL Motives: Differential-algebraic Program Model

differential-algebraic dynamic logic

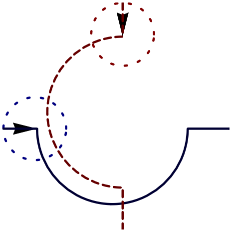
$$\text{DAL} = \text{FOL}_{\mathbb{R}} + \text{DL} + \text{DAP}$$



$[d_1' \leq -\omega d_2 \wedge d_2' \leq \omega d_1 \vee d_1' \leq 4] \, \|d\| \geq 1$



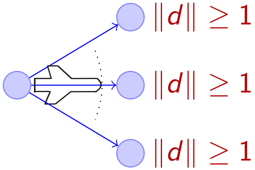
DAL Motives: Differential-algebraic Program Model



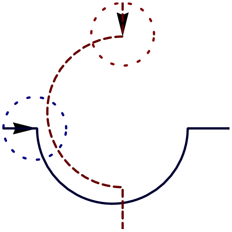
differential-algebraic dynamic logic

$\mathrm{DAL} = \mathrm{FOL}_{\mathbb{R}} + \mathrm{DL} + \mathrm{DAP}$

$[d_1 := -d_2;\ d_1' \leq -\omega d_2 \wedge d_2' \leq \omega d_1 \vee d_1' \leq 4]\,\|d\| \geq 1$

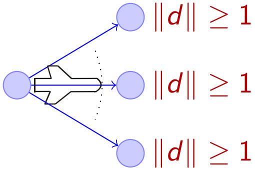


DAL Motives: Differential-algebraic Program Model



differential-algebraic dynamic logic

$\mathrm{DAL} = \mathrm{FOL}_{\mathbb{R}} + \mathrm{DL} + \mathrm{DAP}$



$$[\underbrace{d_1 := -d_2;\ d_1' \leq -\omega d_2 \wedge d_2' \leq \omega d_1 \vee d_1' \leq 4}_{\text{differential-algebraic program}}]\, \|d\| \geq 1$$

differential-algebraic program
= first-order completion of
hybrid programs

General Discrete Jumps/Change

Example (Discrete Jump / DJ constraints)

- $d_1 := -d_2$

General Discrete Jumps/Change

Example (Discrete Jump / DJ constraints)

- $d_1 := -d_2$
- $d_1 := -d_2 \land d_2 := d_1$

General Discrete Jumps/Change

Example (Discrete Jump / DJ constraints)

- $d_1 := -d_2$
- $d_1 := -d_2 \wedge d_2 := d_1$ simultaneous effect, rotate left by $\frac{\pi}{2}$

General Discrete Jumps/Change

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- $d_1 := -d_2 \land d_2 := d_1$ simultaneous effect, rotate left by $\frac{\pi}{2}$
- $d_1 := -d_2 \land d_1 := 0$

General Discrete Jumps/Change

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- $d_1 := -d_2$
- $d_1 := -d_2 \land d_2 := d_1$ simultaneous effect, rotate left by $\frac{\pi}{2}$

× $d_1 := -d_2 \land d_1 := 0$ incompatible jump

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- $d_1 := -d_2 \land d_2 := d_1$ simultaneous effect, rotate left by $\frac{\pi}{2}$
- × $d_1 := -d_2 \land d_1 := 0$ incompatible jump
- $(d_1 > 0 \rightarrow d_1 := -d_2 \land d_2 := d_1) \land (d_1 \leq 0 \rightarrow d_1 := d_2 \land d_2 := -d_1)$

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- $d_1 := -d_2 \wedge d_2 := d_1$ simultaneous effect, rotate left by $\frac{\pi}{2}$
- × $d_1 := -d_2 \wedge d_1 := 0$ incompatible jump
- $(d_1 > 0 \rightarrow d_1 := -d_2 \wedge d_2 := d_1) \wedge (d_1 \leq 0 \rightarrow d_1 := d_2 \wedge d_2 := -d_1)$
- $\neg(d_1 := 5)$

Example (Discrete Jump / DJ constraints)

- $d_1 := -d_2$
- $d_1 := -d_2 \land d_2 := d_1$ simultaneous effect, rotate left by $\frac{\pi}{2}$
- × $d_1 := -d_2 \land d_1 := 0$ incompatible jump
- $(d_1 > 0 \rightarrow d_1 := -d_2 \land d_2 := d_1) \land (d_1 \leq 0 \rightarrow d_1 := d_2 \land d_2 := -d_1)$
- × $\neg(d_1 := 5)$ what is really assigned to d_1 now?

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- × $\neg(d_1 := 5)$ what is really assigned to d_1 now?
- $\exists a\,(\omega := a^2 \land a < 5)$

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- $d_1 := -d_2 \land d_2 := d_1$ simultaneous effect, rotate left by $\frac{\pi}{2}$

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- $(d_1 > 0 \to d_1 := -d_2 \land d_2 := d_1) \land (d_1 \leq 0 \to d_1 := d_2 \land d_2 := -d_1)$

× $\neg(d_1 := 5)$ what is really assigned to d_1 now?

- $\exists a\, (\omega := a^2 \land a < 5)$
- $\forall a\, \omega := a^2$

General Discrete Jumps/Change

Example (Discrete Jump / DJ constraints)

- $d_1 := -d_2$
- $d_1 := -d_2 \wedge d_2 := d_1$ simultaneous effect, rotate left by $\frac{\pi}{2}$

× $d_1 := -d_2 \wedge d_1 := 0$ incompatible jump

- $(d_1 > 0 \rightarrow d_1 := -d_2 \wedge d_2 := d_1) \wedge (d_1 \leq 0 \rightarrow d_1 := d_2 \wedge d_2 := -d_1)$

× $\neg(d_1 := 5)$ what is really assigned to d_1 now?

- $\exists a\, (\omega := a^2 \wedge a < 5)$

× $\forall a\, \omega := a^2$ ω can hardly assume all those values at once

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- × $\forall a\,\omega := a^2$ ω can hardly assume all those values at once
- $\exists a\, a := d_1$

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- × $\forall a\, \omega := a^2$ ω can hardly assume all those values at once
- × $\exists a\, a := d_1$ is just *true* as no visible effects or constraints

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- × $\forall a \, \omega := a^2$ ω can hardly assume all those values at once
- × $\exists a \, a := d_1$ is just *true* as no visible effects or constraints
- $d_1 > 0 \rightarrow \exists a \, (a < 5 \land d_1 := a^2 + 1)$

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- $d_1 := -d_2 \wedge d_2 := d_1$ simultaneous effect, rotate left by $\frac{\pi}{2}$
- × $d_1 := -d_2 \wedge d_1 := 0$ incompatible jump
- $(d_1 > 0 \rightarrow d_1 := -d_2 \wedge d_2 := d_1) \wedge (d_1 \leq 0 \rightarrow d_1 := d_2 \wedge d_2 := -d_1)$
- × $\neg(d_1 := 5)$ what is really assigned to d_1 now?
- $\exists a\,(\omega := a^2 \wedge a < 5)$
- × $\forall a\, \omega := a^2$ ω can hardly assume all those values at once
- × $\exists a\, a := d_1$ is just *true* as no visible effects or constraints
- ? $d_1 > 0 \rightarrow \exists a\,(a < 5 \wedge d_1 := a^2 + 1)$ what happens if $d_1 \leq 0$

General Continuous Change

Example (Differential-algebraic / DA constraints)

- $x_1' = d_1 \land x_2' = d_2$

General Continuous Change

Example (Differential-algebraic / DA constraints)

- $x_1' = d_1 \land x_2' = d_2$
- $x_1' = d_1 \land x_2' = d_2 \land d_1' = -\omega d_2 \land d_2' = \omega d_1$

General Continuous Change

Example (Differential-algebraic / DA constraints)

- $x_1' = d_1 \land x_2' = d_2$
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- $d_1' = -\omega d_2 \land d_2' = \omega d_1 \land d_1 \geq 0$
- $d_1' = -d_2 \land d_1' = 1$

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× $d_1' = -d_2 \land d_1' = 1$ incompatible slope

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- $x_1' = d_1 \land x_2' = d_2 \land d_1' = -\omega d_2 \land d_2' = \omega d_1$
- $d_1' = -\omega d_2 \land d_2' = \omega d_1 \land d_1 \geq 0$

× $d_1' = -d_2 \land d_1' = 1$ incompatible slope

- $(d_1 > 0 \rightarrow d_1' = -d_2 \land d_2' = d_1) \land (d_1 \leq 0 \rightarrow d_1' = d_2 \land d_2' = -d_1)$

General Continuous Change

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- $\neg(d_1' = 5)$

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General Continuous Change

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? $d_1 > 0 \rightarrow x_1' = d_1 \wedge x_2' = d_2 \wedge d_1' = -\omega d_2$ what happens if $d_1 \leq 0$

We only allow change $x := \theta$ or x' in affirmative subformulas:

Definition (Affirmative subformula)

Formula G is *affirmative subformula* of first-order formula F, iff:

1. G is a positive subformula of F, i.e., occurs with an even number of negations, and
2. no variable y that occurs in G is in the scope of a universal quantifier $\forall y$ of a positive subformula of F (or $\exists y$ of negative subformula of F)

Discrete Jump Constraints

Definition (DJ constraint)

- DJ constraint: $\mathrm{FOL}_{\mathbb{R}}$ formula $\mathcal{J}$ over Σ with additional atomic formulas of the form $x := \theta$ where $x \in \Sigma$, $\theta \in \mathrm{Trm}(\Sigma)$.
- The latter are called assignments and are only allowed in affirmative subformulas of DJ-constraints that are not in the scope of a quantifier for x of $\mathcal{J}$.
- DJ-constraint without assignments is called *jump-free*.
- Variable x is (possibly) *changed* in $\mathcal{J}$ iff an assignment of the form $x := \theta$ occurs in $\mathcal{J}$.

Discrete Jump Constraints

$$(x_1 := \theta_1 \wedge .. \wedge x_n := \theta_n \wedge x_1 > 0) \vee (x_1 := \vartheta_1 \wedge .. \wedge x_n := \vartheta_n \wedge x_1 < 0)$$

- if $x_1 > 0$, then simultaneously changes x_i to the respective θ_i, and

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- Overlapping cases as in $(x := x - 1 \land x \geq 0) \lor x := 0$ allow any disjunct to take effect by a nondeterministic choice.
- Quantifiers express unbounded discrete nondeterministic choices.

$$\exists u_1 \exists u_2 \; (e_1 := u_1 \land e_2 := u_2 \land \exists \lambda{>}0 \, \exists \mu{>}0 \, (\lambda d_1 = \mu u_1 \land \lambda d_2 = \mu u_2))$$

Differential-algebraic Constraints

Definition (DA constraints)

- DA-constraint: $\mathrm{FOL}_{\mathbb{R}}$ formula $\mathcal{D}$ over $\Sigma \cup \Sigma'$, in which symbols of Σ' only occur in affirmative subformulas that are not in the scope of a quantifier of $\mathcal{D}$ for that symbol.
- Σ' is the set of all *differential symbols* $x^{(n)}$ with $n \in \mathbb{N}$ for state variables $x \in \Sigma$. Write x' for $x^{(1)}$ ...
- DA-constraint without differential symbols is called *non-differential*.
- Variable x is (possibly) *changed* in $\mathcal{D}$ iff $x^{(n)}$ occurs in $\mathcal{D}$ for an $n \geq 1$.
- $\mathrm{ord}_x \mathcal{D}$ is the highest order $n \in \mathbb{N}$ of a differential symbol $x^{(n)}$ occurring in $\mathcal{D}$, otherwise not defined.

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- Quantifiers express continuous nondeterministic choices:

$$\exists u \, (d_1' = -(\omega + u) d_2 \wedge d_2' = (\omega + u) d_1 \wedge -0.1 \leq u \leq 0.1)$$

expresses that the system follows a continuous evolution in which, at each time, the differential equations are respected for *some* choice of u in $-0.1 \leq u \leq 0.1$, possibly different at each point in time.

Simplify: Fully Homogeneous Constraints

Free nondeterministic change of y is expressible:

- $\exists a\, y := a$, or
- $\exists a\, y' = a$

Convention (Fully Homogeneous)

Expect changes of all changed variables are specified explicitly in all cases of the constraints to improve readability: A DA-constraint or DJ-constraint $\mathcal{C}$ is *homogeneous* iff, in each disjunct of a disjunctive normal form of $\mathcal{C}$, every changed variable of $\mathcal{C}$ is changed exactly once.

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- × $d_1 > 0 \to \exists a\,(a < 5 \land d_1 := a^2 + 1)$ what happens if $d_1 \leq 0$
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Differential-algebraic Programs: Syntax

Definition (Differential-algebraic program α)

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\begin{array}{ll}
\mathcal{D} & \text{(continuous DA-constraint)} \\
\mathcal{J} & \text{(discrete DJ-constraint)} \\
\alpha;\beta & \text{(seq. composition)} \\
\alpha \cup \beta & \text{(nondet. choice)} \\
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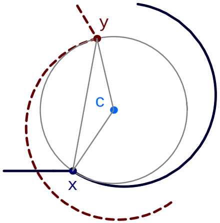
$$\begin{aligned}
TRM &\equiv (ctrl\,;fly)^* \\
ctrl &\equiv (?\|x-y\| \geq 5;\ \exists a\, \omega := a) \\
&\quad \cup\ (?\|x-y\| < 5;\ d := \omega(x-c)^{\perp}) \\
fly &\equiv x' = d' \wedge d_1' = -\omega d_2 \wedge d_1' = \omega d_1 \\
&\quad \wedge \dots
\end{aligned}$$



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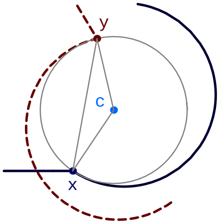
ctrl &\equiv (?\|x - y\| \geq 5;\ \exists a\, \omega := a) \\
&\quad \cup\ (?\|x - y\| < 5;\ d := \omega(x - c)^{\perp}) \\
fly &\equiv x' = d' \wedge d_1' = -\omega d_2 \wedge d_1' = \omega d_1 \\
&\quad \wedge y' = e' \wedge e_1' = -\omega e_2 \wedge e_1' = \omega e_1 \\
&\quad \wedge \ldots
\end{aligned}
$$



Differential-algebraic Programs: Syntax

Definition (Differential-algebraic program α)

$\mathcal{D}$	(continuous DA-constraint)	
$\mathcal{J}$	(discrete DJ-constraint)	
$\alpha; \beta$	(seq. composition)	Kleene algebra
$\alpha \cup \beta$	(nondet. choice)	Kleene algebra
α^*	(nondet. repetition)	Kleene algebra

$$
\begin{aligned}
TRM &\equiv (ctrl\,; fly)^* \\
ctrl &\equiv (?\|x-y\| \geq 5;\ \exists a\, \omega := a) \\
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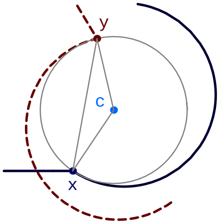

$\mathcal{R}$ Classification & Correspondence

DA-program class	System class
conjunctive DA-constraints	continuous dynamical systems
DA-constraints	switched continuous dynamical systems
no DA-constraints	discrete dynamical systems
no DA-constraints, over $\mathbb{N}$	discrete while programs
general DA-programs	hybrid dynamical systems
	$+$ first-order dynamics

Differential-algebraic Programs: Transition Semantics

Definition (Differential-algebraic programs α: transition semantics)

$$v \xrightarrow{\mathcal{J}} w$$

$v \models \mathcal{J}_{x:=}^{w(x)=}$ $\qquad$ $v = w$ when not changed

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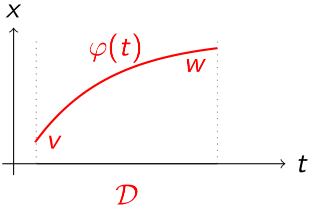

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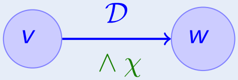

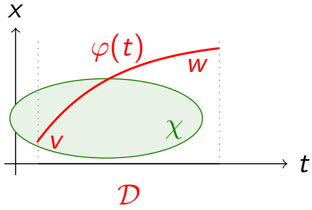

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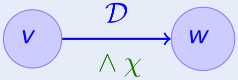

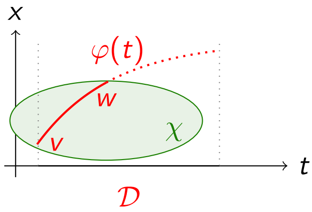

Differential-algebraic Programs: Transition Semantics

Definition (Differential-algebraic programs α: transition semantics)

$$\begin{tikzcd} v \arrow[r, "\alpha"'] \arrow[rr, bend left, "\alpha; \beta"] & s \arrow[r, "\beta"'] & w \end{tikzcd}$$

Differential-algebraic Programs: Transition Semantics

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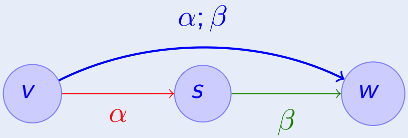

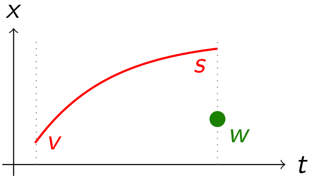

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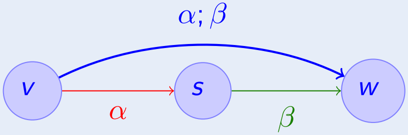

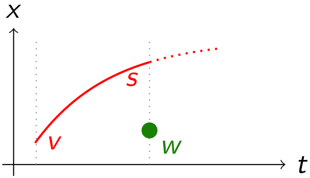

Differential-algebraic Programs: Transition Semantics

Definition (Differential-algebraic programs α: transition semantics)

$$
\begin{tikzcd}
v \arrow[r, "\alpha"'] \arrow[rrrr, bend left, "\alpha^*"] & s_1 \arrow[r, "\alpha"'] & s_2 \arrow[r, dotted, no head] & s_n \arrow[r, "\alpha"'] & w
\end{tikzcd}
$$

Differential-algebraic Programs: Transition Semantics

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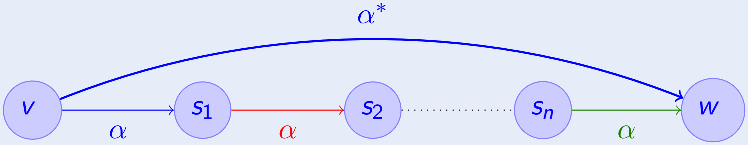

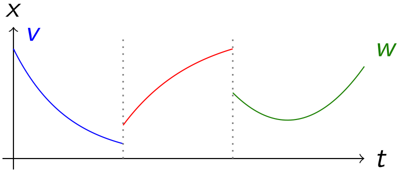

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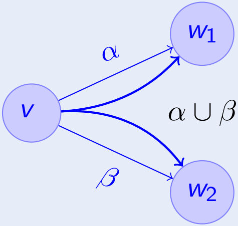

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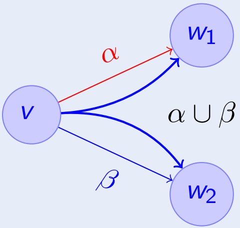

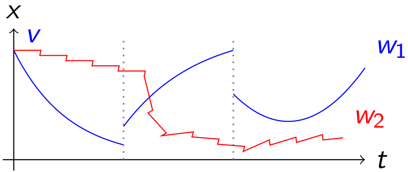

Differential-algebraic Constraints: Semantics

- In which state can we interpret $d_1' = -\omega d_2 \land d_2' = \omega d_1$?

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- Along flow, d_1' makes sense and DA-constraint can be interpreted locally.

Definition (Differential state flow $\varphi : [0, r] \to \text{States}$)

φ componentwise continuous on $[0, r]$: $\varphi(\zeta)(x)$ continuous in ζ for $x \in \Sigma$. *Differentially augmented state* $\bar{\varphi}(\zeta)$ of φ at $\zeta \in [0, r]$ agrees with $\varphi(\zeta)$ except that it assigns values to some $x^{(n)} \in \Sigma'$:

$$\bar{\varphi}(\zeta)\big(x^{(n)}\big) = \begin{cases} \frac{\mathrm{d}^n \varphi(t)(x)}{\mathrm{d}t^n}(\zeta) & \text{if } n\text{-times continuously differentiable in } t \text{ at } \zeta \\ \text{undefined} & \text{otherwise} \end{cases}$$

φ is *state flow of the order of $\mathcal{D}$*, iff value of each differential symbol occurring in DA-constraint $\mathcal{D}$ is defined on $[0, r]$

$\mathcal{R}$ Differential-algebraic Constraints: Semantics

Definition (Interpretation of differential-algebraic constraints)

Let φ state flow of the order of DA-constraint $\mathcal{D}$ and duration $r \geq 0$.
$\varphi \models \mathcal{D}$ iff, for all $\zeta \in [0, r]$,

1. $\bar{\varphi}(\zeta) \models_{\mathbb{R}} \mathcal{D}$ using standard semantics $\models_{\mathbb{R}}$ of first-order real arithmetic, and
2. $[\![z]\!]_{\bar{\varphi}(\zeta)} = [\![z]\!]_{\bar{\varphi}(0)}$ for all variables z that are not changed by $\mathcal{D}$.

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- Only variables whose differential symbols occur in $\mathcal{D}$ need continuously differentiable values.
- Quantified variables can change arbitrarily, even discontinuously
- In $\exists u\, x' = u^2$, the value of u^2 (not u) varies continuously, because x' does.
- For $r = 0$, atomic formulas with differential symbols take no effect (*true* as positive).

Discrete Jump Constraints: Semantics

Definition (Interpretation of discrete jump constraints)

$(v, w) \models \mathcal{J}$ defined as follows:

$$
\begin{aligned}
(v, w) \models x := \theta \quad &:\iff \quad [\![x]\!]_w = [\![\theta]\!]_v \\
(v, w) \models \theta_1 \geq \theta_2 \quad &:\iff \quad [\![\theta_1]\!]_v \geq [\![\theta_2]\!]_v \\
(v, w) \models \phi \land \psi \quad &:\iff \quad (v, w) \models \phi \text{ and } (v, w) \models \psi \\
(v, w) \models \neg\phi \quad &:\iff \quad (v, w) \models \phi \text{ does not hold} \\
(v, w) \models \forall x\, \phi \quad &:\iff \quad (v_x, w) \models \phi \text{ for all } v_x \text{ that agree with } v \\
&\qquad\qquad \text{except for the value of } x \\
(v, w) \models \exists x\, \phi \quad &:\iff \quad (v_x, w) \models \phi \text{ for some } v_x \text{ that agrees with } v \\
&\qquad\qquad \text{except for the value of } x
\end{aligned}
$$

where $[\![z]\!]_w = [\![z]\!]_v$ for all variables z that are not changed in $\mathcal{J}$.

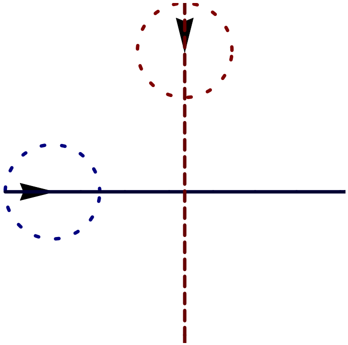
Differential-algebraic Programs: Formal Semantics

Definition (Differential-algebraic programs α)

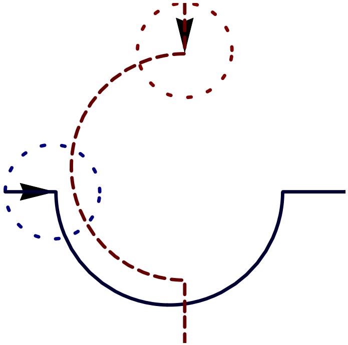
$$
\begin{aligned}
\rho(\mathcal{D}) &= \{(\varphi(0), \varphi(r)) \ : \ \varphi \models \mathcal{D} \text{ of order of } \mathcal{D}\text{, duration } r \geq 0\} \\
\rho(\mathcal{J}) &= \{(v, w) \ : \ (v, w) \models \mathcal{J}\} \\
\rho(\alpha \cup \beta) &= \rho(\alpha) \cup \rho(\beta) \\
\rho(\alpha; \beta) &= \rho(\alpha) \circ \rho(\beta) \\
\rho(\alpha^*) &= \left\{ (v, w) \ : \ \text{there is } \ v \xrightarrow{\rho(\alpha)} v_1 \xrightarrow{\rho(\alpha)} v_2 \cdots\cdots \xrightarrow{\rho(\alpha)} w \right\}
\end{aligned}
$$

Outline

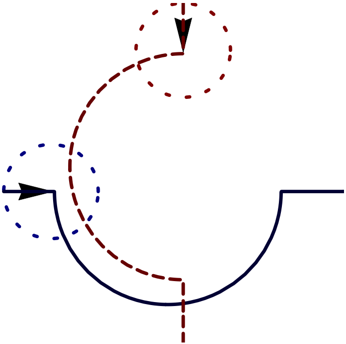
Air Traffic Control

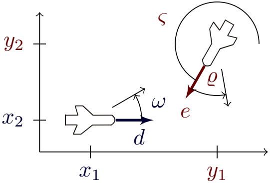


Air Traffic Control

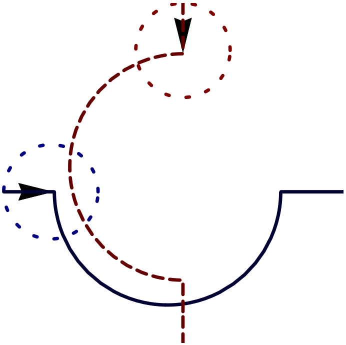


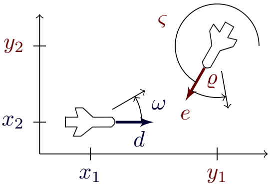
Air Traffic Control





$$\begin{bmatrix} x_1' = v\cos\vartheta & & y_1' = u\cos\varsigma \\ x_2' = v\sin\vartheta & & y_2' = u\sin\varsigma \end{bmatrix}$$





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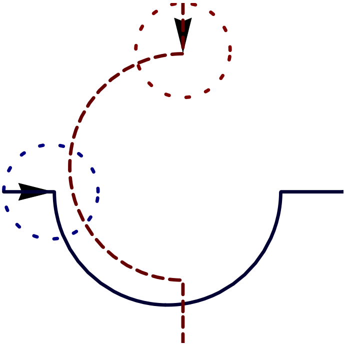

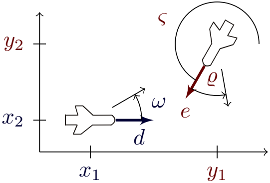

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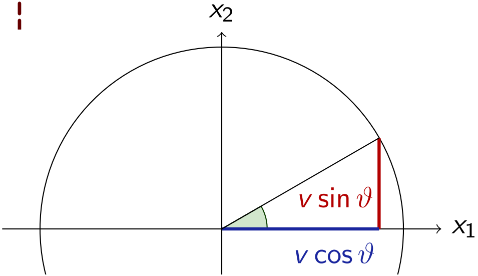
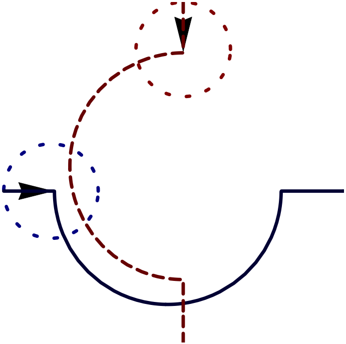
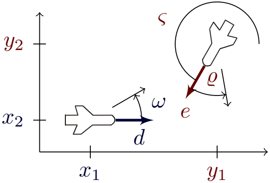
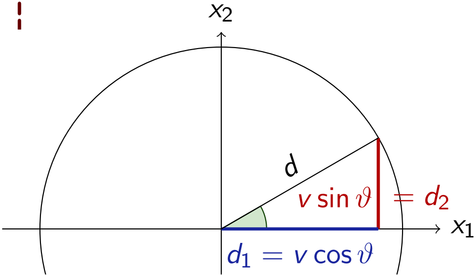


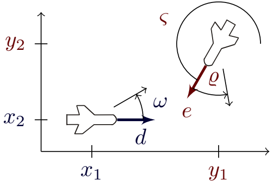
Air Traffic Control





$$\begin{bmatrix} x_1' = v\cos\vartheta = d_1 & y_1' = u\cos\varsigma \\ x_2' = v\sin\vartheta = d_2 & y_2' = u\sin\varsigma \end{bmatrix}$$

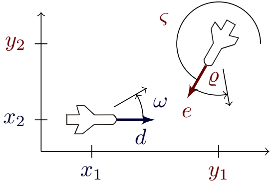




$$\begin{bmatrix} x_1' = v\cos\vartheta = d_1 & y_1' = u\cos\varsigma = e_1 \\ x_2' = v\sin\vartheta = d_2 & y_2' = u\sin\varsigma = e_2 \\ d_1' = & e_1' = \\ d_2' = & e_2' = \end{bmatrix}$$

$$d_1' =$$
$$d_2' =$$

Differential Axiomatization of Flight Dynamics



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$$d_1' = (v\cos\vartheta)'$$
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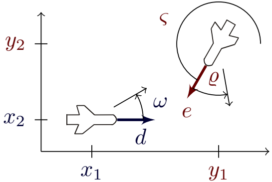

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$$d_1' = (v\cos\vartheta)' = v'\cos\vartheta + v(-\sin\vartheta)\vartheta'$$
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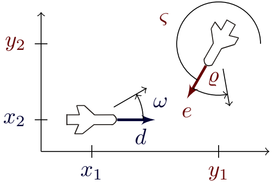

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$$d_1' = (v\cos\vartheta)' = v'\cos\vartheta + v(-\sin\vartheta)\vartheta' = -(v\sin\vartheta)\omega$$
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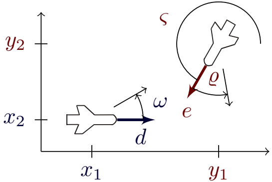
Differential Axiomatization of Flight Dynamics



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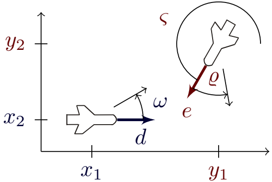
Differential Axiomatization of Flight Dynamics



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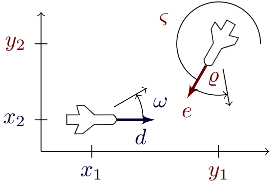
Differential Axiomatization of Flight Dynamics



$$\begin{bmatrix} x_1' = v\cos\vartheta = d_1 & y_1' = u\cos\varsigma = e_1 \\ x_2' = v\sin\vartheta = d_2 & y_2' = u\sin\varsigma = e_2 \\ d_1' = -\omega d_2 & e_1' = -\varrho e_2 \\ d_2' = \omega d_1 & e_2' = \varrho e_1 \end{bmatrix}$$

$$d_1' = (v\cos\vartheta)' = v'\cos\vartheta + v(-\sin\vartheta)\vartheta' = -(v\sin\vartheta)\omega = -\omega d_2$$

$$d_2' = (v\sin\vartheta)' = v'\sin\vartheta + v(\cos\vartheta)\vartheta' = (v\cos\vartheta)\omega = \omega d_1$$

$$v = \|d\| = \sqrt{d_1^2 + d_2^2}$$

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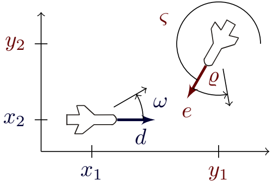

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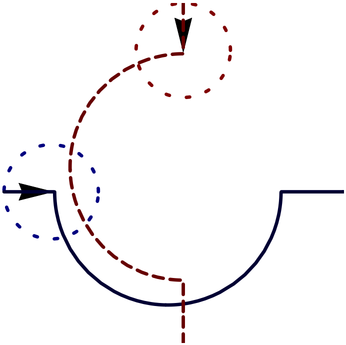
Air Traffic Control Roundabout Maneuvers



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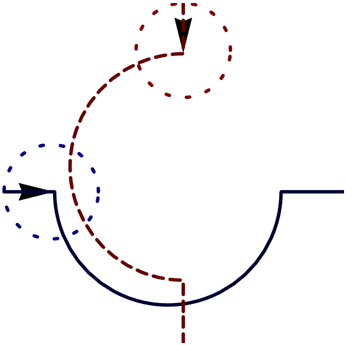

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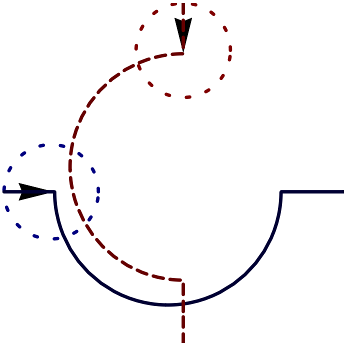

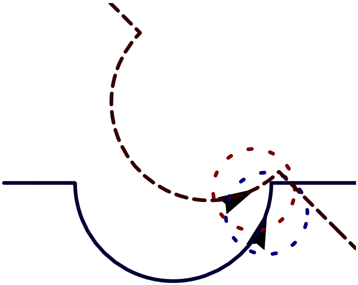

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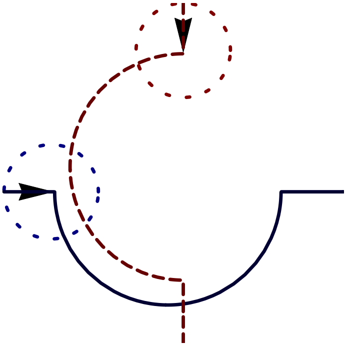
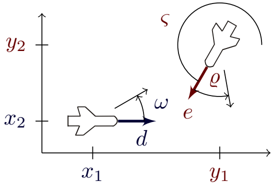
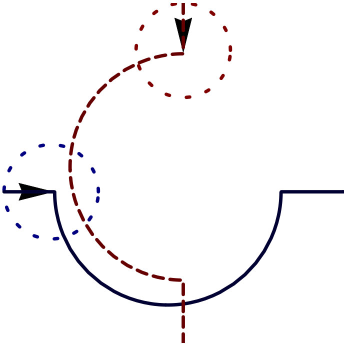
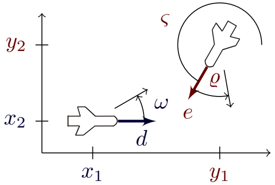




$$\begin{bmatrix} x_1' = -v_1 + v_2 \cos\vartheta + \omega x_2 \\ x_2' = v_2 \sin\vartheta - \omega x_1 \\ \vartheta' = \varpi - \omega \end{bmatrix}$$

Air Traffic Control Roundabout Maneuvers





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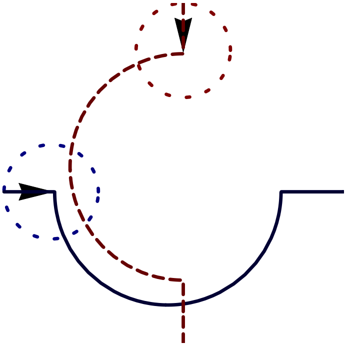
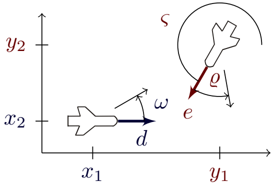
Falsification versus Verification

no more counterexamples but how to verify?

Air Traffic Control Roundabout Maneuvers





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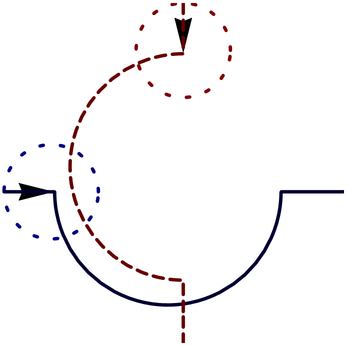
Example ("Solving" differential equations)

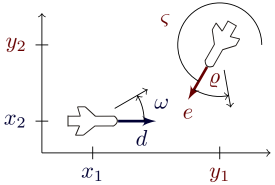
$$x_1(t) = \frac{1}{\omega\varpi}\big(x_1\omega\varpi \cos t\omega - v_2\omega \cos t\omega \sin\vartheta + v_2\omega \cos t\omega \cos t\varpi \sin\vartheta - v_1\varpi \sin t\omega$$

$$+ x_2\omega\varpi \sin t\omega - v_2\omega \cos\vartheta \cos t\varpi \sin t\omega - v_2\omega\sqrt{1-\sin\vartheta^2}\sin t\omega$$

$$+ v_2\omega \cos\vartheta \cos t\omega \sin t\varpi + v_2\omega \sin\vartheta \sin t\omega \sin t\varpi\big)\ \ldots$$

Air Traffic Control Roundabout Maneuvers



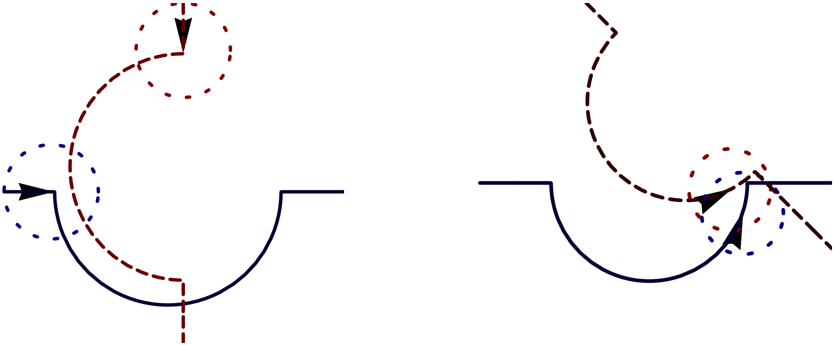


$$\begin{bmatrix} x_1' = -v_1 + v_2 \cos\vartheta + \omega x_2 \\ x_2' = v_2 \sin\vartheta - \omega x_1 \\ \vartheta' = \varpi - \omega \end{bmatrix}$$

Example ("Solving" differential equations)

$$\forall t \geq 0 \quad \frac{1}{\omega\varpi}\big(x_1\omega\varpi\cos t\omega - v_2\omega\cos t\omega\sin\vartheta + v_2\omega\cos t\omega\cos t\varpi\sin\vartheta - v_1\varpi\sin t\omega$$
$$+ x_2\omega\varpi\sin t\omega - v_2\omega\cos\vartheta\cos t\varpi\sin t\omega - v_2\omega\sqrt{1-\sin\vartheta^2}\sin t\omega$$
$$+ v_2\omega\cos\vartheta\cos t\omega\sin t\varpi + v_2\omega\sin\vartheta\sin t\omega\sin t\varpi\big)\ldots$$

Air Traffic Control Roundabout Maneuvers



$$\begin{aligned}
\phi &\equiv \|x-y\|^2 \geq p^2 \equiv (x_1-y_1)^2+(x_2-y_2)^2 \geq p^2\\
\mathit{trm} &\equiv \mathit{free};\quad \mathit{entry};\quad \mathcal{F}(\omega)\wedge\mathcal{G}(\omega)\\
\mathit{free} &\equiv \exists\omega\,\mathcal{F}(\omega)\wedge\exists\varpi\,\mathcal{G}(\varpi)\wedge\phi\\
\mathit{entry} &\equiv \text{will be derived later}
\end{aligned}$$

Hybrid Automata Embedding Theorem

Proposition (Hybrid automata embedding)

There is an effective mapping ι such that the following diagram commutes:

$$
\begin{tikzcd}
HA \arrow[r, hook, "\iota"] \arrow[d, "\overset{*}{\curvearrowright}"'] \arrow[dr, phantom, "\circlearrowright"] & \mathrm{DAP}(\Sigma) \arrow[d] \\
Q^2 \arrow[r, bend left] & \mathrm{States}^2 \arrow[l, bend left]
\end{tikzcd}
$$

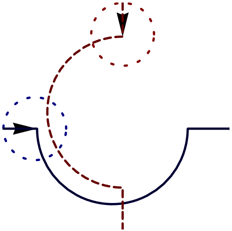
Outline

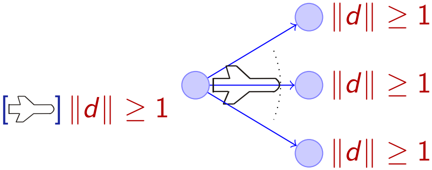
1. Motivation
2. Differential-algebraic Programs
   - Design Motives
   - Syntax
     - Discrete Jump Constraints
     - Differential-algebraic Constraints
     - Differential-algebraic Programs
   - Semantics
     - Differential-algebraic Constraints
     - Discrete Jump Constraints
     - Differential-algebraic Programs
3. Air Traffic Control
4. Differential-algebraic Dynamic Logic DAL
   - Syntax
   - Semantics

DAL Motives: State Transitions in Dynamic Logic

differential-algebraic dynamic logic

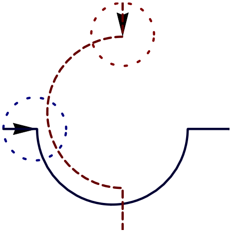
$$\mathrm{DAL} = \mathrm{FOL}_{\mathbb{R}} + \mathrm{DL}$$



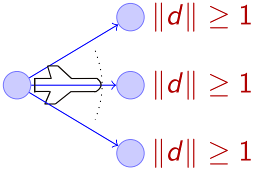


differential-algebraic dynamic logic

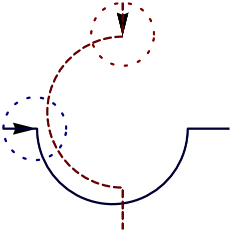
$$\mathrm{DAL} = \mathrm{FOL}_{\mathbb{R}} + \mathrm{DL} + \mathrm{DAP}$$



$[d_1' \leq -\omega d_2 \wedge d_2' \leq \omega d_1 \vee d_1' \leq 4]\, \|d\| \geq 1$



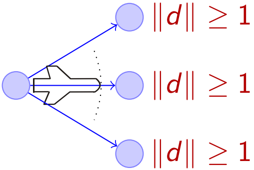
DAL Motives: Differential-algebraic Program Model



differential-algebraic dynamic logic

$$\mathrm{DAL} = \mathrm{FOL}_{\mathbb{R}} + \mathrm{DL} + \mathrm{DAP}$$

$$[d_1 := -d_2;\ d_1' \leq -\omega d_2 \land d_2' \leq \omega d_1 \lor d_1' \leq 4]\ \|d\| \geq 1$$



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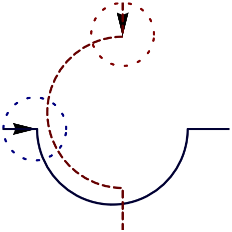
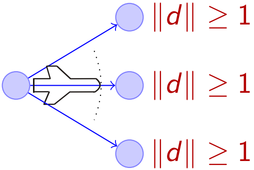


differential-algebraic dynamic logic

$\mathrm{DAL} = \mathrm{FOL}_{\mathbb{R}} + \mathrm{DL} + \mathrm{DAP}$

$$[\underbrace{d_1 := -d_2;\ d_1' \leq -\omega d_2 \wedge d_2' \leq \omega d_1 \vee d_1' \leq 4}]\, \|d\| \geq 1$$



differential-algebraic program

= first-order completion of

hybrid programs

Definition (DAL Signature Σ)

Countable set of predicate or function symbols along with natural numbers as arities containing $0, 1, +, \cdot, /, =, \leq, >, \geq, <$ for reals

Differential-algebraic Dynamic Logic DAL: Syntax

Definition (DAL Signature Σ)

Countable set of predicate or function symbols along with natural numbers as arities containing $0, 1, +, \cdot, /, =, \leq, >, \geq, <$ for reals

Definition (DAL Term t)

$t ::=$

x	for variable $x \in V$
$f(t_1, \ldots, t_n)$	for function $f/n \in \Sigma$ of arity $n \geq 0$

Differential-algebraic Dynamic Logic DAL: Syntax

Definition (DAL Signature Σ)

Countable set of predicate or function symbols along with natural numbers as arities containing $0, 1, +, \cdot, /, =, \leq, >, \geq, <$ for reals

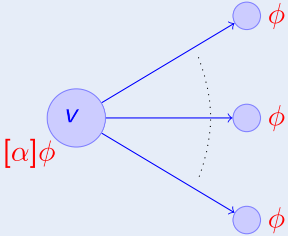
Definition (DAL Formula ϕ, ψ)

$\phi ::=$

$[\alpha]\phi$	"all α reachables"
$\langle\alpha\rangle\phi$	"some α reachable"
$p(t_1, \ldots, t_n)$	for predicate $p/n \in \Sigma$ of arity $n \geq 0$
$\neg\phi$	"not"
$(\phi \land \psi)$	"and"
$(\phi \lor \psi)$	"or"
$(\phi \rightarrow \psi)$	"implies"

Differential-algebraic Dynamic Logic DAL: Semantics

Definition (Formulas ϕ)



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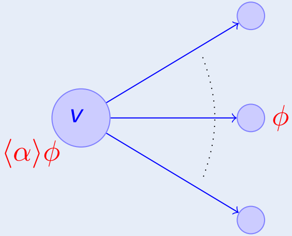
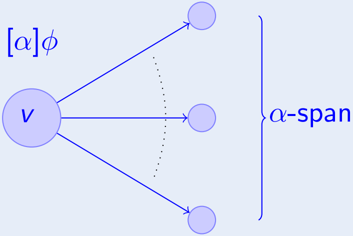


$\mathcal{A}$ Differential-algebraic Dynamic Logic DAL: Semantics

Definition (Formulas ϕ)



Differential-algebraic Dynamic Logic DAL: Semantics

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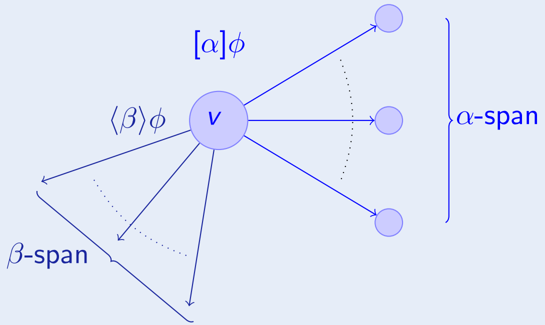
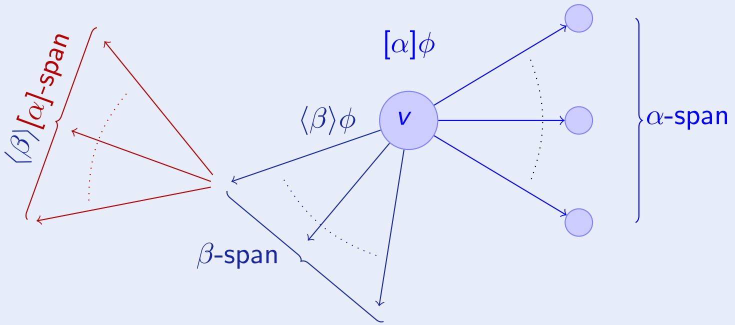


Differential-algebraic Dynamic Logic DAL: Semantics

Definition (Formulas ϕ)



$\mathcal{A}$ Differential-algebraic Dynamic Logic DAL: Semantics

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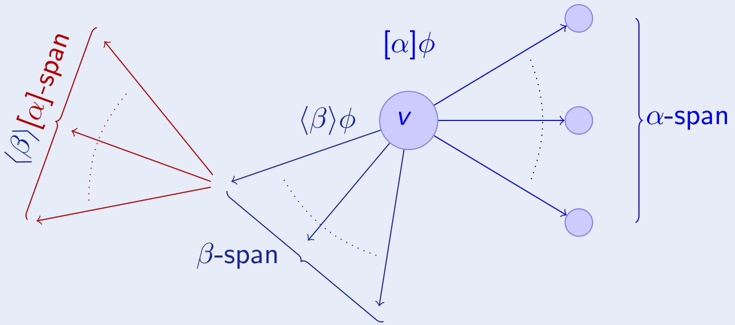

compositional semantics!

Differential-algebraic Dynamic Logic DAL: Semantics

Definition (Formulas ϕ)

$$
\begin{aligned}
v &\models \theta_1 \geq \theta_2 &&:\Longleftrightarrow && [\![\theta_1]\!]_v \geq [\![\theta_2]\!]_v \\
v &\models \phi \land \psi &&:\Longleftrightarrow && v \models \phi \text{ and } v \models \psi \\
v &\models \neg\phi &&:\Longleftrightarrow && v \models \phi \text{ does not hold} \\
v &\models [\alpha]\phi &&:\Longleftrightarrow && w \models \phi \quad \text{for all } w \text{ with } (v, w) \in \rho(\alpha) \\
v &\models \langle\alpha\rangle\phi &&:\Longleftrightarrow && w \models \phi \quad \text{for some } w \text{ with } (v, w) \in \rho(\alpha)
\end{aligned}
$$

Well-definedness and Divisions

Assumption (Well-definedness)

We assume all divisions p/q in any formula are constraint ϕ are taken to mean $\phi \land q \neq 0$.

Time Anomalies and Zeno

Example (Zeno)

- $(a' = -1 \land d \leq a;\ d := d/2)^*$

Time Anomalies and Zeno

Example (Zeno)

- $(a' = -1 \wedge d \leq a;\ d := d/2)^*$
- $(x \geq 0 \rightarrow x'' = -1) \wedge (x < 0 \rightarrow x'' = 1) \wedge y' = 1$

Time Anomalies and Zeno

Example (Zeno)

- $(a' = -1 \land d \leq a;\ d := d/2)^*$
- $(x \geq 0 \rightarrow x'' = -1) \land (x < 0 \rightarrow x'' = 1) \land y' = 1$

Assumption (Non-Zeno)

State flow φ for DA-constraint $\mathcal{D}$ is *non-Zeno*, if there only is a finite number of points in time where some variable needs to obey another differential constraint of $\mathcal{D}$ than before the respective point in time: Let $\mathcal{D}_1 \lor \cdots \lor \mathcal{D}_n$ be a disjunctive normal form of $\mathcal{D}$, then flow $\varphi : [0, r] \rightarrow \text{States}$ is non-Zeno iff there are an $m \in \mathbb{N}$ and $0 = \zeta_0 < \zeta_1 < \cdots < \zeta_m = r$ and indices $i_1, \ldots, i_m \in \{1, \ldots, n\}$ such that φ respects $\mathcal{D}_{i_k}$ on the interval $[\zeta_{k-1}, \zeta_k]$, i.e., $\varphi|_{[\zeta_{k-1}, \zeta_k]} \models \mathcal{D}_{i_k}$ for all $k \in \{1, \ldots, m\}$.

A. Platzer.
Differential-algebraic dynamic logic for differential-algebraic programs.
J. Log. Comput., 2008.
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