15-819M: Data, Code, Decisions

07: Reasoning about While Programs with Dynamic Logic

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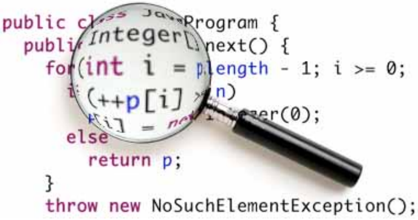

Outline

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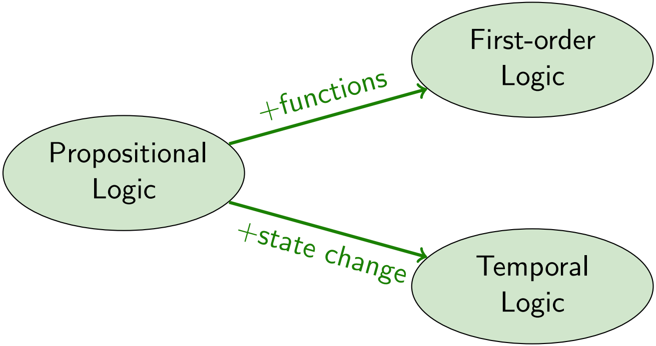
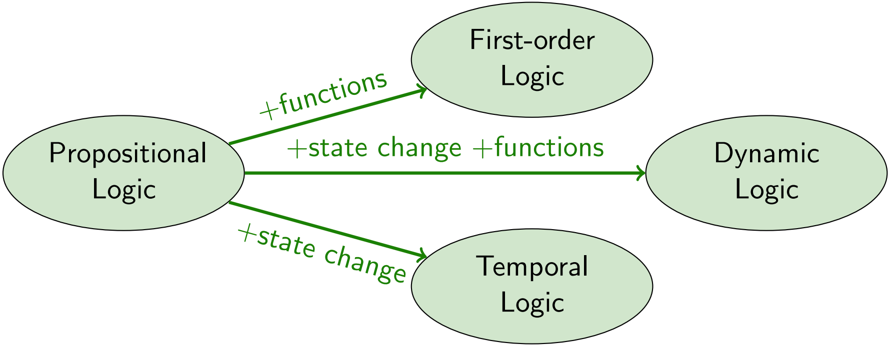
Beyond Propositional Logic



Beyond Propositional Logic



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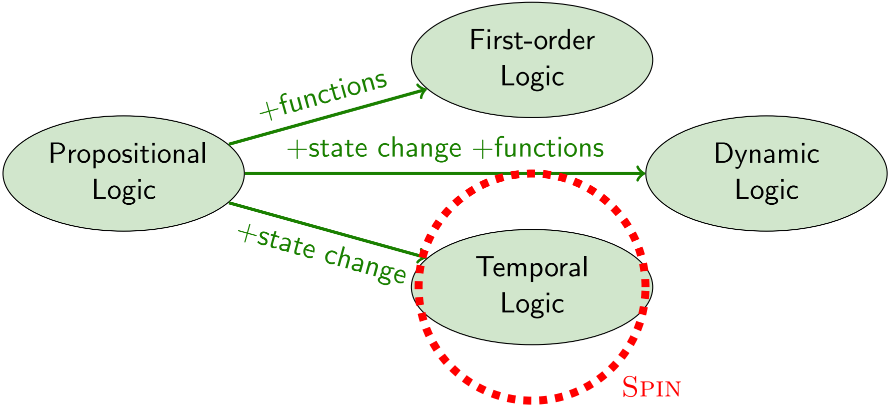

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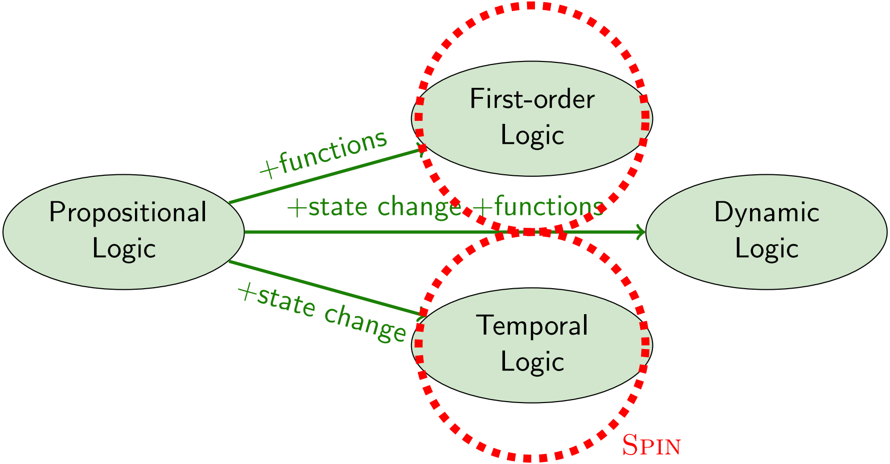

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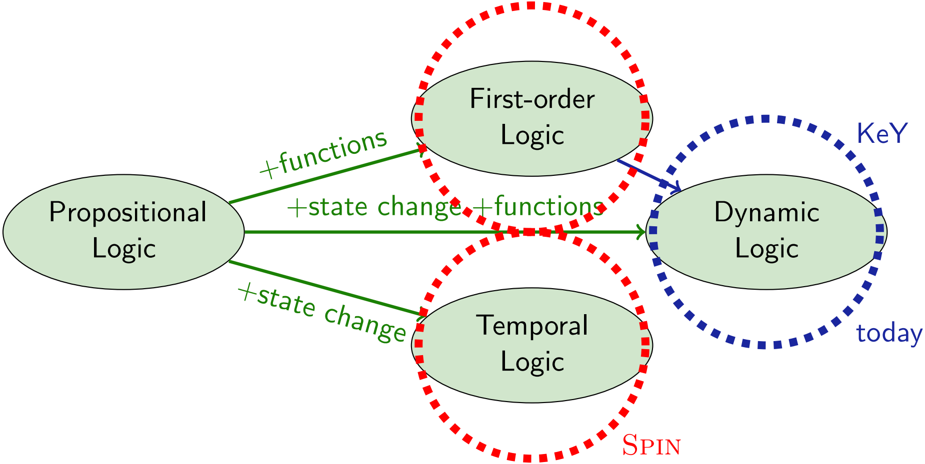

State Dependence of Formula Evaluation

Closed FOL formula either evaluates to true or false in a model $\mathcal{M}$
Consider model $\mathcal{M} = (\mathcal{D}, \delta, \mathcal{I})$ as program state

Let `x` be (local) program variable or attribute
Execution of program `p` may change program state, i.e., value of `x`

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Example

Executing `x=3;` results in $\mathcal{M}$ such that $\mathcal{M} \models \mathtt{x} \doteq 3$
Executing `x=4;` results in $\mathcal{M}$ such that $\mathcal{M} \not\models \mathtt{x} \doteq 3$

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Need a logic to capture state before/after program execution

Rigid versus Flexible Symbols

Signature of program logic defined as in FOL, but:
In addition there are program variables, attributes, etc.

Rigid versus Flexible Symbols

- Rigid symbols have same interpretation in all program states
  - First-order variables (logical variables for quantification)
    Used to hold initial values of program variables
  - Built-in functions and predicates such as `0,1,...,+,*,...,<,...`
- Flexible (or non-rigid) symbols where interpretation depends on state

  Capture side effects on state during program execution
  - Functions modeling program variables and attributes are flexible

Any term containing at least one flexible symbol is also flexible

Signature of Dynamic Logic (Simple Version)

Definition (Dynamic Logic Signature)

First-order signature $\Sigma = (\mathsf{PSym}_r, \mathsf{FSym}_r, \mathsf{FSym}_{nr}, \alpha)$

Rigid predicate symbols	$\mathsf{PSym} = \{>, >=, \ldots\}$
Rigid function symbols	$\mathsf{FSym} = \{+, -, *, 0, 1, \ldots, \mathbf{true}, \mathbf{false}\}$
Flexible function symbols	$\mathsf{FSym} = \{i, j, k, \ldots, p, q, r, \ldots\}$

Type hierarchy

$\mathrm{T} = \{\bot, \mathbf{int}, \mathbf{boolean}, \top\}$ with $\mathbf{int}$, $\mathbf{boolean}$ incomparable

Standard typing: $\mathbf{boolean}$ $\mathbf{true}$; $<(\mathbf{int},\mathbf{int})$;, etc.

Variables

Definition (First-Order/Logical Variables)

Typed logical variables (rigid), declared as `T x;`

Definition

Program Variables Flexible constants **int** `i`; **boolean** `p` used as program variables

Terms

- First-order terms defined as in FOL
- First-order terms may contain rigid and flexible symbols
- $\mathsf{FSym}_r \cap \mathsf{FSym}_{nr} = \emptyset$

Example

Signature for FSym_{nr}: $\mathbf{int}\ \mathtt{j};\ \mathbf{boolean}\ \mathtt{p}$
Variables $\mathbf{int}\ x;\ \mathbf{boolean}\ b;$

- $\mathtt{j}$ and $\mathtt{j}{+}x$ are flexible terms of type $\mathbf{int}$
- $\mathtt{p}$ is a flexible term of type $\mathbf{boolean}$
- $x + x$ is a rigid term of type $\mathbf{int}$
- $\mathtt{j}{+}b$ and $\mathtt{j} + \mathtt{p}$ are not well-typed

Atomic Programs

Definition (Atomic Programs)

The atomic programs Π_0 are assignments of the form $\mathtt{j} = t$ where:

- `T j;` is a program variable (flexible constant)
- t is a first-order term of type `T` without logical variables

Example

Signature for FSym_{nr}: **int** `j`; **boolean** `p`
Variables **int** x; **boolean** b;

- `j=j+1`, `j=0` and `p=`**false** are assignments
- `j=j+`x contains a logical variable on the right
- x`=1` contains a logical variable on the left
- $\mathtt{j} \doteq \mathtt{j}$ is equality, not assignment
- `p=0` is ill-typed

Dynamic Logic Programs (Simple Version)

Definition (Program)

Inductive definition of the set of (DL) programs Π:

- If π is an atomic program, then $\pi;$ is a program
- If p and q are programs, then pq is a program
- If b is a variable-free term of type boolean, p and q programs, then

  if (b) p **else** q; **if** (b) p;

  are programs
- If b is a variable-free term of type boolean, p a program, then

  while (b) p;

  is a program

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  is a program

Programs contain no logical variables!

Dynamic Logic Programs

Example (Admissible Program)

Signature for $FSym_{nr}$: **int** `r;` **int** `i;` **int** `n;`
Signature for $FSym_{r}$: **int** `0;` **int** `+(`**int**`,`**int**`);` **int** `-(`**int**`,`**int**`);`
Signature for $PSym_{r}$: `<(`**int**`,`**int**`);`

```
i=0;
r=0;
while (i<n) {
  i=i+1;
  r=r+i;
};
r=r+r-n;
```

Dynamic Logic Programs

Example (Admissible Program)

Signature for FSym_{nr}: **int** `r`; **int** `i`; **int** `n`;
Signature for FSym_{r}: **int** `0`; **int** `+(`**int**`,`**int**`)`; **int** `-(`**int**`,`**int**`)`;
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```
i=0;
r=0;
while (i<n) {
  i=i+1;
  r=r+i;
};
r=r+r-n;
```

Which value does the program compute in `r`?

Dynamic Logic Formulas (Simple Version)

Definition (Dynamic Logic Formulas (DL Formulas))

- Each FOL formula is a DL formula
- If p is a program and ϕ a DL formula then $\langle \mathrm{p} \rangle \phi$ is a DL formula
- If p is a program and ϕ a DL formula then $[\mathrm{p}]\phi$ is a DL formula
- DL formulas closed under FOL quantifiers and connectives

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- DL formulas closed under FOL quantifiers and connectives

- Program variables are flexible constants: never bound in quantifiers
- Program variables need not be declared or initialized in program
- Programs contain no logical variables
- Modalities can be arbitrarily nested

Dynamic Logic Formulas

Example (Well-formed? If yes, under which signature?)

- $\forall\, \mathbf{int}\ y; ((\langle \texttt{x = 1;} \rangle \texttt{x} \doteq y)\ \texttt{<->}\ (\langle \texttt{x = 1*1;} \rangle \texttt{x} \doteq y))$

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  Well-formed if FSym_{nr} contains $\mathbf{int}\ \texttt{x;}$

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- $\exists\, \mathbf{int}\ x;\ [\mathtt{x\ =\ 1;}](\mathtt{x} \doteq 1)$

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  Well-formed if FSym_{nr} contains $\mathbf{int}$ `x;`
- $\exists\,\mathbf{int}\ x;\ [\texttt{x = 1;}](\texttt{x} \doteq 1)$

  Not well-formed, because logical variable occurs in program
- $\langle \texttt{x = 1;} \rangle ([\mathbf{while}\ (\mathbf{true})\ \texttt{\{\};}]\mathbf{false})$

  Well-formed if FSym_{nr} contains $\mathbf{int}$ `x;`
  program formulas can be nested

Outline

Dynamic Logic Semantics: States

First-order model can be considered as program state

- Interpretation of flexible symbols can change from state to state (program variables, attribute values)
- Interpretation of rigid symbols is the same in all states (built-in functions and predicates)

States as first-order models

From now, consider program state as first-order model $\mathcal{M} = (\mathcal{D}, \delta, \mathcal{I})$

- Only interpretation $\mathcal{I}$ of flexible symbols in FSym_{nr} can change
  $\Rightarrow$ only track values of $f \in \mathsf{FSym}_{nr}$: use s (for state) instead of $\mathcal{M}$
- Set of all states s is S

Dynamic Logic Semantics: Kripke Structure

Definition (Kripke Structure / Labelled Transition System)

Kripke structure or Labelled transition system $K = (S, \rho)$

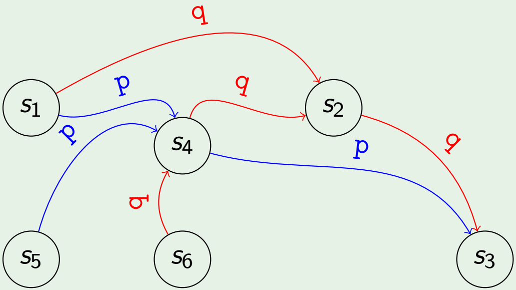
- State (=first-order model) $s = (\mathcal{D}, \delta, \mathcal{I}) \in S$
- Transition relation $\rho : \Pi \to (S \to S)$
  - ρ is the operational semantics of programs Π
  - Each program $\mathrm{p} \in \Pi$ transforms a start state s into end state $\rho(\mathrm{p})(s)$
  - $\rho(\mathrm{p})(s)$ can be undefined: p does not terminate when started in s
  - Our programs are deterministic (unlike PROMELA):
    $\rho(\mathrm{p})$ is a (partial) function (at most one value)

Dynamic Logic Semantics: Kripke Structure

Example (Kripke Structure)

Two programs p and q

Show $\rho(\mathtt{p})$ and $\rho(\mathtt{q})$, states $S = \{s1, \ldots, s_6\}$



When p is started in s_5 it terminates in s_4, etc.

In general, Π and S are infinite!

Semantic Evaluation of Program Formulas

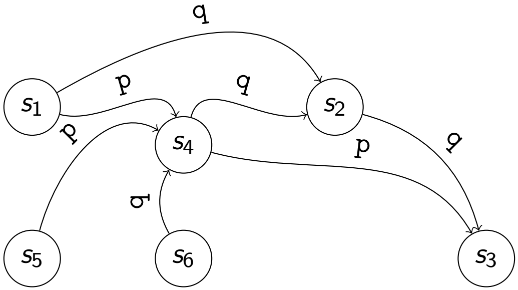
Definition (Validity Relation for Program Formulas)

- $s, \beta \models \langle \mathrm{p} \rangle \phi$ iff $\rho(\mathrm{p})(s), \beta \models \phi$ and $\rho(\mathrm{p})(s)$ is defined

  p terminates and ϕ is true in the final state after execution

- $s, \beta \models [\mathrm{p}] \phi$ iff $\rho(\mathrm{p})(s), \beta \models \phi$ whenever $\rho(\mathrm{p})(s)$ is defined

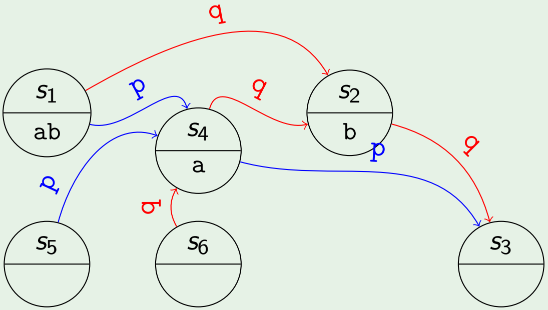
  If p terminates then ϕ is true in the final state after execution



Example (Semantic Evaluation of Program Formulas)

Signature FSym_{nr}: $\mathbf{boolean}\ \mathtt{a};\ \mathbf{boolean}\ \mathtt{b};$

Notation: $\mathcal{I}(x) = T$ iff x appears in node



Question 1: $s_1 \models \langle \mathrm{p} \rangle (\mathtt{a} \doteq \mathbf{true})$?

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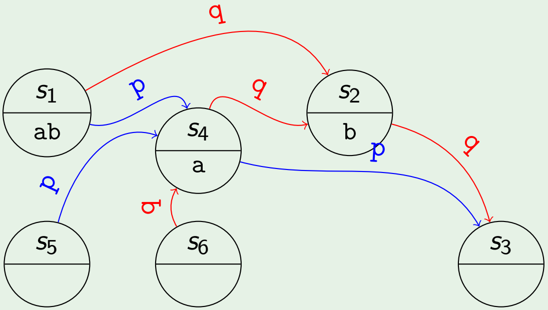

Question 2: $s_1 \models \langle \mathtt{q} \rangle (\mathtt{a} \doteq \mathbf{true})$?

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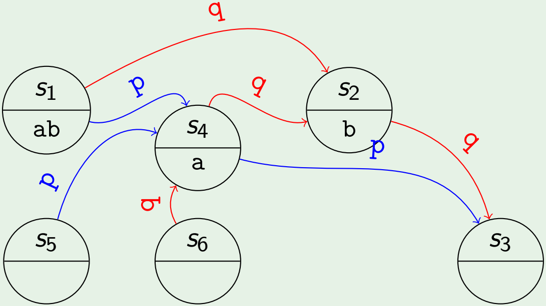

Question 3: $s_5 \models \langle \mathtt{q} \rangle (\mathtt{a} \doteq \mathbf{true})$?

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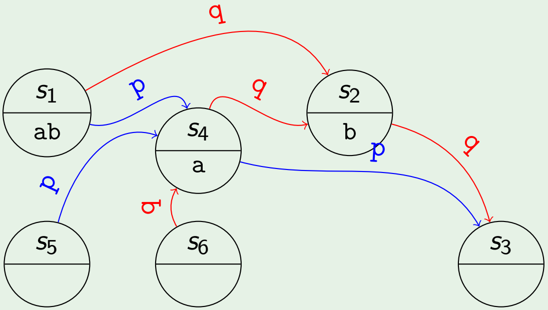

Question 4: $s_5 \models [\mathtt{q}](\mathtt{a} \doteq \mathbf{true})$?

Program Correctness

Definition (Notions of Program Correctness)

- If $s, \beta \models \langle \mathrm{p} \rangle \phi$ then
  p totally correct (with respect to ϕ) in s, β
- If $s, \beta \models [\mathrm{p}]\phi$ then
  p partially correct (with respect to ϕ) in s, β

- Duality $\langle \mathrm{p} \rangle \phi$ iff $![\mathrm{p}]\,!\,\phi$
  Exercise: justify this using the semantics
- Implication if $\langle \mathrm{p} \rangle \phi$ then $[\mathrm{p}]\phi$
  Total correctness implies partial correctness
  - converse is false
  - holds only for deterministic programs!

Semantics of Sequents

$\Gamma = \{\phi_1, \ldots, \phi_n\}$ and $\Delta = \{\psi_1, \ldots, \psi_m\}$ sets of program formulas where all logical variables occur bound

Recall: $s \models (\Gamma \Longrightarrow \Delta)$ iff $s \models (\phi_1 \;\&\; \cdots \;\&\; \phi_n) \;\text{->}\; (\psi_1 \mid \cdots \mid \psi_m)$

Define semantics of DL sequents like semantics of FOL sequents

Definition (Validity of Sequents over Program Formulas)

A sequent $\Gamma \Longrightarrow \Delta$ over program formulas is valid iff

$$s \models (\Gamma \Longrightarrow \Delta) \text{ in all states } s$$

Semantics of Sequents

$\Gamma = \{\phi_1, \ldots, \phi_n\}$ and $\Delta = \{\psi_1, \ldots, \psi_m\}$ sets of program formulas where all logical variables occur bound

Recall: $s \models (\Gamma \Longrightarrow \Delta)$ iff $s \models (\phi_1 \mathbin{\&} \cdots \mathbin{\&} \phi_n) \;\text{->}\; (\psi_1 \mid \cdots \mid \psi_m)$

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Consequence for program variables

Initial value of program variables implicitly "universally quantified"

Initial States

Java initial states

KeY prover "starts" programs in initial states according to Java convention:

- Values of array entries initialized to default values: **int**[] to 0, etc.
- Static object initialization
- No objects created

How to restrict validity to set of initial states $S_0 \subseteq S$?

1. Design closed FOL formula Init with
   $s \models \mathrm{Init}$ iff $s \in S_0$
2. Use sequent $\Gamma, \mathrm{Init} \Longrightarrow \Delta$

Later: simple method for specifying initial value of program variables

Outline

Operational Semantics of Programs

In labelled transition system $K = (S, \rho)$:
$\rho : \Pi \to (S \to S)$ is operational semantics of programs $\mathrm{p} \in \Pi$

How is ρ defined for concrete programs and states?

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How is ρ defined for concrete programs and states?

Example (Operational semantics of assignment)

State s interprets flexible symbols f with $\mathcal{I}_s(f)$

$\rho(\mathtt{x=t})(s) = s'$ where s' identical to s except $\mathcal{I}_{s'}(x) = val_s(t)$

Very tedious task to define ρ for JAVA ...
$\Rightarrow$ here we go directly to calculus for program formulas!

Outline

Symbolic Execution of Programs

Sequent calculus decomposes top-level operator in formula
What is "top-level" in a sequential program `p; q; r` ?

Symbolic Execution (King, late 60s)

- Follow the natural control flow when analyzing a program
- Values of some variables unknown: symbolic state representation

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Symbolic Execution (King, late 60s)

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Example

Compute the final state after termination of

```
int x; int y; x=x+y; y=x-y; x=x-y;
```

Symbolic Execution of Programs

General form of rule conclusions in symbolic execution calculus

$$\langle \texttt{stmt; rest} \rangle \phi, \qquad [\texttt{stmt; rest}]\phi$$

- Rules must symbolically execute first statement
- Repeated application of rules in a proof corresponds to symbolic program execution

Symbolic execution of assignment

$$\text{assign}\ \frac{\{\mathtt{x}/\mathtt{x}_{old}\}\Gamma,\ \mathtt{x} \doteq \{\mathtt{x}/\mathtt{x}_{old}\}t \quad \Longrightarrow \quad \langle \mathtt{rest}\rangle\phi,\ \{\mathtt{x}/\mathtt{x}_{old}\}\Delta}{\Gamma \Longrightarrow \langle \mathtt{x\ =\ t;\ rest}\rangle\phi, \Delta}$$

$\mathtt{x}_{old}$ new program variable that "rescues" old value of $\mathtt{x}$

Symbolic Execution of Programs

Symbolic execution of assignment

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Example

Conclusion matching: $\{\mathtt{x}/\mathtt{x}\}$, $\{\mathtt{t}/\mathtt{x+y}\}$, $\{\mathtt{rest}/\mathtt{y=x-y;\ \ x=x-y;}\}$, $\{\phi/(\mathtt{x} \doteq y_0\ \&\ \mathtt{y} \doteq x_0)\}$, $\{\Gamma/\mathtt{x} \doteq x_0, \mathtt{y} \doteq y_0\}$, $\{\Delta/\emptyset\}$

$$\frac{\mathtt{x}_{old} \doteq x_0,\ \mathtt{y} \doteq y_0,\ \mathtt{x} \doteq \mathtt{x}_{old}+\mathtt{y} \Longrightarrow \langle \mathtt{y=x-y;\ \ x=x-y;}\rangle(\mathtt{x} \doteq y_0\ \&\ \mathtt{y} \doteq x_0)}{\mathtt{x} \doteq x_0,\ \mathtt{y} \doteq y_0 \Longrightarrow \langle \mathtt{x=x+y;\ \ y=x-y;\ \ x=x-y;}\rangle(\mathtt{x} \doteq y_0\ \&\ \mathtt{y} \doteq x_0)}$$

Proving Partial Correctness

Partial correctness assertion

If program p is started in a state satisfying Pre and terminates, then its final state satisfies Post

In Hoare logic $\{\text{Pre}\}\ \text{p}\ \{\text{Post}\}$ (Pre, Post must be FOL)

In DL Pre $->$ [p]Post (Pre, Post any DL formula)

Proving Partial Correctness

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In Hoare logic $\{\text{Pre}\}\ p\ \{\text{Post}\}$ (Pre, Post must be FOL)

In DL $\text{Pre} \ {-}{>}\ [p]\text{Post}$ (Pre, Post any DL formula)

Example (In KeY Syntax, automatic proof)

```
\programVariables {
  int x; int y; }
\problem {
  (\forall int x0; \forall int y0; ((x=x0 & y=y0) ->
     \<{x=x+y; y=x-y; x=x-y;}\>(x=y0 & y=x0)))
}

  lect11/swap.key
```

(Demo)

Advanced Properties

Example

$\forall\, T\ \ y;\ ((\langle\mathtt{p}\rangle\mathtt{x} \doteq y)\ \ \texttt{<->}\ \ (\langle\mathtt{q}\rangle\mathtt{x} \doteq y))$

Advanced Properties

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$\forall\, T\ y;\ ((\langle \mathtt{p} \rangle \mathtt{x} \doteq y)\ \ \texttt{<->}\ \ (\langle \mathtt{q} \rangle \mathtt{x} \doteq y))$

Not valid in general

Programs $\mathtt{p}$ behave $\mathtt{q}$ equivalently on variable $T\ \mathtt{x}$

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$\forall\ T\ y;\ (((\langle \mathtt{p} \rangle \mathtt{x} \doteq y)\ \ \texttt{<->}\ \ (\langle \mathtt{q} \rangle \mathtt{x} \doteq y))$

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Example

$\exists\ T\ y;\ (\mathtt{x} \doteq y\ \ \texttt{->}\ \ \langle \mathtt{p} \rangle \mathbf{true})$

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Not valid in general

Programs p behave q equivalently on variable T x

Example

$\exists\, T\ y;\ (\mathrm{x} \doteq y\ \texttt{->}\ \langle \mathrm{p} \rangle \mathbf{true})$

Not valid in general

Program p terminates in all states where x has suitable initial value

Symbolic Execution of Programs

Symbolic execution of conditional

$$\text{if}\ \frac{\Gamma, \mathtt{b} \doteq \mathbf{true} \Longrightarrow \langle \mathtt{p;\ rest} \rangle \phi, \Delta \qquad \Gamma, \mathtt{b} \doteq \mathbf{false} \Longrightarrow \langle \mathtt{q;\ rest} \rangle \phi, \Delta}{\Gamma \Longrightarrow \langle \mathbf{if}\ \mathtt{(b)\ \{\ p\ \}}\ \mathbf{else}\ \mathtt{\{\ q\ \}\ ;\ rest} \rangle \phi, \Delta}$$

Symbolic execution must consider all possible execution branches

Symbolic Execution of Programs

Symbolic execution of conditional

$$\text{if}\ \frac{\Gamma, \mathtt{b} \doteq \mathbf{true} \Longrightarrow \langle \mathtt{p};\ \mathtt{rest} \rangle \phi, \Delta \qquad \Gamma, \mathtt{b} \doteq \mathbf{false} \Longrightarrow \langle \mathtt{q};\ \mathtt{rest} \rangle \phi, \Delta}{\Gamma \Longrightarrow \langle \mathbf{if}\ \mathtt{(b)}\ \mathtt{\{\ p\ \}}\ \mathbf{else}\ \mathtt{\{\ q\ \}\ ;\ rest} \rangle \phi, \Delta}$$

Symbolic execution must consider all possible execution branches

Symbolic execution of loops: unwind

$$\text{unwindLoop}\ \frac{\Gamma \Longrightarrow \langle \mathbf{if}\ \mathtt{(b)}\ \mathtt{\{\ p;}\ \mathbf{while}\ \mathtt{(b)\ p\};\ r} \rangle \phi, \Delta}{\Gamma \Longrightarrow \langle \mathbf{while}\ \mathtt{(b)\ \{p\};\ r} \rangle \phi, \Delta}$$

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As previous: $\forall\, T\ i_0;\ (i_0 \doteq \mathtt{i}\ \texttt{->}\ \langle \ldots \mathtt{i} \ldots \rangle \phi)$

Solution

Use explicit construct to record values in current state

Update $\forall\, T\ i_0;\ (\{\mathtt{i} := i_0\} \langle \ldots \mathtt{i} \ldots \rangle \phi)$

Explicit State Updates

Updates specify computation state where formula is evaluated

Definition (Syntax of Updates)

If v is program variable, t FOL term type-compatible with v,
t' any FOL term, and ϕ any DL formula, then

- $\{\mathrm{v} := t\}t'$ is DL term
- $\{\mathrm{v} := t\}\phi$ is DL formula

Definition (Semantics of Updates)

State s interprets flexible symbols f with $\mathcal{I}_s(f)$
β variable assignment for logical variables in t

$\rho(\{\mathrm{v} := t\})(s) = s'$ where s' identical to s except $\mathcal{I}_{s'}(x) = val_{s,\beta}(t)$

Explicit State Updates

Facts about updates $\{\mathtt{v} := t\}$

- Update semantics identical to assignment
- Value of update depends on logical variables in t: use β
- Updates as "lazy" assignments (no term substitution done)
- Updates are not assignments: right-hand side is FOL term

  $\{\mathtt{x} := n\}\phi$ cannot be turned into assignment (n logical variable)

  $\langle\mathtt{x=i++;}\rangle\phi$ cannot directly be turned into update
- Updates are not equations: change value of flexible terms

Computing Effect of Updates (Automatic)

Rewrite rules for update followed by ...

program variable $\begin{cases} \{\mathrm{x} := t\}\mathrm{y} \rightsquigarrow \mathrm{y} & \text{where } x \neq y \\ \{\mathrm{x} := t\}\mathrm{x} \rightsquigarrow t & \end{cases}$

logical variable $\{\mathrm{x} := t\}w \rightsquigarrow w$

complex term $\{\mathrm{x} := t\}f(t_1, \dots, t_n) \rightsquigarrow f(\{\mathrm{x} := t\}t_1, \dots, \{\mathrm{x} := t\}t_n)$

FOL formula $\begin{cases} \{\mathrm{x} := t\}(\phi \ \& \ \psi) \rightsquigarrow \{\mathrm{x} := t\}\phi \ \& \ \{\mathrm{x} := t\}\psi \\ \dots \\ \{\mathrm{x} := t\}(\forall\, T\ y; \phi) \rightsquigarrow \forall\, T\ y; (\{\mathrm{x} := t\}\phi) \end{cases}$

program formula $\{\mathrm{x} := t\}(\langle \mathrm{p} \rangle \phi) \rightsquigarrow \{\mathrm{x} := t\}(\langle \mathrm{p} \rangle \phi)$ unchanged!

Update computation delayed until p symbolically executed

Assignment Rule Using Updates

Symbolic execution of assignment using updates

$$\text{assign}\ \frac{\Gamma \Longrightarrow \{\mathtt{x} := t\}\langle\mathtt{rest}\rangle\phi, \Delta}{\Gamma \Longrightarrow \langle\mathtt{x\ =\ t;\ rest}\rangle\phi, \Delta}$$

- Avoids renaming of program variables
- Works as long as t has no side effects (ok in simple DL)

Demo

`Examples/lect11/swap.key`

Example Proof

Example

```
\programVariables {
  int x;
}
\problem {
  (\exists int y;
    ({x := y}\<{while (x > 0) {x = x-1;}}\> x=0 ))
}
```

Intuitive Meaning? Satisfiable? Valid?

Demo

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Examples/lect11/term.key

What to do when we cannot determine a concrete loop bound?

Parallel Updates

How to apply updates on updates?

Example

Symbolic execution of

```
int x; int y; x=x+y; y=x-y; x=x-y;
```

yields:

```
{x := x+y}{y := x-y}{x := x-y}
```

Need to compose three sequential state changes into a single one!

Parallel Updates

Definition (Parallel Update)

A parallel update is expression of the form $\{l_1 := v_1 || \cdots || l_n := v_n\}$ where each $\{l_i := v_i\}$ is simple update

- All v_i computed in old state before update is applied
- Updates of all locations l_i executed simultaneously
- Upon conflict $\quad l_i = l_j,\ v_i \neq v_j \quad$ later update $(\max\{i, j\})$ wins

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Definition (Composition Sequential Updates/Conflict Resolution)

$$\{l_1 := r_1\}\{l_2 := r_2\} \;=\; \{l_1 := r_1 || l_2 := \{l_1 := r_1\}r_2\}$$

$$\{l_1 := v_1 || \cdots || l_n := v_n\}\mathrm{x} \;=\; \begin{cases} \mathrm{x} & \text{if } \mathrm{x} \notin \{l_1, \ldots, l_n\} \\ v_k & \text{if } \mathrm{x} = l_k,\ \mathrm{x} \notin \{l_{k+1}, \ldots, l_n\} \end{cases}$$

Parallel Updates

Example

```
({x := x+y}{y := x-y}){x := x-y} =
{x := x+y || y := (x+y)-y}{x := x-y} =
{x := x+y || y := (x+y)-y || x := (x+y)-((x+y)-y)} =
{x := x+y || y := x || x := y} =
{y := x || x := y}
```

KeY automatically deletes overwritten (unnecessary) updates

Demo

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Parallel updates to store intermediate state of symbolic computation

A Warning

First-order rules that substitute arbitrary terms

$$\exists\text{−right}\ \frac{\Gamma \Longrightarrow [x/t']\,\phi,\ \exists\, T\, x;\ \phi, \Delta}{\Gamma \Longrightarrow \exists\, T\, x;\ \phi, \Delta} \qquad \forall\text{−left}\ \frac{\Gamma, \forall\, T\, x;\ \phi,\ [x/t']\,\phi \Longrightarrow \Delta}{\Gamma, \forall\, T\, x;\ \phi \Longrightarrow \Delta}$$

$$\text{applyEq}\ \frac{\Gamma, t \doteq t', [t/t']\,\psi \Longrightarrow [t/t']\,\phi, \Delta}{\Gamma, t \doteq t', \psi \Longrightarrow \phi, \Delta}$$

t, t' must be rigid, because all occurrences must have the same value

Example

$$\frac{\Gamma, \mathtt{i} \doteq 0 \mathrel{-\!\!>} \langle \mathtt{i++} \rangle \mathtt{i} \doteq 0 \Longrightarrow \Delta}{\Gamma, \forall\, T\, x;\ (x \doteq 0 \mathrel{-\!\!>} \langle \mathtt{i++} \rangle x \doteq 0) \Longrightarrow \Delta}$$

Logically valid formula would result in unsatisfiable antecedent!

KeY prohibits unsound substitutions

Outline

Literature for this Lecture

Essential

KeY Book Verification of Object-Oriented Software, Chapter 10: Using KeY

KeY Book Verification of Object-Oriented Software, Chapter 3: Dynamic Logic (Sections 3.1, 3.2, 3.4, 3.5, 3.6.1, 3.6.3, 3.6.4)