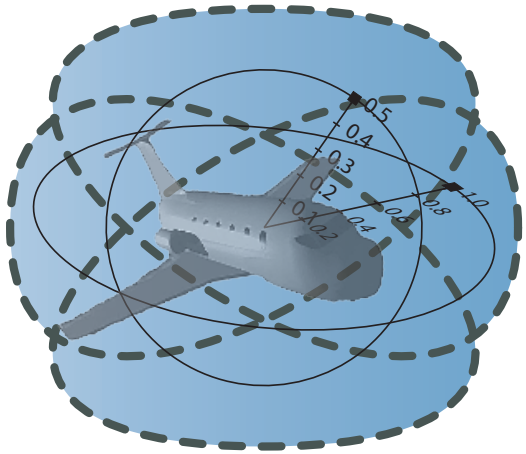
15-819/18-879: Hybrid Systems Analysis & Theorem Proving

02: Propositional and First-order Logic

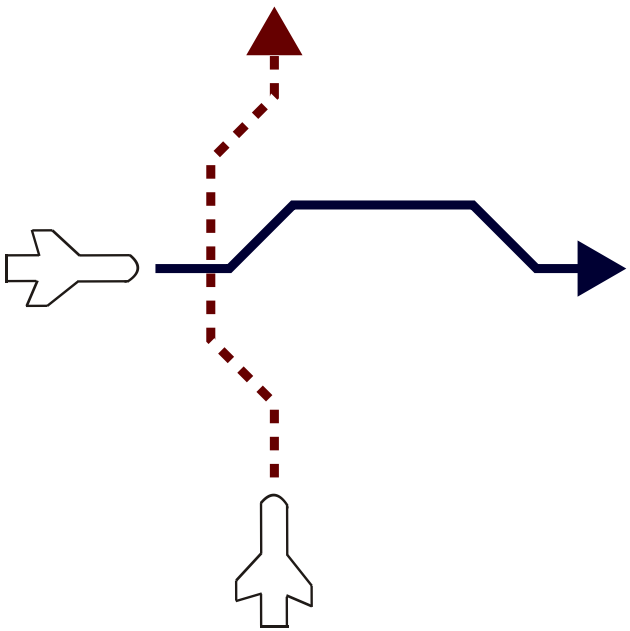
André Platzer

aplatzer@cs.cmu.edu
Carnegie Mellon University, Pittsburgh, PA



Outline

Hybrid Automaton for Collision Avoidance



Example (Property)

If the aircraft are far apart and have compatible speed, then—when following the protocol—they will never crash?

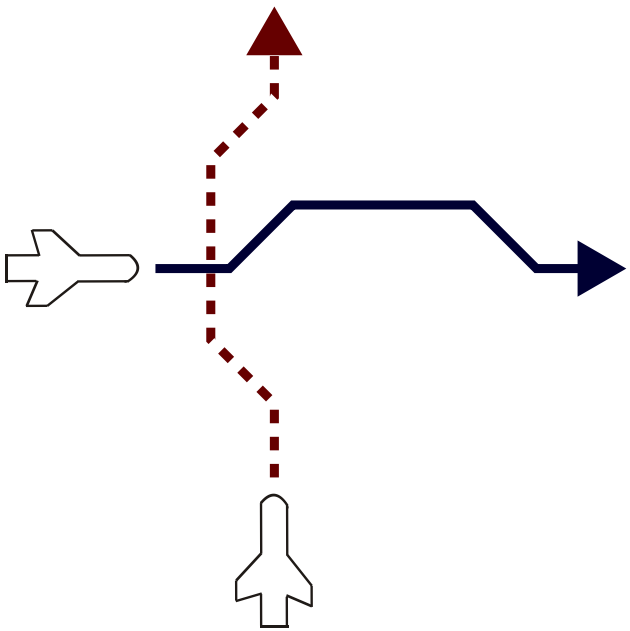
Example (Property)

If the aircraft enter collision avoidance, then—when following the protocol—will they ever leave again, i.e. follow their old route?

Example (System behavior)

If the aircraft are coming too close and I am flying northbound, then we will initiate left evasive actions.

Hybrid Automaton for Collision Avoidance



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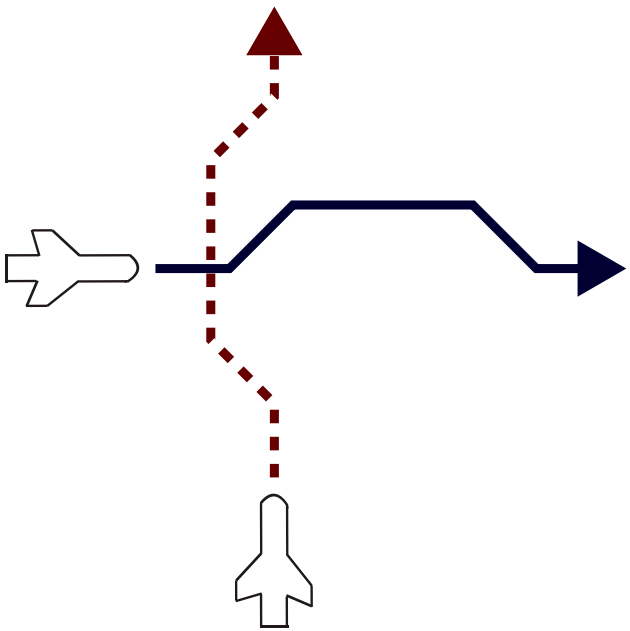

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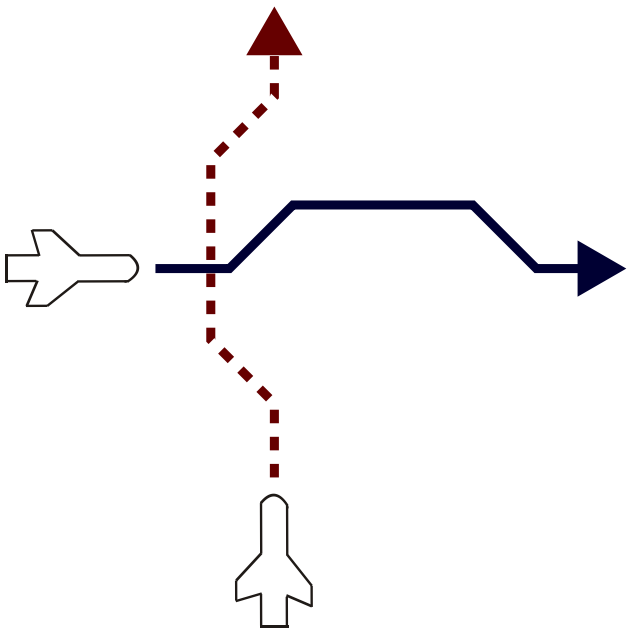

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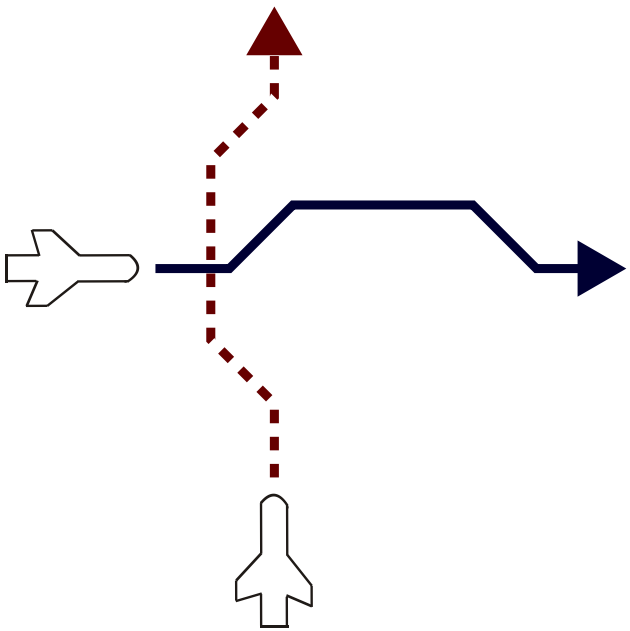

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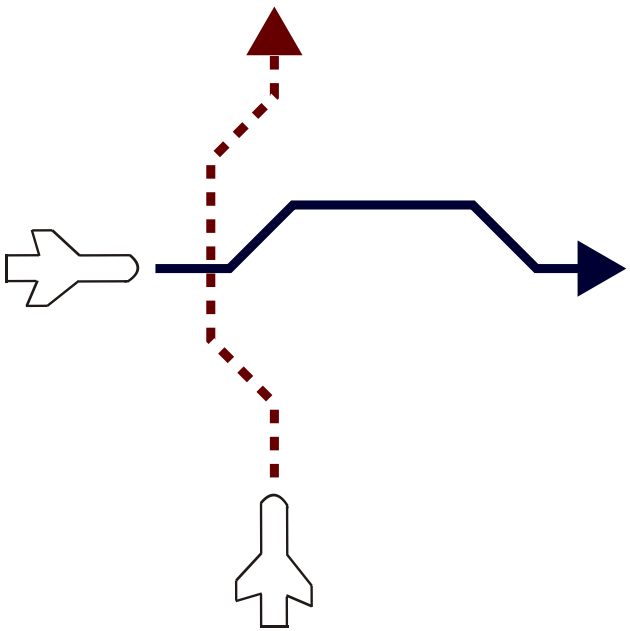

Example (Informal&incomplete reasoning)

If I turn left then so will he, thus we cannot come closer. Further, he will not turn back unless we have sufficient distance. Since we do not change course early later on, we will keep the distance and can never crash.

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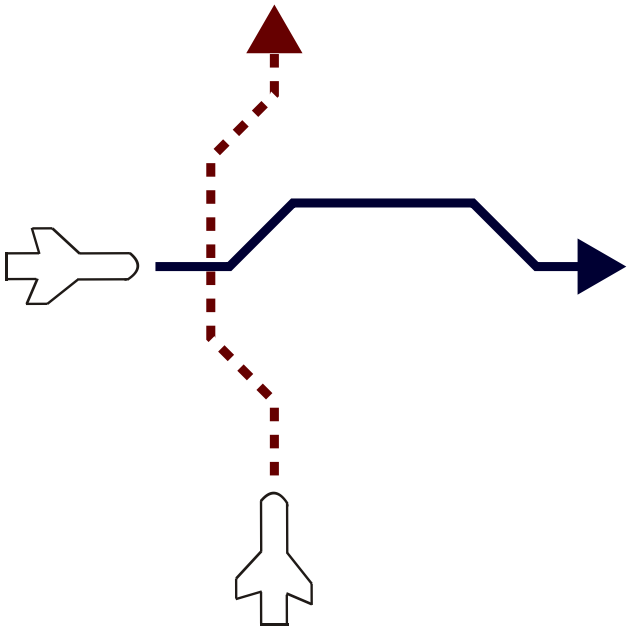

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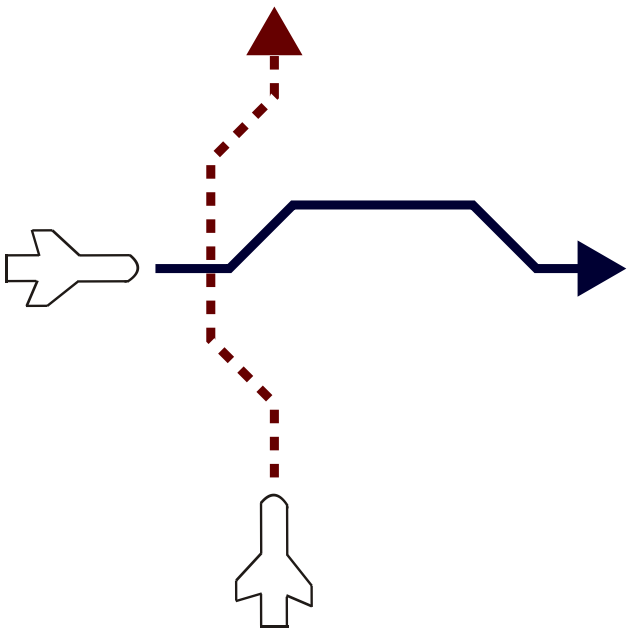

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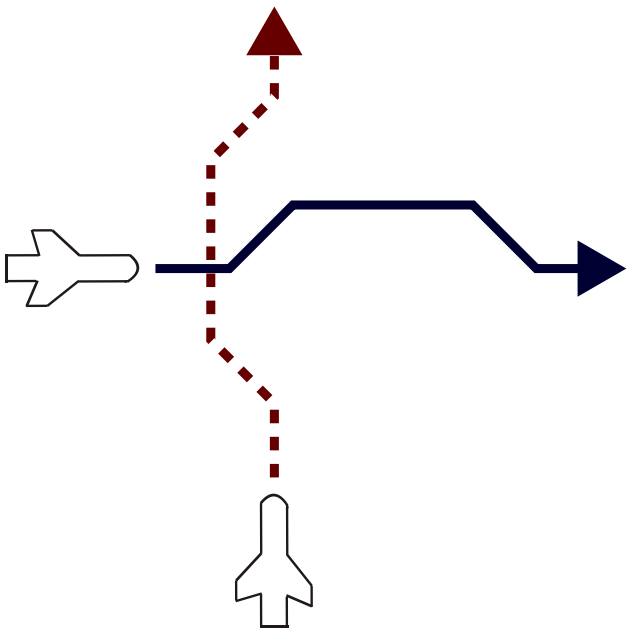

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In hybrid systems, there is significant logical structure in the properties, the system, the reasoning, ...

Outline

Propositional Logic: Syntax

Definition (PL_0 Vocabulary V)

A set V of propositional variables / letters

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A set V of propositional variables / letters

Example

$$V = \{\mathsf{nb}_x, \mathsf{nb}_y, \mathsf{compat}, p_1, p_2, p_3, \dots\}$$

Propositional Logic: Syntax

Definition (PL_0 Vocabulary V)

A set V of propositional variables / letters

Definition (PL_0 Formula F, G)

$F ::=$

p	for any $p \in V$
$\neg F$	"not"
$(F \wedge G)$	"and"
$(F \vee G)$	"or"
$(F \rightarrow G)$	"implies"
$(F \leftrightarrow G)$	"equivalent/bi-implies"

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Example (Northbound flight of aircraft x and y is compatible)

$$(\text{nb}_x \wedge \text{nb}_y) \rightarrow \text{compat}$$

Quiz

Is this a propositional formula?

1. $F \vee (G \wedge (H \leftrightarrow \neg F) \rightarrow J)$
2. $A \rightarrow (B \wedge (\neg F \leftrightarrow$
3. $x > y \leftrightarrow x - y > 0$
4. $A \rightarrow B \vee (\neg B \wedge C)$

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Propositional Logic: Semantics

Definition (PL_0 Truth-assignment I)

Assignment $I : V \rightarrow \{true, false\}$ of truth-values to propositional variables

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Example

v	nb_x	nb_y	compat
$I(v)$	*false*	*true*	*true*

Propositional Logic: Semantics

Definition (PL_0 Truth-assignment I)

Assignment $I : V \to \{true, false\}$ of truth-values to propositional variables

Definition (PL_0 Interpretation $[\![\cdot]\!]_I$)

$[\![\cdot]\!]_I$ is inductively defined as

$$
\begin{aligned}
[\![p]\!]_I &= I(p) \text{ for any } p \in V \\
[\![\neg F]\!]_I &= true \text{ iff } [\![F]\!]_I = false \\
[\![F \wedge G]\!]_I &= true \text{ iff } [\![F]\!]_I = true \text{ and } [\![G]\!]_I = true \\
[\![F \vee G]\!]_I &= true \text{ iff } [\![F]\!]_I = true \text{ or } [\![G]\!]_I = true \\
[\![F \to G]\!]_I &= true \text{ iff } [\![F]\!]_I = false \text{ or } [\![G]\!]_I = true \\
[\![F \leftrightarrow G]\!]_I &= true \text{ iff, either, } [\![F]\!]_I = true \text{ and } [\![G]\!]_I = true \\
&\qquad \text{or, instead, } [\![F]\!]_I = false \text{ and } [\![G]\!]_I = false
\end{aligned}
$$

Propositional Logic: Semantics

Example (Truth tables)

$I(\text{nb}_x)$	$I(\text{nb}_y)$	$I(\text{compat})$	$[\![\text{nb}_x \wedge nb_y]\!]_I$

Propositional Logic: Semantics

Example (Truth tables)

$I(\text{nb}_x)$	$I(\text{nb}_y)$	$I(\text{compat})$	$[\![\text{nb}_x \wedge nb_y]\!]_I$	$[\![\text{nb}_x \wedge nb_y \rightarrow \text{compat}]\!]_I$
true	*true*	*true*	*true*	

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Validity

Definition (Validity / Tautology)

Formula F is *valid* iff $[\![F]\!]_I = \mathit{true}$ for all assignments I, otherwise *invalid*

Example

Is this formula valid: $\mathrm{nb}_x \wedge \mathit{nb}_y \rightarrow \mathrm{separate}$?

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Is this formula valid: $A \rightarrow (B \vee C \leftrightarrow (B \rightarrow \neg C \vee A))$?

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Is this formula valid: F ?

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How do you check validity of a formula?

Proposition

Validity and satisfiability for propositional logic are decidable (by exhaustive enumeration) in exponential time. Better: SAT solving!

Quiz

1. Example for a valid formula?
2. Example for an invalid formula?
3. Example for an unsatisfiable formula?

Quiz

Is this a valid/invalid/satisfiable/unsatisfiable propositional formula?

1. $A \rightarrow (B \rightarrow A)$
2. $\neg(A \wedge B) \leftrightarrow \neg A \vee \neg B$
3. $A \rightarrow B \vee (\neg B \wedge C)$
4. $(A \rightarrow (\neg A \rightarrow B)) \rightarrow B$
5. $\big((A \leftrightarrow B) \leftrightarrow C\big) \leftrightarrow \big(A \leftrightarrow (B \leftrightarrow C)\big)$
6. $x < 0 \rightarrow x^2 > 0$
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Outline

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Definition (FOL Signature Σ)

Countable set of predicate or function symbols along with natural numbers as arities

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Countable set of predicate or function symbols along with natural numbers as arities

Example (Aircraft vocabulary)

$$\Sigma = \{\text{cruise}/1, \text{separate}/2, \text{faster}/2, \text{ground}/1\} \cup \{\ldots\}$$

First-order Logic: Syntax

Definition (FOL Signature Σ)

Countable set of predicate or function symbols along with natural numbers as arities

Definition (FOL Term t)

$t ::=$

x	for variable $x \in V$
$f(t_1, \ldots, t_n)$	for function $f/n \in \Sigma$ of arity $n \geq 0$

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Definition (FOL Formula F, G)

$F ::=$

$p(t_1, \ldots, t_n)$	for predicate $p/n \in \Sigma$ of arity $n \geq 0$
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$\forall x\, F$	"universal quantifier/forall" for $x \in V$
$\exists x\, F$	"existential quantifier/exists" for $x \in V$

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Example (Aircraft vocabulary)

$$\Sigma = \{\text{cruise}/1, \text{separate}/2, \text{faster}/2, \text{ground}/1\} \cup \{\dots\}$$

Example (Cruising aircraft cannot crash)

$$(\forall x\, \text{cruise}(x)) \rightarrow \neg\exists x\, \exists y\, (\neg\text{separate}(x, y))$$

Quiz

Is this a first-order formula?

1. $F \vee (G \wedge (H \leftrightarrow \neg F) \rightarrow J)$
2. $\forall x\,(p(x) \rightarrow \exists y\,(p(y) \wedge \exists x\,\neg r(x,y)))$
3. $\forall x\,\exists y\,p(x \rightarrow f(y))$
4. $\forall x\,\forall y\,(x > y \leftrightarrow x - y > 0)$
5. $\forall x\,(\mathrm{human}(x) \rightarrow \mathrm{mortal}(x)) \wedge \mathrm{human}(\mathit{Sokrates}) \rightarrow \mathrm{mortal}(\mathit{Sokrates})$
6. $\forall x\,\forall y\,(e(x,y) \leftrightarrow \forall p\,(r(x) \wedge p(x) \leftrightarrow p(y)))$
7. $\forall x\,e(f(f(x)),x) \rightarrow \exists y\,e(f(f(f(y))),f(y))$

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× $\forall x\,\forall y\,(x > y \leftrightarrow x - y > 0)$

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Is this a first-order formula?

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Semantics of Formulas

Example (Cruising aircraft cannot crash)

$$(\forall x\, \text{cruise}(x)) \rightarrow \neg\exists x\, \exists y\, (\neg\text{separate}(x, y))$$

Is this formula true over the following signature?

Example (Aircraft vocabulary)

$$\Sigma = \{\text{cruise}/1, \text{separate}/2\} \cup \emptyset$$

Truth depends on the interpretation of symbols

First-order Logic: Semantics

Definition (FOL Interpretation I)

1. D non-empty set (*domain/universe*)
2. I assigns relations and functions on D to all symbols in Σ
   - function $I(f) : D^n \rightarrow D$ for each function symbol f of arity n
   - relation $I(p) \subseteq D^n$ for each predicate symbol p of arity n
   - element $I(c) \in D$ for each constant symbol (function of arity 0)
   - truth-value $I(p) \in \{true, false\}$ for each predicate symbol of arity 0

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Definition (FOL Variable Assignment β)

Assignment $\beta : V \to D$ of an element in domain D to each variable in V

Definition (FOL Valuation of terms $[\![\cdot]\!]_{I,\beta}$)

$[\![\cdot]\!]_{I,\beta}$ is inductively defined as

$$[\![x]\!]_{I,\beta} = \beta(x) \text{ for variable } x \in V$$
$$[\![f(t_1, \ldots, t_n)]\!]_{I,\beta} = I(f)([\![t_1]\!]_{I,\beta}, \ldots, [\![t_n]\!]_{I,\beta})$$

Definition (FOL Valuation of formulas $[\![\cdot]\!]_{I,\beta}$)

$[\![\cdot]\!]_{I,\beta}$ is inductively defined as

$$
\begin{aligned}
[\![p(t_1,\ldots,t_n)]\!]_{I,\beta} &= \mathit{true} \text{ iff } ([\![t_1]\!]_{I,\beta},\ldots,[\![t_n]\!]_{I,\beta}) \in I(p) \text{ for pred. } p \in \Sigma \\
[\![\neg F]\!]_{I,\beta} &= \mathit{true} \text{ iff } [\![F]\!]_{I,\beta} = \mathit{false} \\
[\![F \wedge G]\!]_{I,\beta} &= \mathit{true} \text{ iff } [\![F]\!]_{I,\beta} = \mathit{true} \text{ and } [\![G]\!]_{I,\beta} = \mathit{true} \\
[\![F \vee G]\!]_{I,\beta} &= \mathit{true} \text{ iff } [\![F]\!]_{I,\beta} = \mathit{true} \text{ or } [\![G]\!]_{I,\beta} = \mathit{true} \\
[\![F \rightarrow G]\!]_{I,\beta} &= \mathit{true} \text{ iff } [\![F]\!]_{I,\beta} = \mathit{false} \text{ or } [\![G]\!]_{I,\beta} = \mathit{true} \\
[\![\forall x\, F]\!]_{I,\beta} &= \mathit{true} \text{ iff } [\![F]\!]_{I,\beta'} = \mathit{true} \text{ for all } \beta' \text{ that are like } \beta \\
&\qquad \text{except for the value of } x \\
[\![\exists x\, F]\!]_{I,\beta} &= \mathit{true} \text{ iff } [\![F]\!]_{I,\beta'} = \mathit{true} \text{ for some } \beta' \text{ that is like } \beta \\
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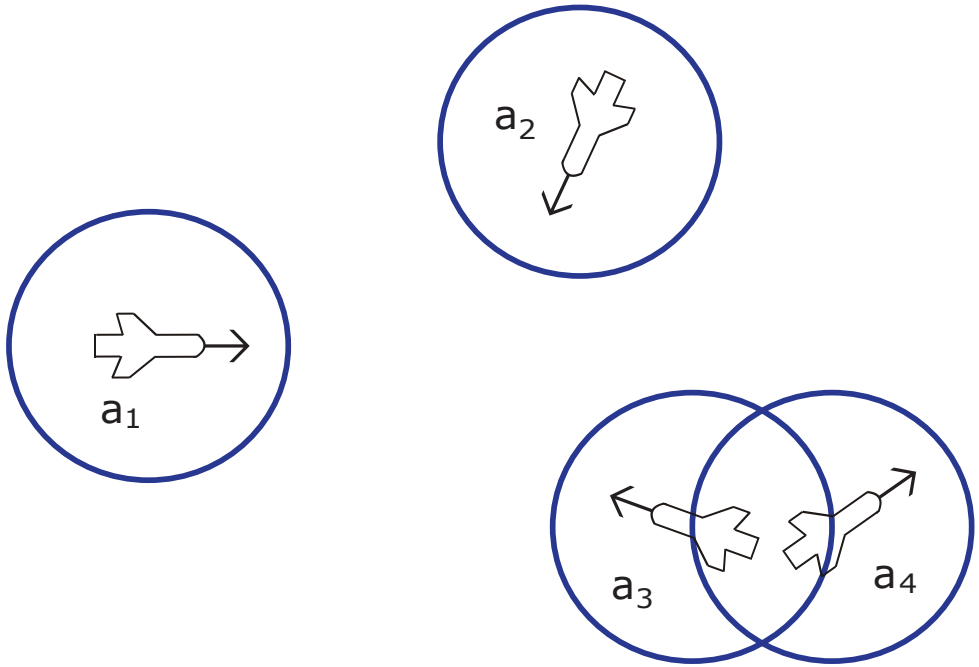
\end{aligned}
$$

One Example for an Interpretation



Example (Cruising aircraft cannot crash)

$$(\forall x\, \text{cruise}(x)) \rightarrow \neg\exists x\, \exists y\, (\neg\text{separate}(x, y))$$

$$I_4(\text{cruise}) = \{a_1, a_3, a_4\}$$
$$I_4(\text{separate}) = \{(a_1, a_2), (a_1, a_3), (a_1, a_4), (a_2, a_3), (a_2, a_4)\}$$

Quiz

Is this a valid first-order formula?

1. $F \vee (G \wedge (H \leftrightarrow \neg F) \rightarrow J)$
2. $\forall x\,(p(x) \rightarrow \exists y\,(p(y) \wedge \exists x\,\neg r(x,y)))$
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5. $\forall x\,e(f(f(x)),x) \rightarrow \exists y\,e(f(f(f(y))),f(y))$
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Outline

Interpreted First-order Logic

- In a formula like

$$(x_1 - y_1)^2 + (x_2 - y_2)^2 > p^2$$

how do we get "+" and "−" and "^2" and ">" to mean what we want?

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Interpreted First-order Logic: Syntax

Definition (Interpreted $\text{FOL}_{\mathbb{R}}$ Term t)

$t ::=$

x	for variable $x \in V$
r	for rational number r
$t_1 + t_2$	(infix notation)
$t_1 - t_2$	(infix notation)
$t_1 \cdot t_2$	(infix notation)
$f(t_1, \ldots, t_n)$	for function $f/n \in \Sigma$ of arity $n \geq 0$

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Definition (Interpreted $\text{FOL}_{\mathbb{R}}$ Formula F, G)

$F ::=$

$t_1 \geq t_2$	(infix notation)
$t_1 > t_2$	(infix notation)
$t_1 = t_2$	(infix notation)
$p(t_1, \ldots, t_n)$	for predicate $p/n \in \Sigma$ of arity $n \geq 0$
$\neg F$	"not"
$(F \wedge G)$	"and"
$(F \vee G)$	"or"
$(F \rightarrow G)$	"implies"
$(F \leftrightarrow G)$	"equivalent/bi-implies"
$\forall x\, F$	"universal quantifier/forall" for $x \in V$
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4. $x < y \wedge \exists z\,x > z^2$
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1. $D = \mathbb{R}$
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   - function $I(f) : \mathbb{R}^n \to \mathbb{R}$ for each function symbol f of arity n
   - relation $I(p) \subseteq \mathbb{R}^n$ for each predicate symbol p of arity n
   - element $I(c) \in \mathbb{R}$ for each constant symbol (function of arity 0)
   - truth-value $I(p) \in \{true, false\}$ for each predicate symbol of arity 0

   such that
   - $I(+)$ is addition on $\mathbb{R}$
   - $I(-)$ is subtraction on $\mathbb{R}$
   - $I(\cdot)$ is multiplication on $\mathbb{R}$
   - $I(=)$ is equality on $\mathbb{R}$
   - $I(>)$ is the greater relation on $\mathbb{R}$
   - $I(\geq)$ is the greater-equals relation on $\mathbb{R}$
   - $I(r) = r$ for all numbers $r \in \mathbb{Q}$

Quiz

(Validity of) which of the following logics is decidable/semidecidable/undecidable/not semidecidable?

1. PL_0
2. FOL
3. $\mathrm{FOL}_{\mathbb{N}}[+, \cdot, =]$
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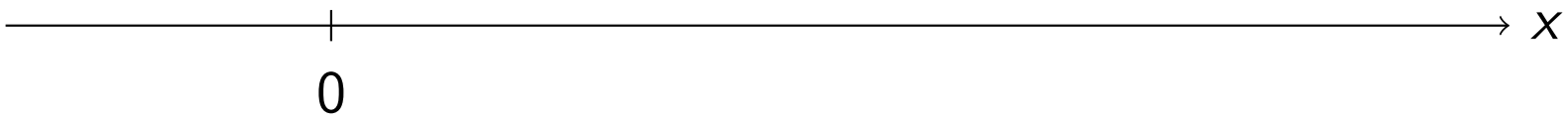
Quantifier Elimination by Example



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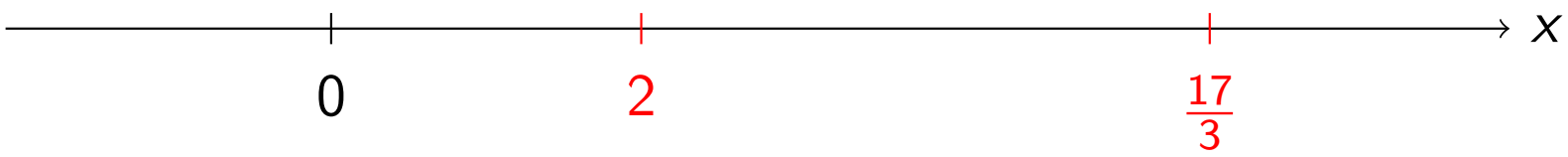

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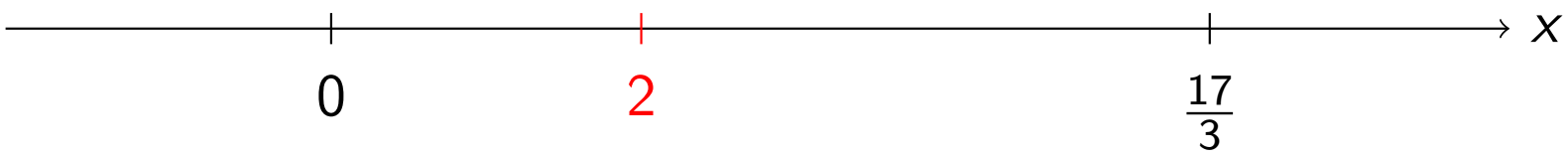

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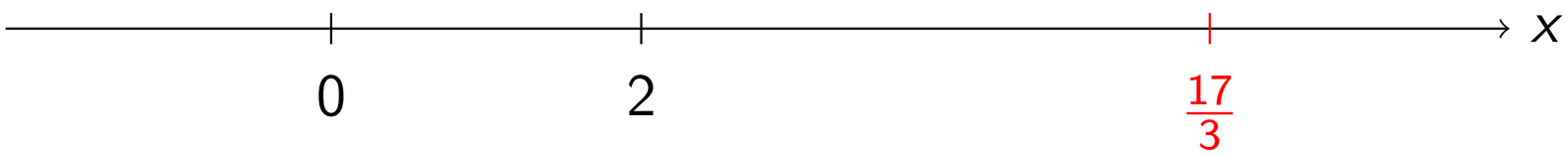

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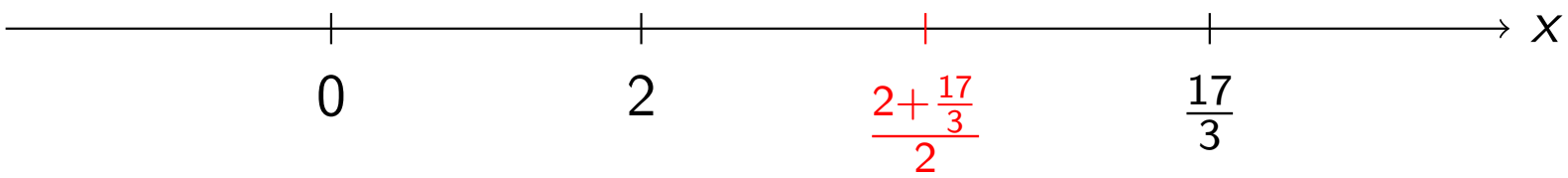

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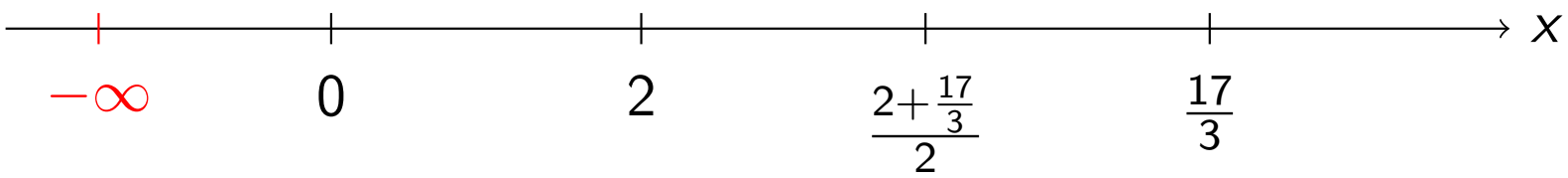

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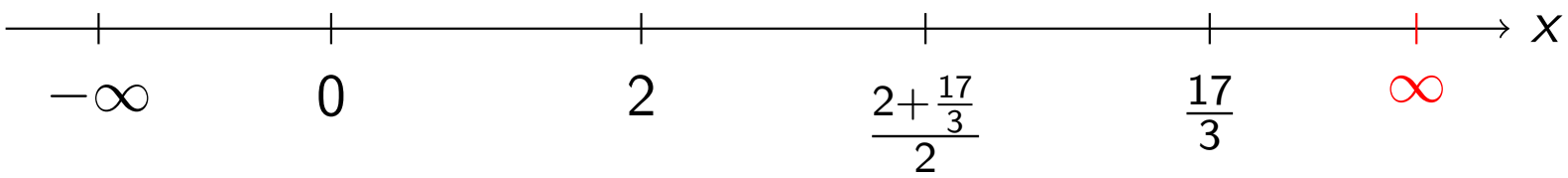

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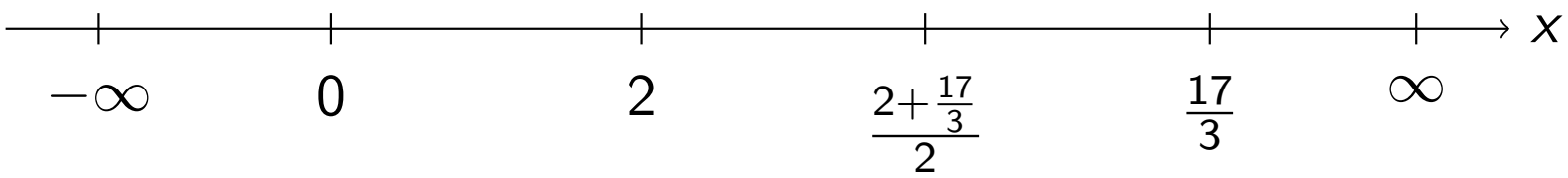

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\equiv & \mathit{true} & \text{evaluate}
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Quantifier Elimination in Real-Closed Fields

Definition (Quantifier elimination)

A first-order theory admits *quantifier elimination* if to each formula ϕ, a quantifier-free formula $\mathrm{QE}(\phi)$ can be effectively associated that is equivalent (i.e., $\phi \leftrightarrow \mathrm{QE}(\phi)$ is valid) and has no other free variables. The operation QE is further assumed to evaluate ground formulas (i.e., without variables), yielding a decision procedure for this theory.

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Theorem (Complexity, Davenport&Heintz'88,Weispfenning'88)

(Time and space) complexity of QE *for* $\mathbb{R}$ *is doubly exponential in the number of quantifier (alternations).*

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(Validity of) which of the following logics is decidable/semidecidable/undecidable/not semidecidable?

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Axioms of Reals

1. Commutative group $(\mathbb{R}, +)$: $\forall x \forall y \forall z\, x + (y + z) = (x + y) + z$
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3. Inverse $\forall x \exists y\, x + y = 0$
4. Abelian $\forall x \forall y\, (x + y = y + x)$
5. Commutative group $(\mathbb{R} \setminus \{0\}, \cdot)$: $\forall x \forall y \forall z\, x \cdot (y \cdot z) = (x \cdot y) \cdot z$
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7. Inverse $\forall x\, (x \neq 0 \rightarrow \exists y\, x \cdot y = 1)$
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9. Distributive $\forall x \forall y \forall z\, (x \cdot (y + z) = (x \cdot y) + (x \cdot z))$
10. Transitive $\forall x \forall y \forall z\, (x \geq y \wedge y \geq z \rightarrow x \geq z)$
11. Antisym. $\forall x \forall y\, (x \geq y \wedge y \geq x \rightarrow x = y)$
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13. Additive $\forall x \forall y \forall z\, (x \geq y \rightarrow x + z \geq y + z)$
14. Positive $\forall x \forall y\, (x \geq 0 \wedge y \geq 0 \rightarrow xy \geq 0)$
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But the first-order "view" of the reals is still fairly amazing

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	×	$\mathrm{FOL}_{\mathbb{Q}}$ undecidable	$\mathrm{FOL}_{\mathbb{R}}$ decidable

Relaxing Reals to Real Fields

Definition (Formally real field)

Field R is a *(formally) real field* iff, any of the following equivalent conditions holds:

1. -1 is not a sum of squares in R.
2. For every $x_1, \ldots, x_n \in R$, $\sum_{i=1}^{n} x_i^2 = 0$ implies $x_1 = \cdots = x_n = 0$.
3. R admits an ordering that makes R an ordered field.

Real-Closed Fields

Definition (Real-closed field)

Field R is *real-closed field* iff, equivalently:

1. R is an ordered field where every positive element is a square and every univariate polynomial in $R[X]$ of odd degree has a root in R (then this order is, in fact, unique).
2. R is not algebraically closed but its field extension $R[\sqrt{-1}] = R[i]/(i^2 + 1)$ is algebraically closed.
3. R is not algebraically closed but its algebraic closure is a finite extension, i.e., finitely generated over R.
4. R has the *intermediate value property*, i.e., R is an ordered field such that for any polynomial $p \in R[X]$ with $a, b \in R, a < b$ and $p(a)p(b) < 0$, there is a ζ with $a < \zeta < b$ such that $p(\zeta) = 0$.
5. R is a real field such that no proper algebraic extension is a formally real field.

Real-Closed Fields beyond the Reals

Example (Real-Closed Fields)

- Real numbers $\mathbb{R}$.

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- Real algebraic numbers $\bar{\mathbb{Q}} \cap \mathbb{R}$, that is, real numbers in the algebraic closure of $\mathbb{Q}$, i.e., real numbers that are roots of a non-zero polynomial with rational or integer coefficients

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- ZFC-Definable numbers, i.e., those real numbers $a \in \mathbb{R}$ for which there is a first-order formula φ in the set theory with one free variable such that a is the unique real number for which φ holds true.

$$\mathcal{I}, \beta \models \varphi \text{ iff } \beta(x) = a$$

The advantages of implicit definition over construction are roughly those of theft over honest toil [Russell]

First-Order Axiom Schemes of Real-Closed Fields

1. Commutative group $(\mathbb{R}, +)$: $\forall x \, \forall y \, \forall z \, x + (y + z) = (x + y) + z$
2. Neutral $\forall x \, x + 0 = x$
3. Inverse $\forall x \, \exists y \, x + y = 0$
4. Abelian $\forall x \, \forall y \, (x + y = y + x)$
5. Commutative group $(\mathbb{R} \setminus \{0\}, \cdot)$: $\forall x \, \forall y \, \forall z \, x \cdot (y \cdot z) = (x \cdot y) \cdot z$
6. Neutral $\forall x \, x \cdot 1 = x$
7. Inverse $\forall x \, (x \neq 0 \rightarrow \exists y \, x \cdot y = 1)$
8. Abelian $\forall x \, \forall y \, (x \cdot y = y \cdot x)$
9. Distributive $\forall x \, \forall y \, \forall z \, (x \cdot (y + z) = (x \cdot y) + (x \cdot z))$
10. $\neg \exists x_1 \, \ldots \exists x_n \, (-1 = x_1^2 + \cdots + x_n^2)$ for any n
11. $\forall x \, \forall y \, (x < y \wedge p(x)p(y) < 0 \rightarrow \exists z \, (x < z < y \wedge p(z) = 0))$ for polynomial p (intermediate value property)

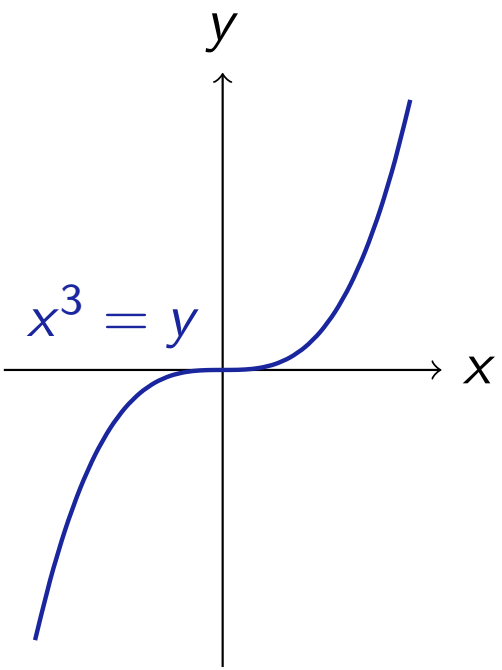
Definition (Real Affine Algebraic Variety)

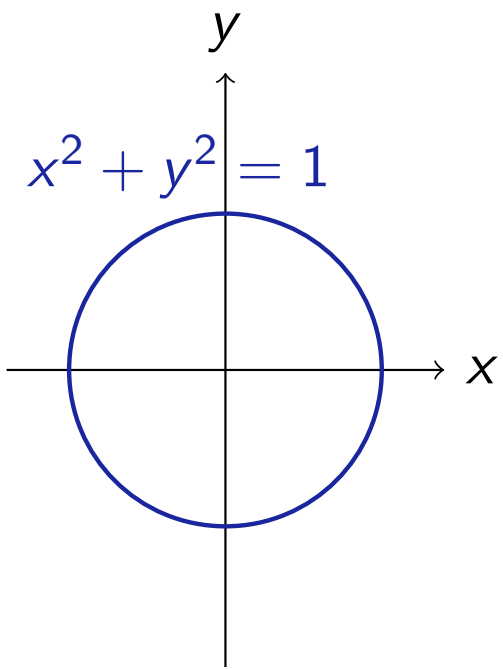
$V \subseteq \mathbb{R}^n$ is an *affine variety* iff, for some set $F \subseteq \mathbb{R}[X_1, \ldots, X_n]$ of polynomials over $\mathbb{R}$:

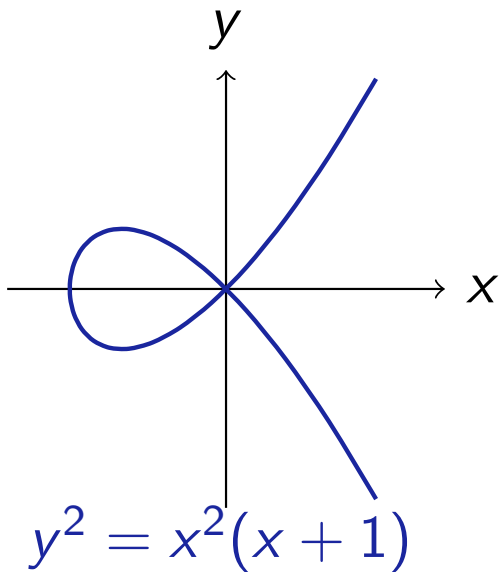
$$V = V(F) := \{x \in \mathbb{R}^n \ : \ f(x) = 0 \text{ for all } f \in F\}$$

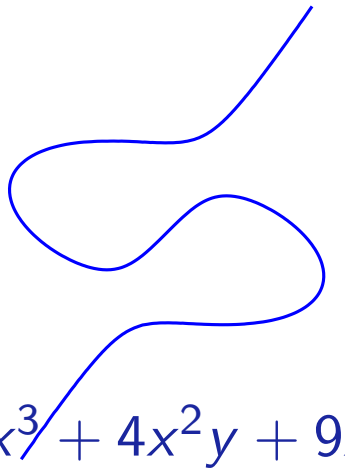
i.e., affine varieties are subsets of $\mathbb{R}^n$ that are definable by a set of polynomial equations.

Algebraic Variety Examples

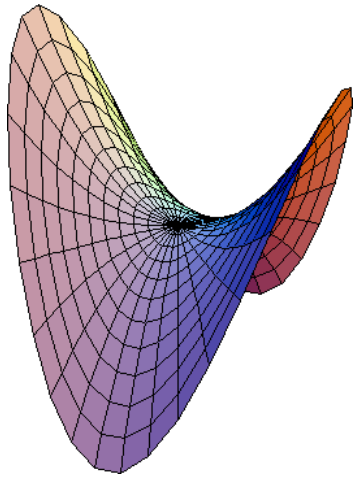








$z = x^2 - y^2$



Definition (Semialgebraic Set)

$S \subseteq \mathbb{R}^n$ is an *semialgebraic set* iff it is defined by a finite intersection of polynomial equations and inequalities or any finite union of such sets.

$$S = \bigcup_{i=1}^{t} \bigcap_{j=1}^{s} \{x \in \mathbb{R}^n \ : \ p(x) \sim 0\} \quad \text{with any } \sim \in \{=, \geq, >\}$$

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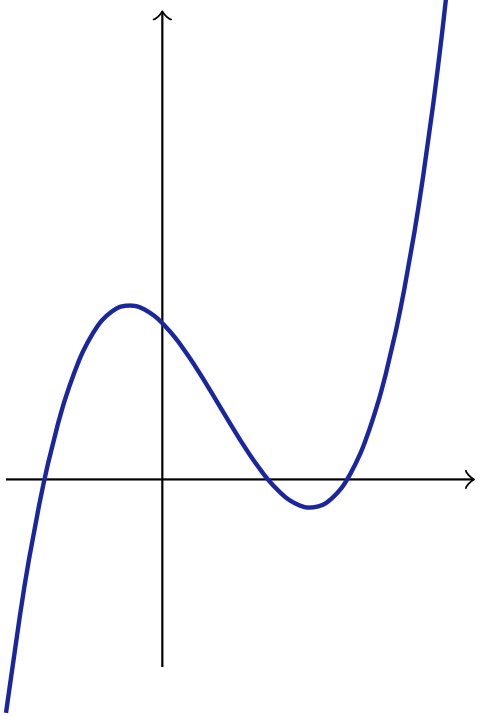
Theorem (Tarski'30,'51, Seidenberg'54)

Semialgebraic sets are closed under finite unions, finite intersections, complements and projection to linear subspaces.

Quantifier Elimination and Projection

$$\exists y\,(y \geq 0 \wedge 1 - x - 1.83x^2 + 1.66x^3 > y)$$



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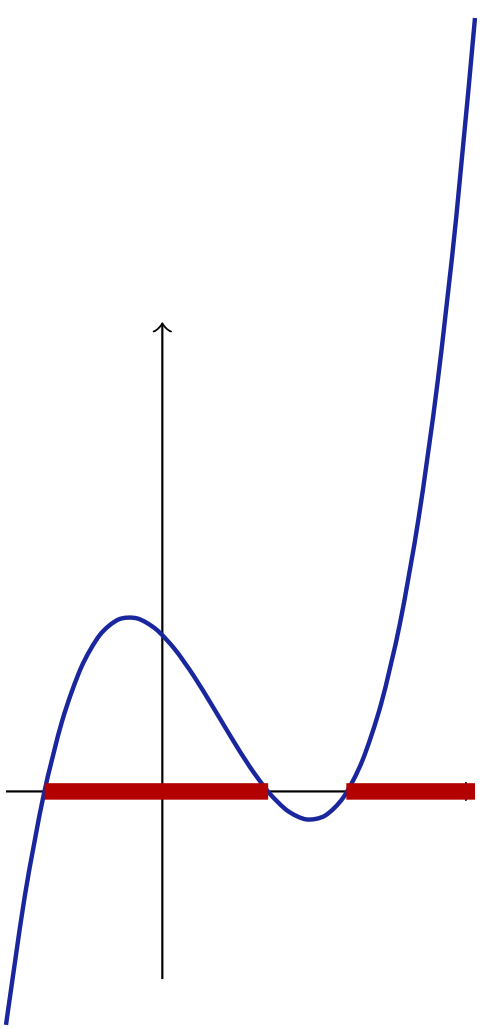

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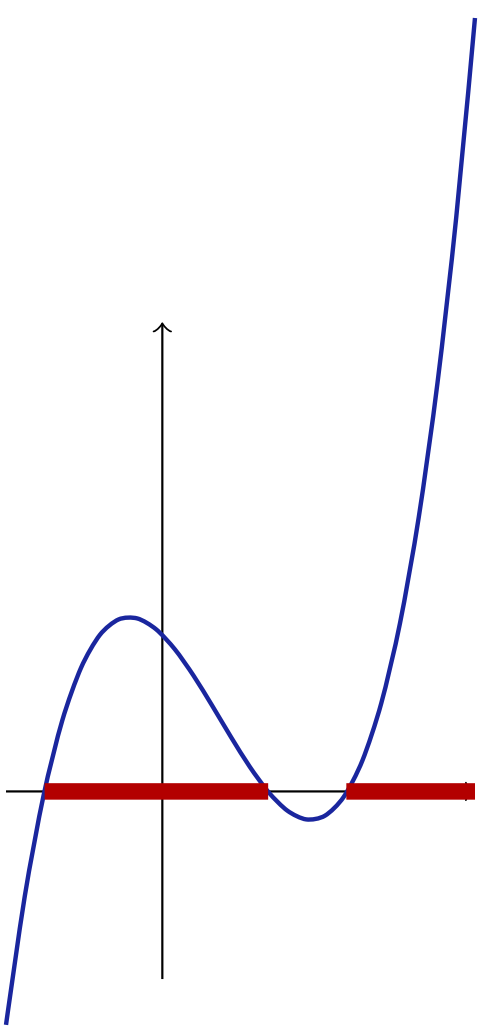

$$0.75 < x \wedge x < 0.68 \vee x > 1.17$$


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