15-819M: Data, Code, Decisions

11: Proving Loop Properties

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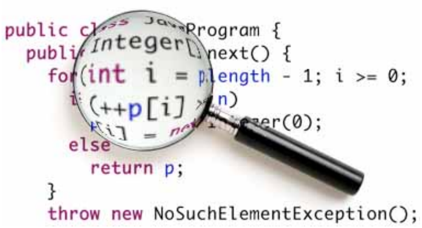

Outline

1. Motivation
2. Basic Invariant Rule
3. Anonymising Update
4. Improved Invariant Rule
5. Literature

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Proving Loop

Symbolic execution of loops: unwind

$$\text{unwindLoop}\ \frac{\Gamma \Longrightarrow \mathcal{U}[\pi\ \textbf{if}\ \texttt{(b)}\ \{\texttt{p;}\ \textbf{while}\ \texttt{(b)}\ \texttt{p}\}\ \omega]\phi, \Delta}{\Gamma \Longrightarrow \mathcal{U}[\pi\ \textbf{while}\ \texttt{(b)}\ \texttt{p}\ \omega]\phi, \Delta}$$

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How to handle a loop with...

- 0 iterations?

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How to handle a loop with...

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How to handle a loop with...

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- 10 iterations?

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- 0 iterations? Unwind $1\times$
- 10 iterations? Unwind $11\times$

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How to handle a loop with...

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  (and don't make any plans for the rest of the day)

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We need an invariant rule (or some other form of induction)

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Loop Invariants

Idea behind loop invariants

- A formula *Inv* whose validity is preserved by loop guard and body
- Consequence: if *Inv* was valid at start of the loop, then it still holds after arbitrarily many loop iterations
- If the loop terminates at all, then *Inv* holds afterwards
- Encode the desired postcondition after loop into *Inv*

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- Context Γ, Δ, $\mathcal{U}$ must be omitted in 2nd and 3rd premise:

  Γ, Δ in general don't hold in state defined by $\mathcal{U}$

  2nd premise *Inv* must be invariant for any state, not only $\mathcal{U}$

  3rd premise We don't know the state after the loop exits

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  3rd premise We don't know the state after the loop exits
- But: context contains (part of) precondition and class invariants
- Required context information must be added to loop invariant *Inv*

Example

```
int i = 0;
while(i < a.length) {
    a[i] = 1;
    i++;
}
```

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Precondition: $!\,\mathtt{a} \doteq \mathtt{null}$

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Precondition: $!\,\mathtt{a} \doteq \mathtt{null}$ & *ClassInv*

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& *ClassInv'*

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  Analogous situation: ∀-Right quantifier rule $\Longrightarrow \forall x; \phi$
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  To change value of program location use update, not substitution
- Anonymising updates $\mathcal{V}$ erase information about modified locations

$$\mathcal{V} = \{\mathtt{i} := * \mid\mid \backslash\mathtt{for}\ x;\ \mathtt{a}[x] := *\}$$

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- Context is kept as far as possible
- Invariant does not need to include unmodified locations
- For `assignable \everything` (the default):
  - $\mathcal{V} = \{* := *\}$ wipes out **all** information
  - Equivalent to basic invariant rule
  - Avoid this! Always give a specific `assignable` clause

Example with Improved Invariant Rule

```
int i = 0;
while(i < a.length) {
    a[i] = 1;
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Precondition: $!\, \mathtt{a} \doteq \mathtt{null}$

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Example in JML/JAVA— Demo lect14/Loop.java

```
public int[] a;
/*@ public normal_behavior
  @  ensures (\forall int x; 0<=x && x<a.length; a[x]==1);
  @  diverges true;
  @*/
public void m() {
  int i = 0;
  /*@ loop_invariant
    @  (0 <= i && i <= a.length &&
    @   (\forall int x; 0<=x && x<i; a[x]==1));
    @ assignable i, a[*];
    @*/
    while(i < a.length) {
      a[i] = 1;
      i++;
    }
  }
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Hints

Proving `assignable`

- The invariant rule assumes that `assignable` is correct
  E.g., with `assignable \nothing;` one can prove nonsense
- Invariant rule of KeY generates proof obligation that ensures correctness of `assignable`

Hints

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Setting in the KeY Prover when proving loops

- Loop treatment: Invariant
- Quantifier treatment: No Splits with Progs
- If program contains `*`, `/`:
  Arithmetic treatment: DefOps
- Is search limit high enough (time out, rule apps.)?
- When proving partial correctness, add `diverges true;`

Total Correctness

Find a decreasing integer term v (called variant)

Add the following premisses to the invariant rule:

- $v \geq 0$ is initially valid
- $v \geq 0$ is preserved by the loop body
- v is strictly decreased by the loop body

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- Remove directive `diverges true;`
- Add directive `decreasing v;` to loop invariant
- KeY creates suitable invariant rule and PO (with $\langle \ldots \rangle \phi$)

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Example (Same loop as above)

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@ decreasing
```

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Example (Same loop as above)

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@ decreasing  a.length - i;
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Literature for this Lecture

Essential

KeY Book Verification of Object-Oriented Software (see course web page), Chapter 3: Dynamic Logic (Section 3.7)