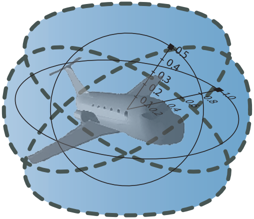
15-819/18-879: Hybrid Systems Analysis & Theorem Proving

05: Sequent Calculus

André Platzer

aplatzer@cs.cmu.edu
Carnegie Mellon University, Pittsburgh, PA



Outline

1. Sequent Calculus
   - Propositional Sequent Calculus
   - First-Order Sequent Calculus
   - Context-free Short Notation
   - Soundness & Completeness

Outline

1. Sequent Calculus
   - Propositional Sequent Calculus
   - First-Order Sequent Calculus
   - Context-free Short Notation
   - Soundness & Completeness

Motivation

- A "general" notation for proof calculi
- Introduced by Gerald Gentzen 1935 as a tool for studying natural deduction.
- Tableaux have a canonical sequent rendition.
- Branch transformations, paths, side conditions, repetitions have a cleaner formulation.

Sequents

A normal form for formulas

Definition (Sequent)

A *sequent* is a pair of finite sets of formulas, with the notation

$$\Gamma \vdash \Delta$$

Γ is called antecedent and Δ succedent. Empty sets are allowed for Γ, Δ and denoted without set brackets.

Definition (Semantics of sequents)

$$val(s, \Gamma \vdash \Delta) = val(s, \bigwedge_{G \in \Gamma} G \to \bigvee_{F \Delta} F)$$

where empty conjunctions are *true* and empty disjunctions *false*

Sequent Calculus: Propositional Rules

$$\frac{}{\Gamma, \phi \land \psi \vdash \Delta}$$

Sequent Calculus: Propositional Rules

$$\frac{\Gamma, \phi, \psi \vdash \Delta}{\Gamma, \phi \wedge \psi \vdash \Delta}$$

Sequent Calculus: Propositional Rules

$$\frac{\Gamma, \phi, \psi \vdash \Delta}{\Gamma, \phi \land \psi \vdash \Delta}$$

$$\frac{}{\Gamma \vdash \phi \land \psi, \Delta}$$

Sequent Calculus: Propositional Rules

$$\frac{\Gamma, \phi, \psi \vdash \Delta}{\Gamma, \phi \wedge \psi \vdash \Delta}$$

$$\frac{\Gamma \vdash \phi, \Delta \quad \Gamma \vdash \psi, \Delta}{\Gamma \vdash \phi \wedge \psi, \Delta}$$

Sequent Calculus: Propositional Rules

$$\frac{\Gamma, \phi, \psi \vdash \Delta}{\Gamma, \phi \wedge \psi \vdash \Delta} \qquad \frac{}{\Gamma, \phi \vee \psi \vdash \Delta}$$

$$\frac{\Gamma \vdash \phi, \Delta \quad \Gamma \vdash \psi, \Delta}{\Gamma \vdash \phi \wedge \psi, \Delta}$$

Sequent Calculus: Propositional Rules

$$\frac{\Gamma, \phi, \psi \vdash \Delta}{\Gamma, \phi \land \psi \vdash \Delta}$$

$$\frac{\Gamma, \phi \vdash \Delta \quad \Gamma, \psi \vdash \Delta}{\Gamma, \phi \lor \psi \vdash \Delta}$$

$$\frac{\Gamma \vdash \phi, \Delta \quad \Gamma \vdash \psi, \Delta}{\Gamma \vdash \phi \land \psi, \Delta}$$

Sequent Calculus: Propositional Rules

$$\frac{\Gamma, \phi, \psi \vdash \Delta}{\Gamma, \phi \wedge \psi \vdash \Delta} \qquad \frac{\Gamma, \phi \vdash \Delta \quad \Gamma, \psi \vdash \Delta}{\Gamma, \phi \vee \psi \vdash \Delta}$$

$$\frac{\Gamma \vdash \phi, \Delta \quad \Gamma \vdash \psi, \Delta}{\Gamma \vdash \phi \wedge \psi, \Delta} \qquad \frac{}{\Gamma \vdash \phi \vee \psi, \Delta}$$

Sequent Calculus: Propositional Rules

$$\frac{\Gamma, \phi, \psi \vdash \Delta}{\Gamma, \phi \wedge \psi \vdash \Delta} \qquad \frac{\Gamma, \phi \vdash \Delta \quad \Gamma, \psi \vdash \Delta}{\Gamma, \phi \vee \psi \vdash \Delta}$$

$$\frac{\Gamma \vdash \phi, \Delta \quad \Gamma \vdash \psi, \Delta}{\Gamma \vdash \phi \wedge \psi, \Delta} \qquad \frac{\Gamma \vdash \phi, \psi, \Delta}{\Gamma \vdash \phi \vee \psi, \Delta}$$

Sequent Calculus: Propositional Rules

$$\frac{\Gamma, \phi, \psi \vdash \Delta}{\Gamma, \phi \land \psi \vdash \Delta} \qquad \frac{\Gamma, \phi \vdash \Delta \quad \Gamma, \psi \vdash \Delta}{\Gamma, \phi \lor \psi \vdash \Delta} \qquad \frac{}{\Gamma, \neg\phi \vdash \Delta}$$

$$\frac{\Gamma \vdash \phi, \Delta \quad \Gamma \vdash \psi, \Delta}{\Gamma \vdash \phi \land \psi, \Delta} \qquad \frac{\Gamma \vdash \phi, \psi, \Delta}{\Gamma \vdash \phi \lor \psi, \Delta}$$

Sequent Calculus: Propositional Rules

$$\frac{\Gamma, \phi, \psi \vdash \Delta}{\Gamma, \phi \land \psi \vdash \Delta} \qquad \frac{\Gamma, \phi \vdash \Delta \quad \Gamma, \psi \vdash \Delta}{\Gamma, \phi \lor \psi \vdash \Delta} \qquad \frac{\Gamma \vdash \phi, \Delta}{\Gamma, \neg\phi \vdash \Delta}$$

$$\frac{\Gamma \vdash \phi, \Delta \quad \Gamma \vdash \psi, \Delta}{\Gamma \vdash \phi \land \psi, \Delta} \qquad \frac{\Gamma \vdash \phi, \psi, \Delta}{\Gamma \vdash \phi \lor \psi, \Delta}$$

Sequent Calculus: Propositional Rules

$$\frac{\Gamma, \phi, \psi \vdash \Delta}{\Gamma, \phi \land \psi \vdash \Delta} \qquad \frac{\Gamma, \phi \vdash \Delta \quad \Gamma, \psi \vdash \Delta}{\Gamma, \phi \lor \psi \vdash \Delta} \qquad \frac{\Gamma \vdash \phi, \Delta}{\Gamma, \neg\phi \vdash \Delta}$$

$$\frac{\Gamma \vdash \phi, \Delta \quad \Gamma \vdash \psi, \Delta}{\Gamma \vdash \phi \land \psi, \Delta} \qquad \frac{\Gamma \vdash \phi, \psi, \Delta}{\Gamma \vdash \phi \lor \psi, \Delta} \qquad \frac{}{\Gamma \vdash \neg\phi, \Delta}$$

Sequent Calculus: Propositional Rules

$$\frac{\Gamma, \phi, \psi \vdash \Delta}{\Gamma, \phi \land \psi \vdash \Delta} \qquad \frac{\Gamma, \phi \vdash \Delta \quad \Gamma, \psi \vdash \Delta}{\Gamma, \phi \lor \psi \vdash \Delta} \qquad \frac{\Gamma \vdash \phi, \Delta}{\Gamma, \neg\phi \vdash \Delta}$$

$$\frac{\Gamma \vdash \phi, \Delta \quad \Gamma \vdash \psi, \Delta}{\Gamma \vdash \phi \land \psi, \Delta} \qquad \frac{\Gamma \vdash \phi, \psi, \Delta}{\Gamma \vdash \phi \lor \psi, \Delta} \qquad \frac{\Gamma, \phi \vdash \Delta}{\Gamma \vdash \neg\phi, \Delta}$$

Sequent Calculus: Propositional Rules

$$\frac{\Gamma, \phi, \psi \vdash \Delta}{\Gamma, \phi \land \psi \vdash \Delta} \qquad \frac{\Gamma, \phi \vdash \Delta \quad \Gamma, \psi \vdash \Delta}{\Gamma, \phi \lor \psi \vdash \Delta} \qquad \frac{\Gamma \vdash \phi, \Delta}{\Gamma, \neg\phi \vdash \Delta}$$

$$\frac{\Gamma \vdash \phi, \Delta \quad \Gamma \vdash \psi, \Delta}{\Gamma \vdash \phi \land \psi, \Delta} \qquad \frac{\Gamma \vdash \phi, \psi, \Delta}{\Gamma \vdash \phi \lor \psi, \Delta} \qquad \frac{\Gamma, \phi \vdash \Delta}{\Gamma \vdash \neg\phi, \Delta}$$

$$\frac{}{\Gamma \vdash \phi \to \psi, \Delta}$$

Sequent Calculus: Propositional Rules

$$\frac{\Gamma, \phi, \psi \vdash \Delta}{\Gamma, \phi \land \psi \vdash \Delta} \qquad \frac{\Gamma, \phi \vdash \Delta \quad \Gamma, \psi \vdash \Delta}{\Gamma, \phi \lor \psi \vdash \Delta} \qquad \frac{\Gamma \vdash \phi, \Delta}{\Gamma, \neg\phi \vdash \Delta}$$

$$\frac{\Gamma \vdash \phi, \Delta \quad \Gamma \vdash \psi, \Delta}{\Gamma \vdash \phi \land \psi, \Delta} \qquad \frac{\Gamma \vdash \phi, \psi, \Delta}{\Gamma \vdash \phi \lor \psi, \Delta} \qquad \frac{\Gamma, \phi \vdash \Delta}{\Gamma \vdash \neg\phi, \Delta}$$

$$\frac{\Gamma, \phi \vdash \psi, \Delta}{\Gamma \vdash \phi \to \psi, \Delta}$$

Sequent Calculus: Propositional Rules

$$\frac{\Gamma, \phi, \psi \vdash \Delta}{\Gamma, \phi \land \psi \vdash \Delta}$$

$$\frac{\Gamma, \phi \vdash \Delta \quad \Gamma, \psi \vdash \Delta}{\Gamma, \phi \lor \psi \vdash \Delta}$$

$$\frac{\Gamma \vdash \phi, \Delta}{\Gamma, \neg\phi \vdash \Delta}$$

$$\frac{\Gamma \vdash \phi, \Delta \quad \Gamma \vdash \psi, \Delta}{\Gamma \vdash \phi \land \psi, \Delta}$$

$$\frac{\Gamma \vdash \phi, \psi, \Delta}{\Gamma \vdash \phi \lor \psi, \Delta}$$

$$\frac{\Gamma, \phi \vdash \Delta}{\Gamma \vdash \neg\phi, \Delta}$$

$$\frac{\Gamma, \phi \vdash \psi, \Delta}{\Gamma \vdash \phi \to \psi, \Delta}$$

$$\frac{\quad}{\Gamma, \phi \to \psi \vdash \Delta}$$

Sequent Calculus: Propositional Rules

$$\frac{\Gamma, \phi, \psi \vdash \Delta}{\Gamma, \phi \land \psi \vdash \Delta} \qquad \frac{\Gamma, \phi \vdash \Delta \quad \Gamma, \psi \vdash \Delta}{\Gamma, \phi \lor \psi \vdash \Delta} \qquad \frac{\Gamma \vdash \phi, \Delta}{\Gamma, \neg\phi \vdash \Delta}$$

$$\frac{\Gamma \vdash \phi, \Delta \quad \Gamma \vdash \psi, \Delta}{\Gamma \vdash \phi \land \psi, \Delta} \qquad \frac{\Gamma \vdash \phi, \psi, \Delta}{\Gamma \vdash \phi \lor \psi, \Delta} \qquad \frac{\Gamma, \phi \vdash \Delta}{\Gamma \vdash \neg\phi, \Delta}$$

$$\frac{\Gamma, \phi \vdash \psi, \Delta}{\Gamma \vdash \phi \to \psi, \Delta}$$

$$\frac{\Gamma \vdash \phi, \Delta \quad \Gamma, \psi \vdash \Delta}{\Gamma, \phi \to \psi \vdash \Delta}$$

Sequent Calculus: Propositional Rules

$$\frac{\Gamma, \phi, \psi \vdash \Delta}{\Gamma, \phi \land \psi \vdash \Delta} \qquad \frac{\Gamma, \phi \vdash \Delta \quad \Gamma, \psi \vdash \Delta}{\Gamma, \phi \lor \psi \vdash \Delta} \qquad \frac{\Gamma \vdash \phi, \Delta}{\Gamma, \neg\phi \vdash \Delta}$$

$$\frac{\Gamma \vdash \phi, \Delta \quad \Gamma \vdash \psi, \Delta}{\Gamma \vdash \phi \land \psi, \Delta} \qquad \frac{\Gamma \vdash \phi, \psi, \Delta}{\Gamma \vdash \phi \lor \psi, \Delta} \qquad \frac{\Gamma, \phi \vdash \Delta}{\Gamma \vdash \neg\phi, \Delta}$$

$$\frac{\Gamma, \phi \vdash \psi, \Delta}{\Gamma \vdash \phi \to \psi, \Delta} \qquad \frac{}{\Gamma, \phi \vdash \phi, \Delta}$$

$$\frac{\Gamma \vdash \phi, \Delta \quad \Gamma, \psi \vdash \Delta}{\Gamma, \phi \to \psi \vdash \Delta}$$

Sequent Calculus: Propositional Rules

$$\frac{\Gamma, \phi, \psi \vdash \Delta}{\Gamma, \phi \land \psi \vdash \Delta} \qquad \frac{\Gamma, \phi \vdash \Delta \quad \Gamma, \psi \vdash \Delta}{\Gamma, \phi \lor \psi \vdash \Delta} \qquad \frac{\Gamma \vdash \phi, \Delta}{\Gamma, \neg\phi \vdash \Delta}$$

$$\frac{\Gamma \vdash \phi, \Delta \quad \Gamma \vdash \psi, \Delta}{\Gamma \vdash \phi \land \psi, \Delta} \qquad \frac{\Gamma \vdash \phi, \psi, \Delta}{\Gamma \vdash \phi \lor \psi, \Delta} \qquad \frac{\Gamma, \phi \vdash \Delta}{\Gamma \vdash \neg\phi, \Delta}$$

$$\frac{\Gamma, \phi \vdash \psi, \Delta}{\Gamma \vdash \phi \to \psi, \Delta} \qquad \frac{}{\Gamma, \phi \vdash \phi, \Delta}$$

$$\frac{\Gamma \vdash \phi, \Delta \quad \Gamma, \psi \vdash \Delta}{\Gamma, \phi \to \psi \vdash \Delta}$$

Sequent Calculus: Propositional Rules

$$\frac{\Gamma, \phi, \psi \vdash \Delta}{\Gamma, \phi \wedge \psi \vdash \Delta} \qquad \frac{\Gamma, \phi \vdash \Delta \quad \Gamma, \psi \vdash \Delta}{\Gamma, \phi \vee \psi \vdash \Delta} \qquad \frac{\Gamma \vdash \phi, \Delta}{\Gamma, \neg\phi \vdash \Delta}$$

$$\frac{\Gamma \vdash \phi, \Delta \quad \Gamma \vdash \psi, \Delta}{\Gamma \vdash \phi \wedge \psi, \Delta} \qquad \frac{\Gamma \vdash \phi, \psi, \Delta}{\Gamma \vdash \phi \vee \psi, \Delta} \qquad \frac{\Gamma, \phi \vdash \Delta}{\Gamma \vdash \neg\phi, \Delta}$$

$$\frac{\Gamma, \phi \vdash \psi, \Delta}{\Gamma \vdash \phi \rightarrow \psi, \Delta} \qquad \frac{}{\Gamma, \phi \vdash \phi, \Delta}$$

$$\frac{\Gamma \vdash \phi, \Delta \quad \Gamma, \psi \vdash \Delta}{\Gamma, \phi \rightarrow \psi \vdash \Delta} \qquad \frac{}{\Gamma \vdash \Delta}$$

Sequent Calculus: Propositional Rules

$$\frac{\Gamma, \phi, \psi \vdash \Delta}{\Gamma, \phi \wedge \psi \vdash \Delta} \qquad \frac{\Gamma, \phi \vdash \Delta \quad \Gamma, \psi \vdash \Delta}{\Gamma, \phi \vee \psi \vdash \Delta} \qquad \frac{\Gamma \vdash \phi, \Delta}{\Gamma, \neg\phi \vdash \Delta}$$

$$\frac{\Gamma \vdash \phi, \Delta \quad \Gamma \vdash \psi, \Delta}{\Gamma \vdash \phi \wedge \psi, \Delta} \qquad \frac{\Gamma \vdash \phi, \psi, \Delta}{\Gamma \vdash \phi \vee \psi, \Delta} \qquad \frac{\Gamma, \phi \vdash \Delta}{\Gamma \vdash \neg\phi, \Delta}$$

$$\frac{\Gamma, \phi \vdash \psi, \Delta}{\Gamma \vdash \phi \rightarrow \psi, \Delta} \qquad \frac{}{\Gamma, \phi \vdash \phi, \Delta}$$

$$\frac{\Gamma \vdash \phi, \Delta \quad \Gamma, \psi \vdash \Delta}{\Gamma, \phi \rightarrow \psi \vdash \Delta} \qquad \frac{\Gamma, \phi \vdash \Delta \quad \Gamma \vdash \phi, \Delta}{\Gamma \vdash \Delta}$$

Cuts in Sequent Calculi

- Cuts can do arbitrary case distinctions.

Cuts in Sequent Calculi

- Cuts can do arbitrary case distinctions.
- Do cuts add deductive power to propositional tableaux?

Sequent Calculus: First-Order Rules

$$\frac{}{\Gamma, \forall x\, F(x) \vdash \Delta}$$

$X \notin \Gamma, \Delta$ new variable

Sequent Calculus: First-Order Rules

$$\frac{\Gamma, \forall x\, F(x), F(X) \vdash \Delta}{\Gamma, \forall x\, F(x) \vdash \Delta}$$

$X \notin \Gamma, \Delta$ new variable

Sequent Calculus: First-Order Rules

$$\frac{\Gamma, \forall x\, F(x), F(X) \vdash \Delta}{\Gamma, \forall x\, F(x) \vdash \Delta}$$

$X \notin \Gamma, \Delta$ new variable

$$\frac{}{\Gamma, \exists x\, F(x) \vdash \Delta}$$

$FV(\exists x\, F(x)) = \{X_1, \ldots, X_n\}$, f new

Sequent Calculus: First-Order Rules

$$\frac{\Gamma, \forall x\, F(x), F(X) \vdash \Delta}{\Gamma, \forall x\, F(x) \vdash \Delta}$$

$X \notin \Gamma, \Delta$ new variable

$$\frac{\Gamma, F(f(X_1, \ldots, X_n)) \vdash \Delta}{\Gamma, \exists x\, F(x) \vdash \Delta}$$

$FV(\exists x\, F(x)) = \{X_1, \ldots, X_n\}$, f new

Sequent Calculus: First-Order Rules

$$\frac{\Gamma, \forall x\, F(x), F(X) \vdash \Delta}{\Gamma, \forall x\, F(x) \vdash \Delta}$$

$X \notin \Gamma, \Delta$ new variable

$$\frac{}{\Gamma \vdash \forall x\, F(x), \Delta}$$

$FV(\forall x\, F(x)) = \{X_1, \ldots, X_n\}$, f r

$$\frac{\Gamma, F(f(X_1, \ldots, X_n)) \vdash \Delta}{\Gamma, \exists x\, F(x) \vdash \Delta}$$

$FV(\exists x\, F(x)) = \{X_1, \ldots, X_n\}$, f new

Sequent Calculus: First-Order Rules

$$\frac{\Gamma, \forall x\, F(x), F(X) \vdash \Delta}{\Gamma, \forall x\, F(x) \vdash \Delta}$$

$X \notin \Gamma, \Delta$ new variable

$$\frac{\Gamma \vdash F(f(X_1, \ldots, X_n)), \Delta}{\Gamma \vdash \forall x\, F(x), \Delta}$$

$FV(\forall x\, F(x)) = \{X_1, \ldots, X_n\}$, f r

$$\frac{\Gamma, F(f(X_1, \ldots, X_n)) \vdash \Delta}{\Gamma, \exists x\, F(x) \vdash \Delta}$$

$FV(\exists x\, F(x)) = \{X_1, \ldots, X_n\}$, f new

Sequent Calculus: First-Order Rules

$$\frac{\Gamma, \forall x\, F(x), F(X) \vdash \Delta}{\Gamma, \forall x\, F(x) \vdash \Delta}$$

$X \notin \Gamma, \Delta$ new variable

$$\frac{\Gamma \vdash F(f(X_1, \ldots, X_n)), \Delta}{\Gamma \vdash \forall x\, F(x), \Delta}$$

$FV(\forall x\, F(x)) = \{X_1, \ldots, X_n\}$, f r

$$\frac{\Gamma, F(f(X_1, \ldots, X_n)) \vdash \Delta}{\Gamma, \exists x\, F(x) \vdash \Delta}$$

$FV(\exists x\, F(x)) = \{X_1, \ldots, X_n\}$, f new

$$\frac{}{\Gamma \vdash \exists x\, F(x), \Delta}$$

$X \notin \Gamma, \Delta$ new variable

Sequent Calculus: First-Order Rules

$$\frac{\Gamma, \forall x\, F(x), F(X) \vdash \Delta}{\Gamma, \forall x\, F(x) \vdash \Delta}$$

$X \notin \Gamma, \Delta$ new variable

$$\frac{\Gamma \vdash F(f(X_1, \ldots, X_n)), \Delta}{\Gamma \vdash \forall x\, F(x), \Delta}$$

$FV(\forall x\, F(x)) = \{X_1, \ldots, X_n\}$, f r

$$\frac{\Gamma, F(f(X_1, \ldots, X_n)) \vdash \Delta}{\Gamma, \exists x\, F(x) \vdash \Delta}$$

$FV(\exists x\, F(x)) = \{X_1, \ldots, X_n\}$, f new

$$\frac{\Gamma \vdash \exists x\, F(x), F(X), \Delta}{\Gamma \vdash \exists x\, F(x), \Delta}$$

$X \notin \Gamma, \Delta$ new variable

Sequent Calculus: "Context-free" Notation

$$(\wedge\ \text{left})\ \frac{\phi, \psi \vdash}{\phi \wedge \psi \vdash} \qquad (\vee\ \text{left})\ \frac{\phi \vdash \quad \psi \vdash}{\phi \vee \psi \vdash} \qquad (\neg\ \text{left})\ \frac{\vdash \phi}{\neg\phi \vdash}$$

$$(\wedge\ \text{right})\ \frac{\vdash \phi \quad \vdash \psi}{\vdash \phi \wedge \psi} \qquad (\vee\ \text{right})\ \frac{\vdash \phi, \psi}{\vdash \phi \vee \psi} \qquad (\neg\ \text{right})\ \frac{\phi \vdash}{\vdash \neg\phi}$$

$$(\rightarrow\ \text{right})\ \frac{\phi \vdash \psi}{\vdash \phi \rightarrow \psi} \qquad (\text{axiom})\ \frac{}{\phi \vdash \phi}$$

$$(\rightarrow\ \text{left})\ \frac{\vdash \phi \quad \psi \vdash}{\phi \rightarrow \psi \vdash} \qquad (\text{cut})\ \frac{\phi \vdash \quad \vdash \phi}{\vdash}$$

Provability in Sequent Calculus

Definition (Provability, ground case)

- Formula ψ is provable from a set of formulas Φ, denoted as

$$\Phi \vdash_{\text{seq}} \psi$$

  iff there is a finite subset $\Phi_0 \subseteq \Phi$ for which the sequent $\Phi_0 \vdash \psi$ is derivable.

- Sequent $\Gamma \vdash \Delta$ is *derivable* iff, inductively, there is an instance

$$\frac{\Gamma_1 \vdash \Delta_1 \quad \ldots \quad \Gamma_n \vdash \Delta_n}{\Gamma \vdash \Delta}$$

  of a sequent calculus rule with conclusion $\Gamma \vdash \Delta$ such that all premisses $\Gamma_i \vdash \Delta_i$ of the rule are derivable.

Provability in Sequent Calculus

Definition (Provability, free variable case)

- Formula ψ is provable from a set of formulas Φ, denoted as

$$\Phi \vdash_{\mathsf{seq}} \psi$$

  iff there is a finite subset $\Phi_0 \subseteq \Phi$ such that there is a closed derivation of the sequent $\Phi_0 \vdash \psi$.
- A *derivation* is a tree of nodes labeled with sequents such that: The labels of the successors of a node labelled $\Gamma \vdash \Delta$ are exactly the premisses $\Gamma_i \vdash \Delta_i$ of an instance of a sequent calculus rule

$$\frac{\Gamma_1 \vdash \Delta_1 \quad \dots \quad \Gamma_n \vdash \Delta_n}{\Gamma \vdash \Delta}$$

- A derivation is closed if there is a substitution σ such that for each label $\Gamma \vdash \Delta$ of all its leaves, $\sigma(\Gamma \vdash \Delta)$ is an instance of an axiom.

Sequent Calculus Derivation

$$\frac{\frac{\frac{\frac{\frac{\frac{}{}}{}}{}}{}}{}}{\vdash \exists y\, \forall x\, \mathrm{ctrl}(x, y) \rightarrow \forall x\, \exists y\, \mathrm{ctrl}(x, y)}$$

Sequent Calculus Derivation

$$\dfrac{\exists y\,\forall x\,\mathrm{ctrl}(x,y) \vdash \forall x\,\exists y\,\mathrm{ctrl}(x,y)}{\vdash \exists y\,\forall x\,\mathrm{ctrl}(x,y) \rightarrow \forall x\,\exists y\,\mathrm{ctrl}(x,y)}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{}}{}}{\exists y\, \forall x\, \mathrm{ctrl}(x,y) \vdash \exists y\, \mathrm{ctrl}(s,y)}}{\exists y\, \forall x\, \mathrm{ctrl}(x,y) \vdash \forall x\, \exists y\, \mathrm{ctrl}(x,y)}}{}}{\vdash \exists y\, \forall x\, \mathrm{ctrl}(x,y) \rightarrow \forall x\, \exists y\, \mathrm{ctrl}(x,y)}$$

$\exists y\, \forall x\, \mathrm{ctrl}(x,y) \vdash \exists y\, \mathrm{ctrl}(s,y)$

$\exists y\, \forall x\, \mathrm{ctrl}(x,y) \vdash \forall x\, \exists y\, \mathrm{ctrl}(x,y)$

$\vdash \exists y\, \forall x\, \mathrm{ctrl}(x,y) \rightarrow \forall x\, \exists y\, \mathrm{ctrl}(x,y)$

Sequent Calculus Derivation

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{}}{}}{\exists y\, \forall x\, \mathrm{ctrl}(x,y) \vdash \exists y\, \mathrm{ctrl}(s,y), \mathrm{ctrl}(s,Y)}}{\exists y\, \forall x\, \mathrm{ctrl}(x,y) \vdash \exists y\, \mathrm{ctrl}(s,y)}}{\exists y\, \forall x\, \mathrm{ctrl}(x,y) \vdash \forall x\, \exists y\, \mathrm{ctrl}(x,y)}}{\vdash \exists y\, \forall x\, \mathrm{ctrl}(x,y) \rightarrow \forall x\, \exists y\, \mathrm{ctrl}(x,y)}$$

Sequent Calculus Derivation

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\forall x\, \mathrm{ctrl}(x,t) \vdash \exists y\, \mathrm{ctrl}(s,y), \mathrm{ctrl}(s,Y)}}{\exists y\, \forall x\, \mathrm{ctrl}(x,y) \vdash \exists y\, \mathrm{ctrl}(s,y), \mathrm{ctrl}(s,Y)}}{\exists y\, \forall x\, \mathrm{ctrl}(x,y) \vdash \exists y\, \mathrm{ctrl}(s,y)}}{\exists y\, \forall x\, \mathrm{ctrl}(x,y) \vdash \forall x\, \exists y\, \mathrm{ctrl}(x,y)}}{\vdash \exists y\, \forall x\, \mathrm{ctrl}(x,y) \rightarrow \forall x\, \exists y\, \mathrm{ctrl}(x,y)}$$

Sequent Calculus Derivation

$$
\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\forall x\, \mathrm{ctrl}(x,t), \mathrm{ctrl}(X,t) \vdash \exists y\, \mathrm{ctrl}(s,y), \mathrm{ctrl}(s,Y)}{\forall x\, \mathrm{ctrl}(x,t) \vdash \exists y\, \mathrm{ctrl}(s,y), \mathrm{ctrl}(s,Y)}}{\exists y\, \forall x\, \mathrm{ctrl}(x,y) \vdash \exists y\, \mathrm{ctrl}(s,y), \mathrm{ctrl}(s,Y)}}{\exists y\, \forall x\, \mathrm{ctrl}(x,y) \vdash \exists y\, \mathrm{ctrl}(s,y)}}{\exists y\, \forall x\, \mathrm{ctrl}(x,y) \vdash \forall x\, \exists y\, \mathrm{ctrl}(x,y)}}{\vdash \exists y\, \forall x\, \mathrm{ctrl}(x,y) \rightarrow \forall x\, \exists y\, \mathrm{ctrl}(x,y)}
$$

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{*\{s/X\}\{t/Y\}}{\forall x\, \mathrm{ctrl}(x,t), \mathrm{ctrl}(X,t) \vdash \exists y\, \mathrm{ctrl}(s,y), \mathrm{ctrl}(s,Y)}}{\forall x\, \mathrm{ctrl}(x,t) \vdash \exists y\, \mathrm{ctrl}(s,y), \mathrm{ctrl}(s,Y)}}{\exists y\, \forall x\, \mathrm{ctrl}(x,y) \vdash \exists y\, \mathrm{ctrl}(s,y), \mathrm{ctrl}(s,Y)}}{\exists y\, \forall x\, \mathrm{ctrl}(x,y) \vdash \exists y\, \mathrm{ctrl}(s,y)}}{\exists y\, \forall x\, \mathrm{ctrl}(x,y) \vdash \forall x\, \exists y\, \mathrm{ctrl}(x,y)}}{\vdash \exists y\, \forall x\, \mathrm{ctrl}(x,y) \rightarrow \forall x\, \exists y\, \mathrm{ctrl}(x,y)}
$$

Soundness & Completeness of FOL Sequent Calculus

Theorem (Soundness)

All provable (closed) FOL *formulas* F *are valid:*

$$\vdash_{seq} F \quad \Rightarrow \quad \vDash F$$

Soundness & Completeness of FOL Sequent Calculus

Theorem (Soundness)

All provable (closed) FOL *formulas F are valid:*

$$\vdash_{seq} F \quad \Rightarrow \quad \models F$$

Theorem (Completeness)

All valid (closed) FOL *formulas F are provable:*

$$\models F \quad \Rightarrow \quad \vdash_{seq} F$$

Soundness & Completeness of FOL Sequent Calculus

Theorem (Soundness)

All provable (closed) FOL *formulas F are valid:*

$$\vdash_{seq} F \quad \Rightarrow \quad \vDash F$$

Theorem (Completeness)

All valid (closed) FOL *formulas F are provable:*

$$\vDash F \quad \Rightarrow \quad \vdash_{seq} F$$

Theorem (Strong Soundness & Completeness)

For (closed) FOL *formula F and set of (closed)* FOL *formulas* Γ*:*

$$\Gamma \vdash_{seq} F \quad \Longleftrightarrow \quad \Gamma \vDash F$$

Jean H. Gallier.
Logic for Computer Science.
Longman Higher Education, 1986.

Gerald Gentzen.
Untersuchungen über das logische Schließen.
Math. Zeit., 39:405–431, 1935.