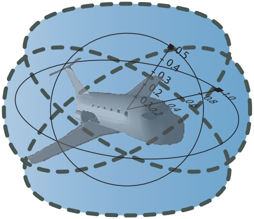
15-819/18-879: Hybrid Systems Analysis & Theorem Proving

09: Train Control Verification

André Platzer

aplatzer@cs.cmu.edu
Carnegie Mellon University, Pittsburgh, PA



Outline

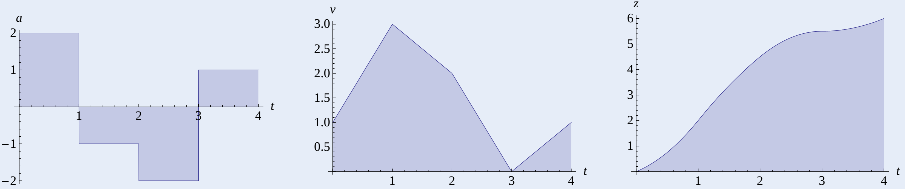
Outline

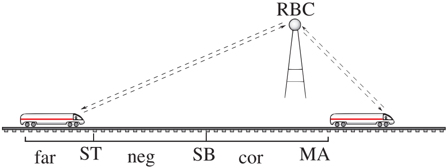
ETCS Control Verification

Problem

Hybrid System

- Continuous evolutions (differential equations)
- Discrete jumps (control decisions)

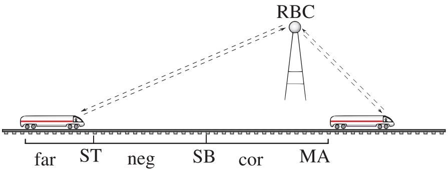




ETCS objectives:

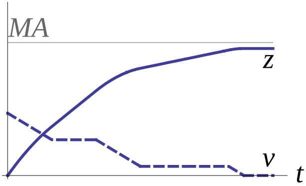
1. Collision free
2. Maximise throughput & velocity (300 km/h)
3. $2.1 * 10^6$ passengers/day

Verifying Parametric Hybrid Systems

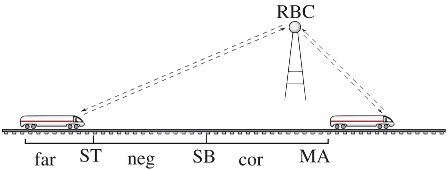


Parametric Hybrid Systems

continuous evolution along differential equations + discrete change



Verifying Parametric Hybrid Systems



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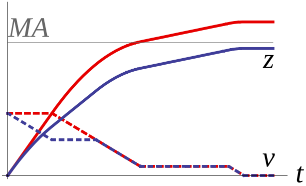

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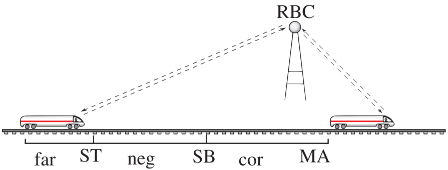

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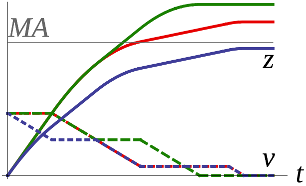

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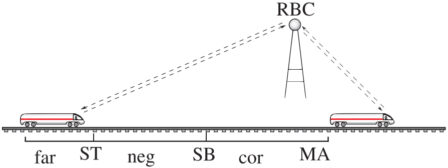
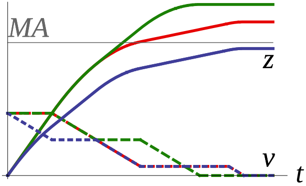


Parametric Hybrid Systems

continuous evolution along differential equations + discrete change

- Parameters have nonlinear influence
- Handle *SB* as free symbolic parameter?
- Challenge: verification (falsifying is "easy")
- Which constraints for *SB*?

$\forall \mathbf{m}\, \exists SB$ "train always safe"



$$
\begin{aligned}
\text{system} &\equiv (\mathit{cor}\,;\mathit{drive})^* \\
\mathit{cor} &\equiv (?\mathbf{m} - z \le SB; a := -b) \cup (?\mathbf{m} - z \ge SB; a := A) \\
\mathit{drive} &\equiv \tau := 0; (z' = v, v' = a, \tau' = 1 \land v \ge 0 \land \tau \le \varepsilon)
\end{aligned}
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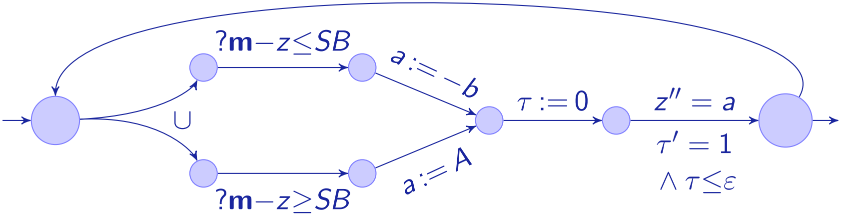
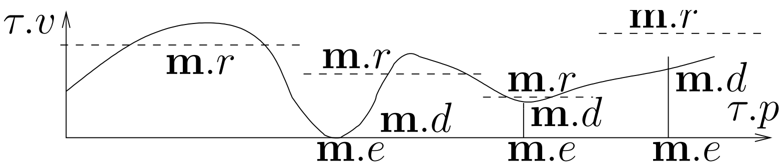


Outline

2D Movement Authorities

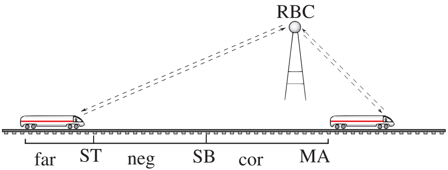


- Vectorial MA $\mathbf{m} = (d, e, r)$:
- Beyond point $\mathbf{m}.e$ train not faster than $\mathbf{m}.d$.
- Train should try not to keep *recommended speed* $\mathbf{m}.r$

Separation Principle

Lemma (Principle of separation by movement authorities)

If each train stays within its MA and, at any time, MAs issued by the RBC form a disjoint partitioning of the track, then trains can never collide.



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- Consider any point in time ζ.

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- This contradicts the assumption of disjoint MA.

Model/State Variables

Train τ:

- $\tau.v$ Position
- $\tau.v$ Speed
- $\tau.a$ Acceleration
- (t model time)

RBC + MA:

- $\mathbf{m}.e$ End of Authority
- $\mathbf{m}.d$ Speed limit
- $\mathbf{m}.r$ Recommended speed
- $rbc.message$ Channel

Parameters:

- SB Start Braking
- ST Start Talking
- b Braking power/deceleration
- A Maximum acceleration
- ε Maximum cycle time
- Δ Maximum expected communication delay

Parametric Skeleton of ETCS Cooperation Protocol

$$
\begin{array}{ll}
ETCS_{skel} & : (train \cup rbc)^* \\
train & : spd;\ atp;\ drive \\
spd & : \quad (?\tau.v \le \mathbf{m}.r;\ \tau.a := *;\ ? -b \le \tau.a \le A) \\
 & \quad \cup (?\tau.v \ge \mathbf{m}.r;\ \tau.a := *;\ ? -b \le \tau.a \le 0) \\
atp & : \mathtt{if}(\mathbf{m}.e - \tau.p \le SB \lor rbc.message = emergency)\, \tau.a := -b \\
drive & : t := 0;\ (\tau.p' = \tau.v, \tau.v' = \tau.a, t' = 1 \land \tau.v \ge 0 \land t \le \varepsilon) \\
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$$[ETCS_{skel}](\tau.p \ge \mathbf{m}.e \to \tau.v \le \mathbf{m}.d)$$

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\end{array}
$$

Verify safety?

$$[ETCS_{skel}](\tau.p \geq \mathbf{m}.e \rightarrow \tau.v \leq \mathbf{m}.d)$$

Lots of counterexamples!

1. *Controllability discovery*: Start with uncontrolled system dynamics. Apply structural d$\mathcal{L}$ decomposition until FOL-formula is obtained revealing controllable state region, which specifies for which parameter combinations the system dynamics can be controlled safely by any control law.

Iterative Control Refinement Process

1. *Controllability discovery*: Start with uncontrolled system dynamics. Apply structural d$\mathcal{L}$ decomposition until FOL-formula is obtained revealing controllable state region, which specifies for which parameter combinations the system dynamics can be controlled safely by any control law.
2. *Control refinement*: Successively add partial control laws to the system while leaving its decision parameters (like SB or $\mathbf{m}$) free. Apply d$\mathcal{L}$ decomposition to discover parametric constraints which maintain controllability under these control laws.

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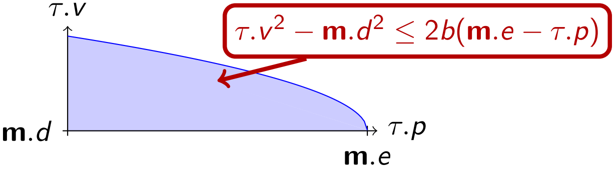
3. *Safety convergence*: Repeat step 2 until resulting system proven safe.
4. *Liveness check*: Prove that discovered parametric constraints do not over-constrain system inconsistently by showing that it remains live.

Outline

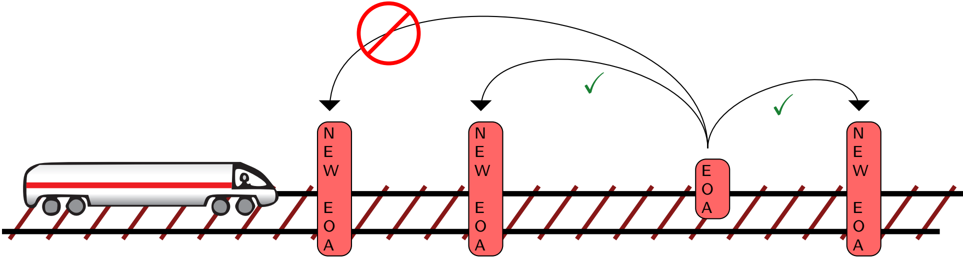
ETCS Controllability



Proposition (Controllability)

$$[\tau.p' = \tau.v, \tau.v' = -b \wedge \tau.v \geq 0](\tau.p \geq \mathbf{m}.e \rightarrow \tau.v \leq \mathbf{m}.d)$$

$$\equiv \mathcal{C} \;\equiv\; \tau.v^2 = \mathbf{m}.d^2 \leq 2b(\mathbf{m}.e - \tau.p)$$

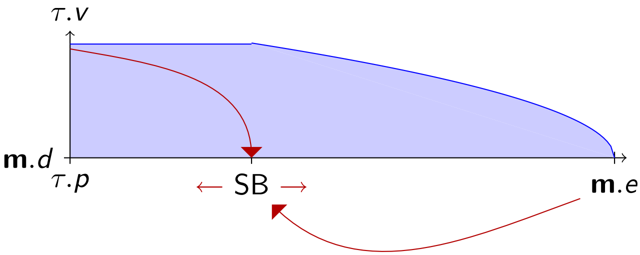


Proposition (RBC Controllability)

$$\mathbf{m}.d \geq 0 \land b > 0 \rightarrow [\mathbf{m}_0 := \mathbf{m};\ rbc] \Big($$

$$\mathcal{M} \equiv \mathbf{m}_0.d^2 - \mathbf{m}.d^2 \leq 2b(\mathbf{m}.e - \mathbf{m}_0.e) \land \mathbf{m}_0.d \geq 0 \land \mathbf{m}.d \geq 0 \leftrightarrow$$

$$\forall \tau \left((\langle \mathbf{m} := \mathbf{m}_0 \rangle \mathcal{C}) \rightarrow \mathcal{C} \right) \Big)$$



Proposition (Reactivity)

$$\Big(\forall \mathbf{m}.e\, \forall \tau.p\, \big(\mathbf{m}.e - \tau.p \geq SB \land \mathcal{C} \rightarrow [\tau.a := A;\ drive]\, \mathcal{C}\big)\Big)$$

$$\equiv SB \geq \frac{\tau.v^2 - \mathbf{m}.d^2}{2b} + \left(\frac{A}{b} + 1\right)\left(\frac{A}{2}\varepsilon^2 + \varepsilon\, \tau.v\right)$$

Refined ETCS Control

$$
\begin{array}{lcl}
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 & & \cup(?\tau.v \ge \mathbf{m}.r;\ \tau.a := *;\ ?0 > \tau.a \ge -b) \\
atp & : & SB := \frac{\tau.v^2 - \mathbf{m}.d^2}{2b} + \left(\frac{A}{b}+1\right)\left(\frac{A}{2}\varepsilon^2 + \varepsilon\ \tau.v\right); \\
 & : & \mathtt{if}(\mathbf{m}.e - \tau.p \le SB \lor rbc.message = emergency)\, \tau.a := -b \\
drive & : & t := 0;\ (\tau.p' = \tau.v, \tau.v' = \tau.a, t' = 1 \land \tau.v \ge 0 \land t \le \varepsilon) \\
rbc & : & (rbc.message := emergency) \\
 & & \cup\ \big(\mathbf{m}_0 := \mathbf{m}; \mathbf{m} := *; \\
 & & \quad ?\mathbf{m}.r \ge 0 \land \mathbf{m}.d \ge 0 \land \mathbf{m}_0.d^2 - \mathbf{m}.d^2 \le 2b(\mathbf{m}.e - \mathbf{m}_0.e)\big)
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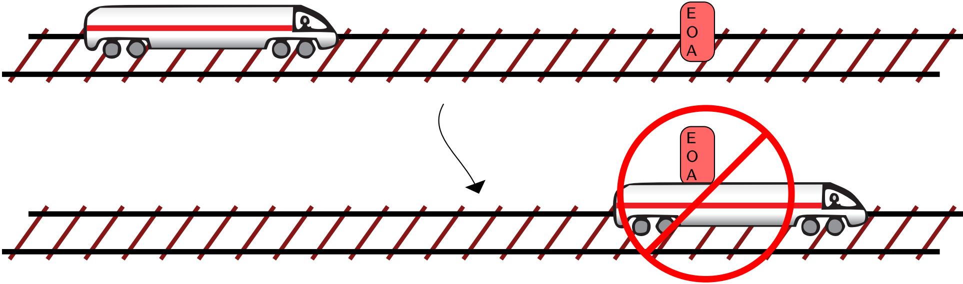
\end{array}
$$

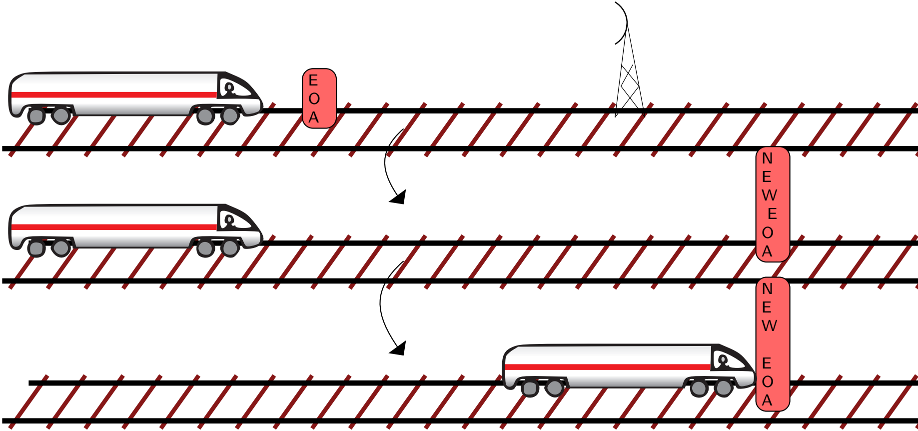
$$\tau.v^2 = \mathbf{m}.d^2 \le 2b(\mathbf{m}.e - \tau.p) \to [ETCS_{aug}](\tau.p \ge \mathbf{m}.e \to \tau.v \le \mathbf{m}.d)$$



Proposition (Safety)

$$\mathcal{C} \to$$
$$[ETCS](\tau.p \geq \mathbf{m}.e \to \tau.v \leq \mathbf{m}.d)$$

ETCS Liveness



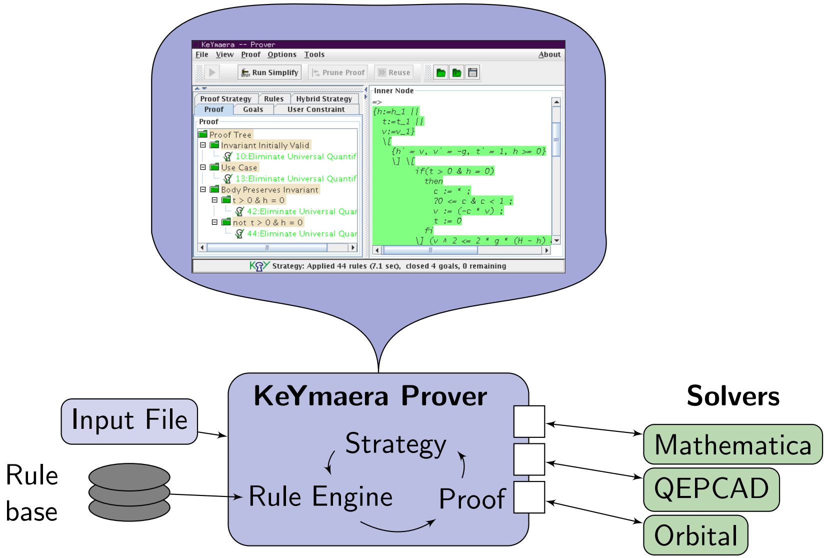
Proposition (Liveness)

$$\tau.v > 0 \land \varepsilon > 0 \rightarrow \forall P \, \langle ETCS \rangle \, \tau.p \geq P$$

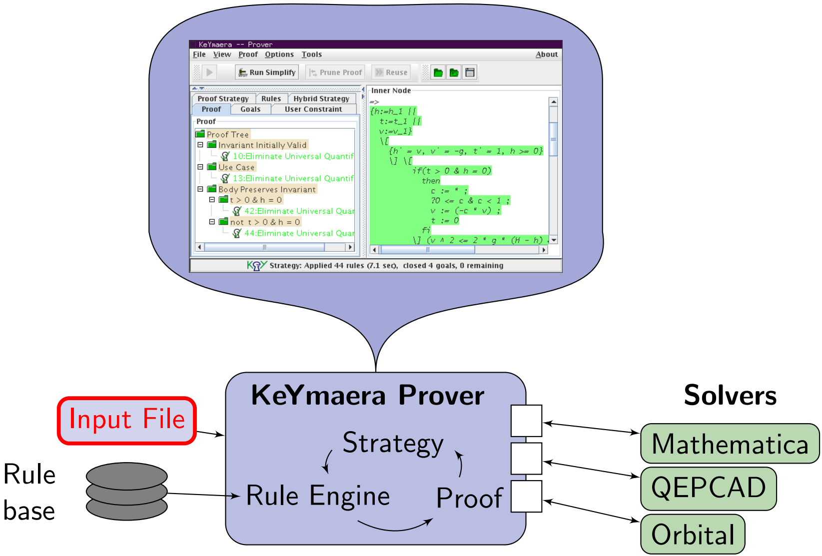
Outline

1. Motivation
2. Train Control
   - Separation Principle
   - Parametric ETCS
3. Parametric European Train Control System
   - Controllability
   - Reactivity
   - Refined Control
   - Safety
   - Liveness
4. Proving ETCS in KeYmaera
   - Architecture
   - KeYmaera Problem Input
   - KeYmaera Rule Base
   - Real Arithmetic, Computer Algebra and Automation
   - Experiments

KeYmaera Architecture



KeYmaera Architecture



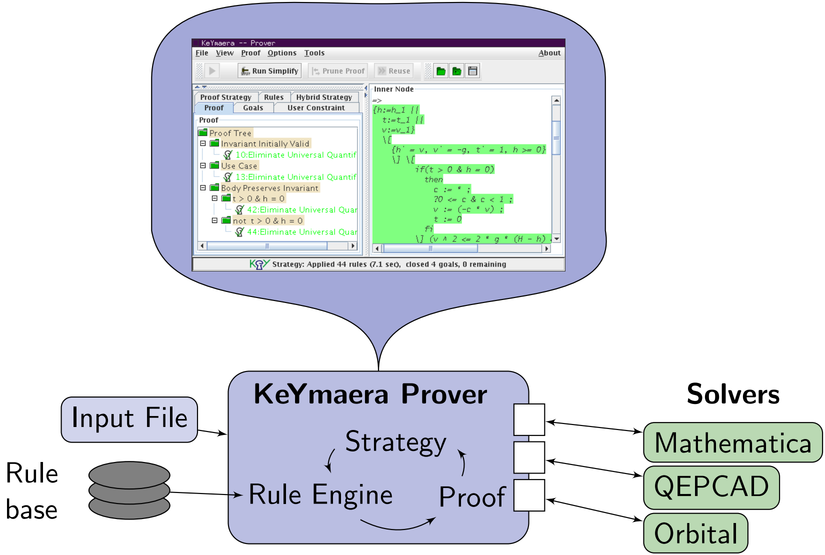
KeYmaera Problem Specification Input File .key

```
\functions {
  R ep;     R b;    R A;
}
\problem {
\[ R SB, a, v, z, t, m; \] (
  ( v^2 <= 2*b*(m-z) & b > 0 & A>=0)
->
 \[(
  SB := (v^2)/(2*b) + ((A/b)+1)*((A/2)*ep^2 + ep*v);
  ((?m - z <= SB; a:= -b)
   ++ (?m - z >= SB; a:=A));
  t:=0;
  {z`=v, v` = a, t`=1, (v >= 0 & t <= ep)}
  )*
  \] (z <= m)
)
}
```

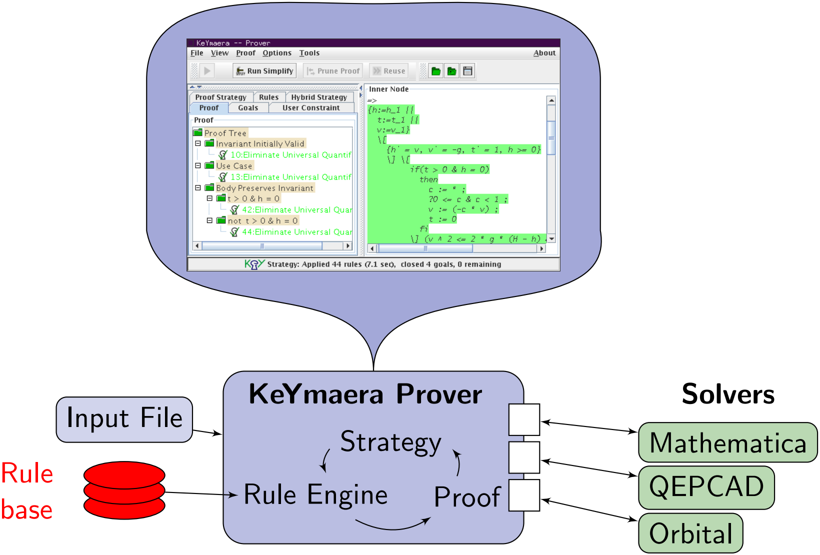
KeYmaera Problem Specification Input File .key

```
\functions { R b;     R A;    R ep; }
\problem {
\[ R r,SB,mo,t,a,v,z,m,d,do,drive,brake,state;drive:=0
   (v^2-d^2 <= 2*b*(m-z) & d>=0 & b>0 & A>=0 & ep>=0)
-> \[((
      (do:=d; mo:=m; m:=*; d:=*; r:=*;
       ?d>=0 & do^2-d^2<=2*b*(m-mo) & r>=0)
      ++ (state := brake)
     ) ++ (
       ((?v <= r; a:=*; ?a >=-b & a <= A)
       ++ (?v >= r; a:=*; ?a <0 & a >= -b));
       SB := (v^2-d^2)/(2*b) + (A/b+1)*(A/2*ep^2+ep*v);
       if (m-z <= SB | state=brake) then a:= b fi;
       (t:=0; {z'=v,v'=a,t'=1, (v>=0 & t<=ep)})
     )
     )*\] (z>=m -> v<=d)    )
}
```

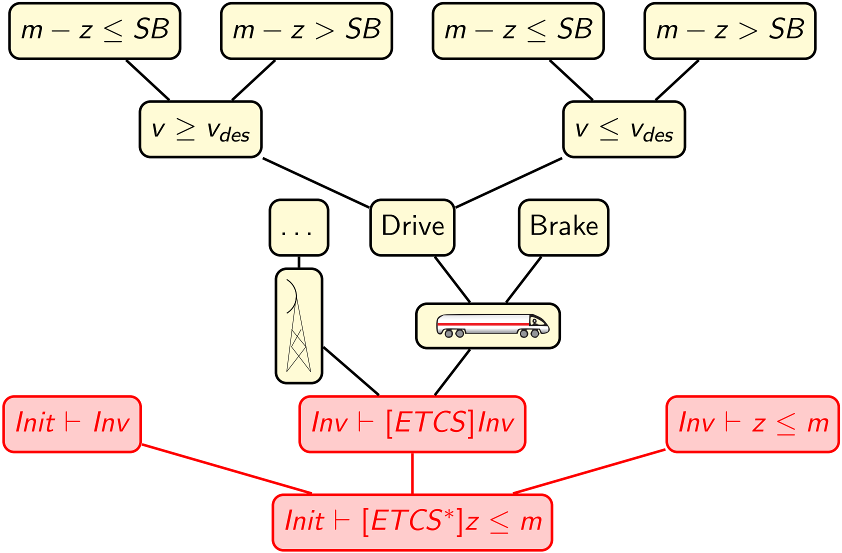
KeYmaera Architecture



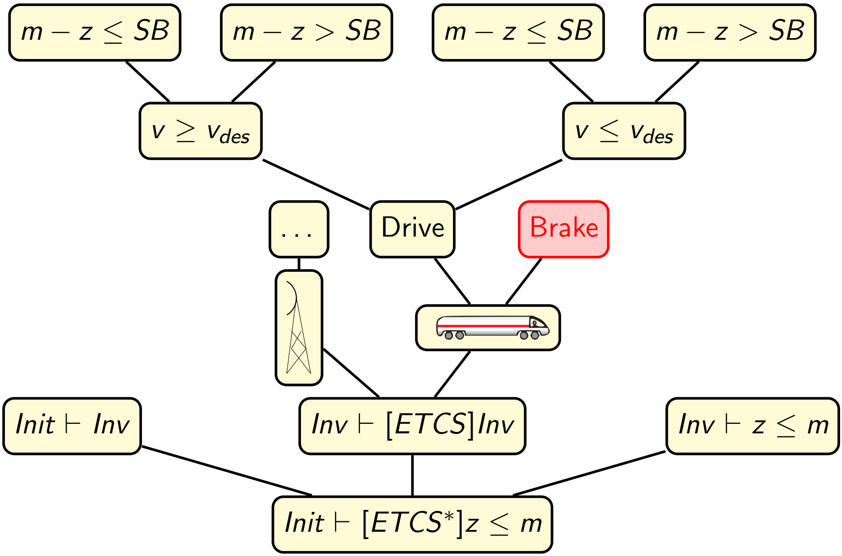
KeYmaera Architecture



Proof Sketch



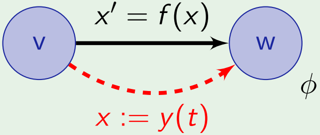
Proof Sketch



Handling Differential Equations

Example

$$\frac{\forall t \geq 0 \ [x := y(t)]\, \phi}{[x' = f(x)]\, \phi}$$

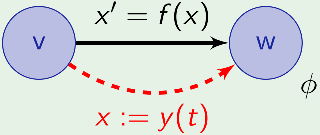


$$\frac{}{\ldots \vdash [z' = v, v' = -b] z \leq m}$$

Handling Differential Equations

Example

$$\frac{\forall t \geq 0\ [x := y(t)]\,\phi}{[x' = f(x)]\,\phi}$$

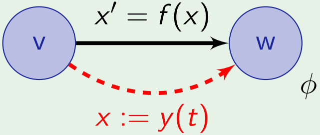


$$\frac{\overline{\ldots \vdash \forall t \geq 0\,[z := -\frac{1}{2}bt^2 + tv + z]z \leq m}}{\ldots \vdash [z' = v, v' = -b]z \leq m}$$

Handling Differential Equations

Example

$$\frac{\forall t \geq 0\ [x := y(t)]\,\phi}{[x' = f(x)]\,\phi}$$



$$\cfrac{\cfrac{\ldots \vdash \forall t \geq 0\ \left(-\frac{1}{2}bt^2 + tv + z \leq m\right)}{\ldots \vdash \forall t \geq 0\,[z := -\frac{1}{2}bt^2 + tv + z]z \leq m}}{\ldots \vdash [z' = v, v' = -b]z \leq m}$$

KeYmaera Rule Base

$$\frac{\Gamma \vdash \phi, \Delta \quad \Gamma \vdash \psi, \Delta}{\Gamma \vdash \phi \land \psi, \Delta}$$

```
and_right {
  \find (==> b & c)
  \replacewith(==> b);
  \replacewith(==> c)
  \heuristics(split,beta)
};
```

KeYmaera Rule Base

$$\frac{\Gamma \vdash [\alpha]\phi, \Delta \quad \Gamma \vdash [\beta]\phi, \Delta}{\Gamma \vdash [\alpha \cup \beta]\phi, \Delta}$$

```
box_choice_right {
  \find (==> \[ #dl ++ #dl2 \](post))
  \replacewith(==> \[#dl\](post));
  \replacewith(==> \[#dl2\](post))
  \heuristics(simplify_prog)
};
```

KeYmaera Rule Base

$$\frac{\Gamma \vdash \langle \mathcal{S}(t) \rangle \phi, \Delta}{\Gamma \vdash [x_1' = \theta_1, .., x_n' = \theta_n]\phi, \Delta}$$

```
ODESolve_right {
  \find (==> \[ #simpleode \](post))
  \replacewith(==> #ODESolve(\[#simpleode\](post)))
  \heuristics(diff_solve, diff_rule)
  \displayname "ODESolve"
};
```

$$\frac{\Gamma \vdash \langle \mathcal{S}(t) \rangle \phi, \Delta}{\Gamma \vdash [x_1' = \theta_1, \ldots, x_n' = \theta_n]\phi, \Delta}$$

```
ODESolve_right {
  \find (==> \[ #simpleode \](post))
  \replacewith(==> #ODESolve(\[#simpleode\](post)))
  \heuristics(diff_solve, diff_rule)
  \displayname "ODESolve"
};
```

Using meta-operator #ODESolve implemented in Java

$$\frac{\phi(X) \vdash}{\forall x\, \phi(x) \vdash} \qquad \frac{\phi(s(X_1, \ldots, X_n)) \vdash}{\exists x\, \phi(x) \vdash}$$

```
all_left {
  \find (\forall u; b ==>)
  \replacewith ({\subst u; q}(b) ==>)
  \heuristics(gamma)
};
ex_left {
  \find (\exists u; b ==>)
  \varcond (\new(sk, \dependingOn(b)))
  \replacewith ({\subst u; sk}b ==>)
  \heuristics(delta)
};
```

KeYmaera Rule Base

$$\frac{\vdash \mathrm{QE}(\forall X\,(\Phi(X) \vdash \Psi(X)))}{\Phi(s(X_1,\ldots,X_n)) \vdash \Psi(s(X_1,\ldots,X_n))}$$

$$\frac{\vdash \mathrm{QE}(\exists X\ \bigwedge_i(\Phi_i \vdash \Psi_i))}{\Phi_1 \vdash \Psi_1 \quad \ldots \quad \Phi_n \vdash \Psi_n}$$

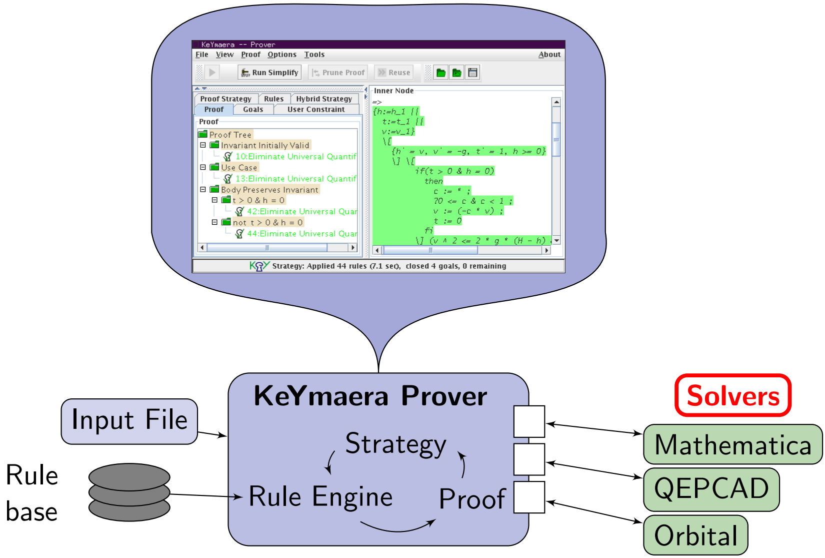
KeYmaera Rule Base

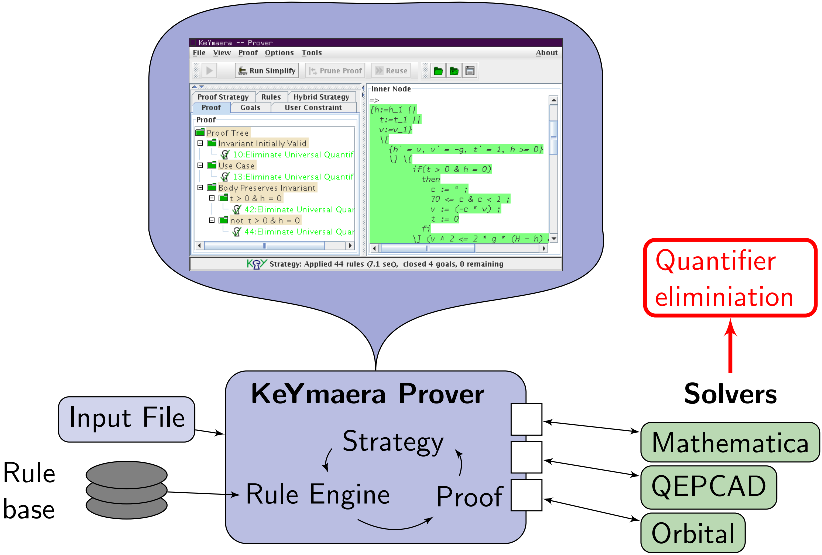
$$\frac{\vdash \mathrm{QE}(\forall X\,(\Phi(X) \vdash \Psi(X)))}{\Phi(s(X_1,\ldots,X_n)) \vdash \Psi(s(X_1,\ldots,X_n))}$$

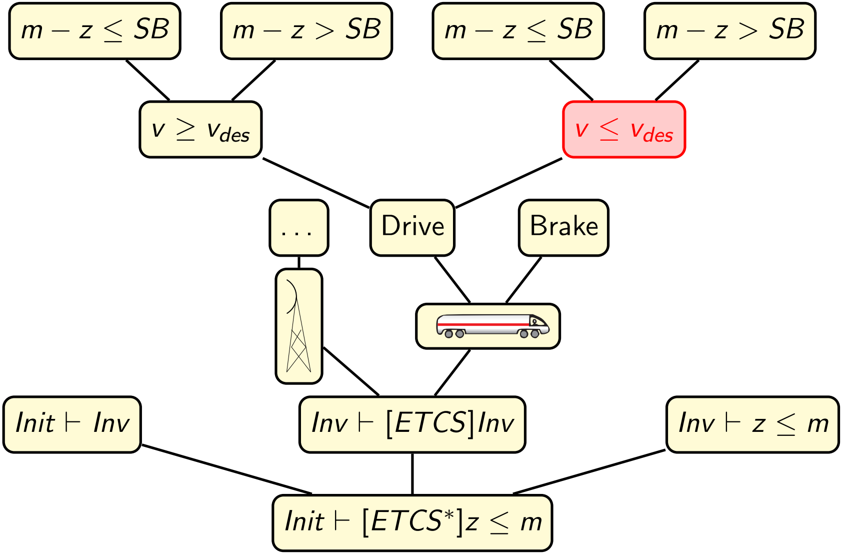
$$\frac{\vdash \mathrm{QE}(\exists X\ \bigwedge_i(\Phi_i \vdash \Psi_i))}{\Phi_1 \vdash \Psi_1 \quad \ldots \quad \Phi_n \vdash \Psi_n}$$

Using built-in rule implemented in Java

KeYmaera Architecture




KeYmaera -- Prover
File View Proof Options Tools About
Run Simplify
Prune Proof
Reuse
Proof Strategy Rules Hybrid Strategy
Proof Goals User Constraint
Proof Tree
Invariant Initially Valid
10:Eliminate Universal Quantif
Use Case
13:Eliminate Universal Quantif
Body Preserves Invariant
t > 0 & h = 0
42:Eliminate Universal Quar
not t > 0 & h = 0
44:Eliminate Universal Quar
Inner Node
Strategy: Applied 44 rules (7.1 sec), closed 4 goals, 0 remaining
Quantifier eliminiation
KeYmaera Prover
Input File
Strategy
Rule base
Rule Engine
Proof
Solvers
Mathematica
QEPCAD
Orbital


$m - z \le SB$
$m - z > SB$
$m - z \le SB$
$m - z > SB$
$v \ge v_{des}$
$v \le v_{des}$
...
Drive
Brake
$Init \vdash Inv$
$Inv \vdash [ETCS]Inv$
$Inv \vdash z \le m$
$Init \vdash [ETCS^*]z \le m$

Proof Sketch

$m - z \leq SB$ $m - z > SB$ $m - z \leq SB$ $m - z > SB$

$v \geq v_{des}$ $v \leq v_{des}$

Example

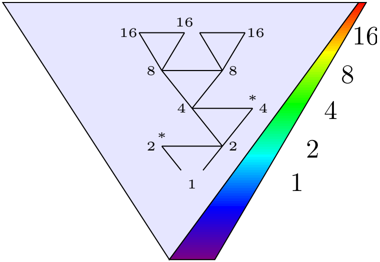
$$m - z \geq \left(\frac{A}{b} + 1\right)\left(\varepsilon v + \frac{A}{2}\varepsilon^2\right) + \frac{v^2 - d^2}{2b} \land 0 \leq a \leq A \land 0 \leq v \leq vdes$$
$$\land v^2 - d^2 \leq 2b(m - z) \land d \geq 0 \land \varepsilon > 0 \land b > 0 \land A > 0$$
$$\vdash$$
$$\forall t \geq 0\, ((\forall 0 \leq \tilde{t} \leq t\ (a\tilde{t} + v \geq 0 \land \tilde{t} \leq \varepsilon))$$
$$\rightarrow (at + v)^2 - d^2 \leq 2b(m - (\tfrac{1}{2}at + tv + z)) \land at + v \geq 0 \land d \geq 0)$$

$Init \vdash [ETCS^*]z \leq m$

Quantifier Elimination and Proof Strategies

- Quantifier elimination is doubly exponential
- Choice conflict:
  1. Apply quantifier elimination
  2. Split using

$$\frac{\vdash A \quad \vdash B}{\vdash A \land B}$$



Experimental Results

Case Study	Interact	Steps	IBC(s)	Eager QE(s)
ETCS essentials	0	46	47.8	∞
	1	46	6.6	8.8
ETCS complete	0	163	2045.2	∞
	1	168	23.3	∞
ETCS reactivity	0	49	76.2	∞
ETCS liveness	3	112	17.6	16.0
Aircraft TRM	0	94	10.9	∞
	1	94	1.2	1.2
TRM 3 Planes	0	187	171.8	∞
	1	187	21.2	∞
TRM 4 Planes	0	255	704.3	∞
	1	255	170	∞
Water tank	1	375	2.0	2.0

$\infty \mathrel{\widehat{=}}$ more than five hours

Experimental Results

Case Study	Interact	Steps	IBC(s)	Eager QE(s)
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$\infty \hat{=}$ more than five hours

A. Platzer.
Differential dynamic logic for hybrid systems.
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