Ghosts & Differential Ghosts

Nathan Fulton

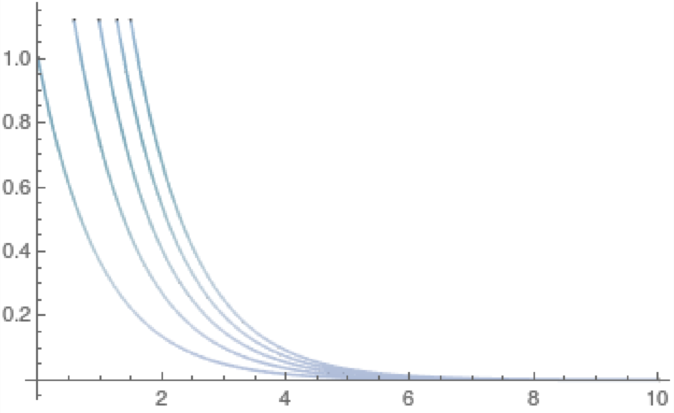
February 23, 2017

Differential Ghosts

A Failure of Differential Invariants and Cuts

Consider the system:

$$x > 0 \to [\{x' = -x\}]x > 0$$



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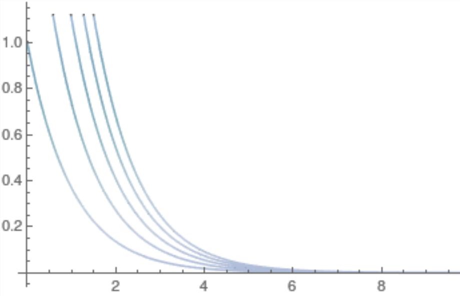
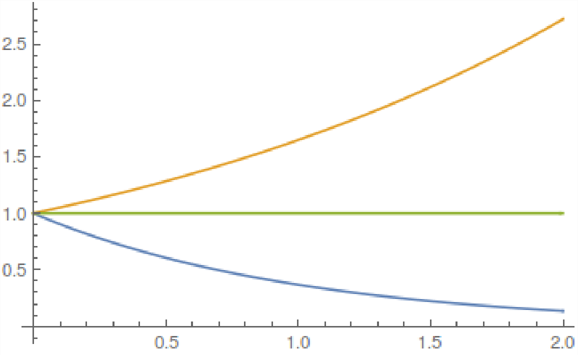


A Spooky Invariant



$x' = -x$

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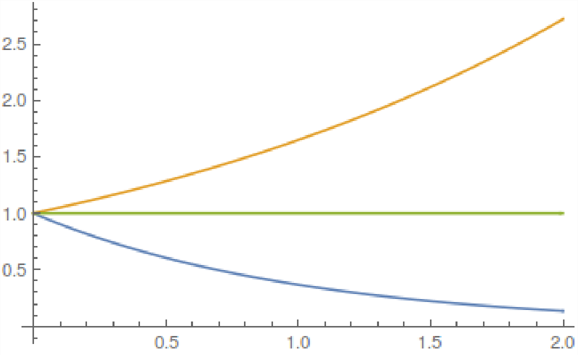

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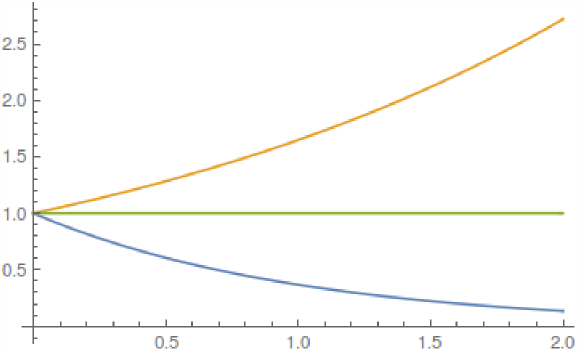

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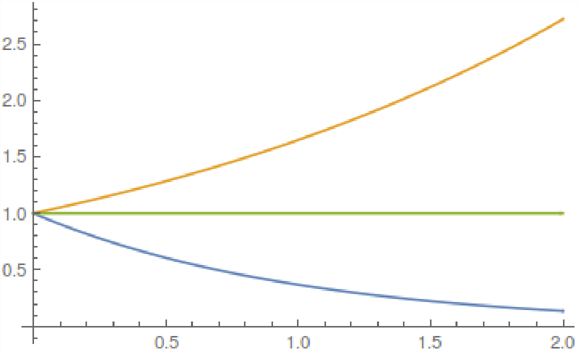

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$$\cfrac{\cfrac{y^2x - xy^2 = 0}{[x' := -x][y' := \frac{1}{2}y]2yy'x + y^2x' = 0}}{xy^2 = 1 \rightarrow [x' = -x, y' = \frac{1}{2}y]xy^2 = 1}$$

Differential Ghosts and Differential Auxiliaries

DG:

$$[x' = f(x) \& Q]P \leftrightarrow \exists y [x' = f(x), y' = a(x)y + b(x) \& Q]P$$

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Exercise: Derive DA from DG and MR.

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Exercise: Derive DA from DG and MR.

Example:

$$\frac{x > 0 \leftrightarrow \exists y . xy^2 = 1 \qquad xy^2 = 1 \vdash [x' = -x, y' = \frac{1}{2}y + 0]xy^2 = 1}{x > 0 \vdash [x' = -x]x > 0}$$

Finding a Ghost

To find a ghost argument for $x > 0 \rightarrow [\{x' = -x\}]x > 0$:

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$$\begin{aligned} (xy^2 = 1)' \quad x'y^2 + x2yy' &= 0 \\ x2yy' &= -x'y^2 \\ x2yy' &= -(-x)y^2 \\ y' &= \frac{xy^2}{2xy} \\ y' &= \frac{1}{2}y \end{aligned}$$

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We won't always get so lucky with y':

- Division by zero.
- $y' = \frac{1}{2}y$ has form $y' = ax + b$.

Using the Differential Ghost Axiom

Theorem

$$x > 0 \to [\{x' = -x\}]x > 0$$

Proof.

$$\text{DA}\ \dfrac{\dfrac{*}{x > 0 \leftrightarrow \exists y(xy^2 = 1)} \qquad \dfrac{\dfrac{*}{xy^2 = 1 \vdash xy^2 = 1} \qquad \dfrac{*}{2\frac{1}{2}yyx - xy^2 = 0}}{xy^2 = 1 \vdash [x' = -x, y' = \frac{1}{2}y]xy^2 = 1}\ \text{DI}}{x > 0 \to [\{x' = -x\}]x > 0}$$

$\square$

Tactic: implyR(1); ODE(1)

Proving Fixed Points using Ghosts

Theorem

$$x = 0 \to [\{x' = -x\}]x = 0$$

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$$x = 0 \leftrightarrow \exists y(xy = 0 \land y > 0).$$

2. Compute ghost: $xy = 0 \leftrightarrow x'y + y'x = 0 \leftrightarrow \cdots \leftrightarrow$ **y'=y**.

Proving Fixed Points using Ghosts

$$
\dfrac{\dfrac{*}{x = 0 \leftrightarrow \exists y (xy = 0 \land y > 0)} \qquad \dfrac{\dfrac{\vdash xy - xy = 0 \land {\color{red} y > 0}}{\vdash [x' := -x][y' := y](xy = 0 \land y > 0)}}{xy = 0 \land y > 0 \vdash [\{x' = -x, y' = y\}](xy = 0 \land y > 0)}}{x = 0 \rightarrow [\{x' = -x\}]x = 0}
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But we know how to prove properties like this already!

Proving Fixed Points using (two!) Ghosts

$$\dfrac{\cdots \qquad \text{BOXAND}\ \dfrac{\dfrac{xy - xy = 0}{xy = 0 \vdash [x' = -x, y' = y]xy = 0} \qquad \dfrac{\mathit{TODO}}{y > 0 \vdash [x' = -x, y' = y]y > 0}}{xy = 0 \land y > 0 \vdash [x' = -x, y' = y](xy = 0 \land y > 0)}}{x = 0 \to [\{x' = -x\}]x = 0}$$

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1. Choose an R: $y > 0 \leftrightarrow \exists z. z^2 y = 1$.
2. Calculate

$$(z^2 y = 1)' \quad 2zz'y + y'z^2 = 0$$
$$z' = \frac{-yz^2}{2zy}$$
$$z' = -\frac{1}{2}z$$

Proving Fixed Points using (two!) Ghosts

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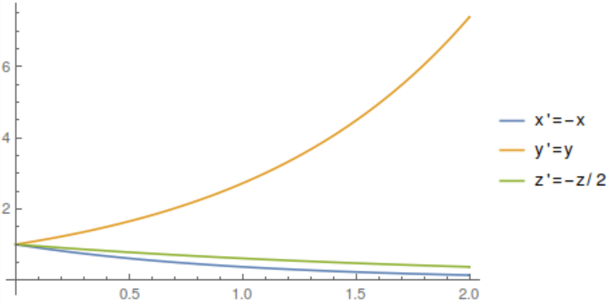
Completed Proof



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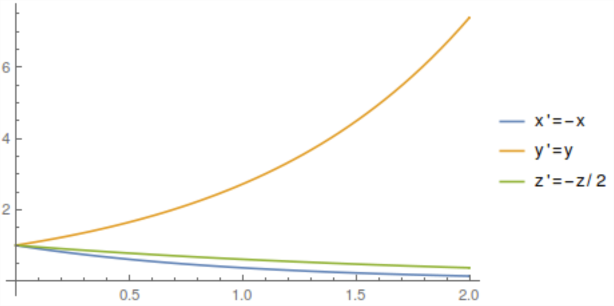

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Proving Weak Inequalities Using Ghosts

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$$\dfrac{\dfrac{*}{x \geq 0 \vdash x > 0 \lor x = 0} \qquad \dfrac{\dfrac{\dfrac{\text{Lemma 1}}{x > 0 \vdash [\{x' = -x\}]x > 0}}{x \geq 0, x > 0 \vdash [\{x' = -x\}]x \geq 0} \qquad \dfrac{\dfrac{\text{Lemma 2}}{x = 0 \vdash [\{x' = -x\}]x = 0}}{x \geq 0, x = 0 \vdash [\{x' = -x\}]x \geq 0}}{x \geq 0, x > 0 \lor x = 0 \vdash [\{x' = -x\}]x \geq 0}}{x \geq 0 \rightarrow [\{x' = -x\}]x \geq 0}$$

Arithmetic Ghosts and Fancier Equilibrium Points

Theorem (An Example from the 1D Blue Sky Bifurcation)
$x' = r + x^2$ *has an equilibrium point at* $-\sqrt{-r}$ *when* $r < 0$.

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Problem: d$\mathcal{L}$ does not have $\sqrt{\cdot}$.

Solution: *Arithmetic* ghosts!

Theorem (Example with $s = \sqrt{-r}$)

$$r < 0 \land r = -(s * s) \land x = -s \rightarrow [x' = r + x^2]x = -s$$

Arithmetic Ghosts and Fancier Equilibrium Points

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Proof.

$$\dfrac{\dfrac{\dfrac{\dfrac{*}{-y(x-s)(x+s) + y(r+x^2) = 0}}{y(x+s) = 0 \vdash [x', y']y(x+s) = 0} \qquad \dfrac{\dfrac{\dfrac{*}{z^2(x-s)y - z^2(x-s)y = 0}}{z^2 y = 1 \vdash [x', y' = -(x-s)y, z' = (x-s)z/2]z^2 y = 1}}{y > 0 \vdash [x' = r + x^2, y' = -(x-s)y]y > 0}}{y(x+s) = 0 \land y > 0 \vdash [x' = r + x^2, y' = -(x-s)y]y(x+s) = 0 \land y > 0}}{r < 0 \land r = -(s * s) \land x = -s \rightarrow [x' = r + x^2]x = -s}$$

□

Axiomatic ODE Solver

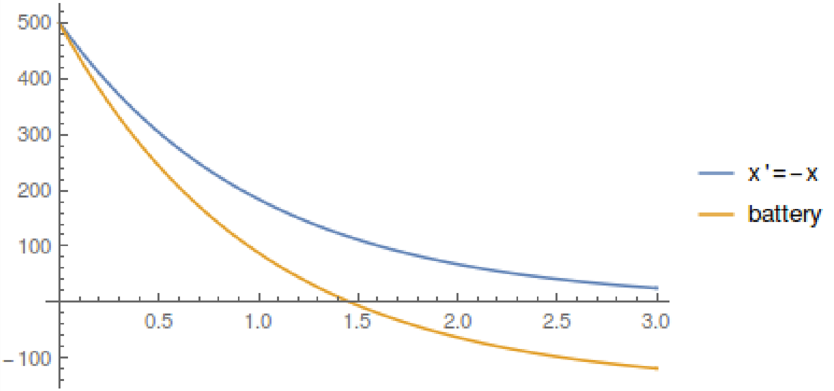
Axiomatic ODE Solver

See solvable ghosts in http://symbolaris.com/course/fcps16/12-diffghost-slides.pdf

Case Study: Battery Depletion

Theorem (Battery lasts at least T time units)

$a > 0 \land T > 0 \land x > 0 \land b \geq axT \land t = 0 \rightarrow$
$[\{x' = -x, b' = -ax, t' = 1 \& t \leq T\}]b \geq 0$



Battery Depletion Model

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$[\{x' = -x, b' = -ax, t' = 1 \& t \leq T\}] b \geq 0$

Proof Idea:

1) $x > 0$	Diff Aux
2) $b \geq b_0 - ax_0 t$ is invariant	dI, (1), $\{b_0, x_0, a\} \subset \mathbb{R}_+$
3) $b_0 - ax_0 t \geq b_0 - ax_0 T$ is invariant	dI and $t \leq T$
4) $b_0 - ax_0 T \geq 0$ is invariant	trivial
5) $b \geq 0$	2,3,4